# ACM201 a - 2 - Four important Linear Partial Differential Equations

## 1 Overview

- The transport Equation $u_t + b.Du = 0$.
- Laplace's equation $\Delta u = 0$.
- The heat equation $u_t - \Delta u = 0$
- The wave equation $u_{tt} - \Delta u = 0$

## 2 Transport Equation

### 2.1 Homogeneous Problem

The simplest PDE. Let $b \in \mathbb{R}^n$ and $g \in \mathbb{R}^n \to \mathbb{R}$ (we assume $g$ to be $C^1$).

Find $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} u_t + b.Du = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1}$$

Set $z(s) := u(x + sb, t + s)$. Observe that $\dot{z}(t) = 0$. Deduce that

**Theorem 2.1.** *The solution of (1) is*

$$u(x, t) = g(x - tb) \tag{2}$$

### 2.2 Nonhomogeneous Problem

Let $b \in \mathbb{R}^n$, $g \in \mathbb{R}^n \to \mathbb{R}$ (we assume $g$ to be $C^1$) and $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ (we assume $f$ to be continuous)

Find $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} u_t + b.Du = f \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{3}$$

Set $z(s) := u(x + sb, t + s)$. Observe that $\dot{z}(t) = f(x + sb, t + s)$. Deduce that

**Theorem 2.2.** *The solution of (3) is*

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds \tag{4}$$

*Remark* 2.3. We have solved the PDE by converting them into ODE. This is a special case of the method of characteristics

**Exercise 2.4.** Write down an explicit formula for a function $u$ solving the initial value problem

$$\begin{cases} u_t + b.Du + cu = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{5}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

# 3 Laplace's Equation

Let $U$ be an open subset of $\mathbb{R}^d$. $f : U \to \mathbb{R}$ continuous. $u : \bar{U} \to \mathbb{R}$ is the unknown

- The Laplace equation[1]

$$\Delta u = 0 \tag{6}$$

[1]Pierre-Simon Laplace was born of poor parents in 1749. His mathematical ability early won him good teaching posts;as a political opportunist, he ingratiated himself with whichever party happened to be in power during the uncertain days of the French Revolution.

During the years 1784-1787 he produced some memoirs of exceptional power. Prominent among these is one read in 1784, and reprinted in the third volume of the Méchanique céleste, in which he completely determined the attraction of a spheroid on a particle outside it. This is memorable for the introduction into analysis of spherical harmonics or Laplace's coefficients, as also for the development of the use of the potential - a name first given by Green in 1828.

If the co-ordinates of two points be $(r, \mu, \omega)$ and $(r', \mu', \omega')$, and if $r' < r$, then the reciprocal of the distance between them can be expanded in powers of r/r', and the respective coefficients are Laplace's coefficients. Their utility arises from the fact that every function of the co-ordinates of a point on the sphere can be expanded in a series of them. It should be stated that the similar coefficients for space of two dimensions, together with some of their properties, had been previously given by Legendre in a paper sent to the French Academy in 1783. Legendre had good reason to complain of the way in which he was treated in this matter.

This paper is also remarkable for the development of the idea of the potential, which was appropriated from Lagrange, who had used it in his memoirs of 1773, 1777 and 1780. Laplace showed that the potential always satisfies the differential equation

$$\Delta V = 0$$

and on this result his subsequent work on attractions was based. The quantity $\Delta V$ has been termed the concentration of $V$, and its value at any point indicates the excess of the value of V there over its mean value in the neighborhood of the point.

The five-volume Traite de mecanique celeste, which earned him the title of "the Newton of France," embraced all previous discoveries in this field along with Laplace's own contributions, and marked the author as the unrivaled master in the subject. It may be of interest to repeat a couple of anecdotes often told in connection with this word. When Napoleon teasingly remarked that God was not mentioned in his treatise, Laplace replied, "Sir, I did not need that hypothesis." Laplace died in 1827, exactly one hundred years after the death of Isaac Newton.

- The Poisson Equation[2]

$$\Delta u = f \tag{7}$$

The Laplace's equation is used to describe gravitational potential in absence of mass, to define electrostatic potential in absence of charges, and to describe temperature in a steady state heat flow. The Laplace's equation is often called the potential equation because $u$ defines the potential equation. The Poisson equation is an inhomogeneous Laplace equation and hence it is used to study all phenomena described by Laplace equation in the presence of external sources or sinks.

**Definition 3.1.** A function satisfying (6) is called a harmonic function

**Physical Interpretation** Equilibrium phenomenon: $u$ concentration of some chemical. $V$ smooth bounded open subset of $U$. $\mathbf{F}$ flux density of the chemical component. If the concentration is at equilibrium (we write $d\mathbf{S}$ the surface measure and $\nu$ the unit outer normal field)

$$\int_{\partial V} \mathbf{F}.\nu dS = 0$$

by the Green theorem

$$\int_{\partial V} \mathbf{F}.\nu dS = \int_V \operatorname{div} \mathbf{F} \, dx$$

According to one report, his last words were :"What we know is slight;what we don't know is immense. " The following story about Laplace is of interest and offers a valuable suggestion to one applying for a position. When Laplace arrived as a young man in Paris seeking a professorship of mathematics. He submitted his recommendations by prominent people to d'Alembert, but he was not received. Returning to his lodgings, Laplace wrote d'Alembert a brilliant letter on the general principles of mechanics. This opened the door and d'Alembert replied: "Sir, you notice that I paid little attention to your recommendations. You don't need any; you have introduced yourself better. " A few days later, Laplace was appointed professor of mathematics at the Military School of Paris.

[2] Poisson, Siméon-Denis (1781-1840). French mathematician and student of Laplace whose devotion to mathematics led him to make the statement that life was only good for two things: to do mathematics and to teach it (Boyer 1968, p. 569). Poisson extended the theory of physical mechanics to replace analytical mechanics, publishing Traité de mécanique (2 vols., 1811, 1833). This theory was based on the hypothesis of molecular and forces. He showed that a particle placed in the interior of two similarly oriented ellipsoidal shells (a homeoid ) experiences no force. He also applied mathematics to electricity and magnetism, formulating the extension of Laplace's equation known as poisson's equation. He noted that the potential must be constant on the surface of a conductor, and analytically found the charge density on two spherical conductors in terms of digamma functions. He also formulated the theory of bound surface current and volume magnetization.

In 1819, during the judging of Fresnel's paper on diffraction at the Paris Academy, Poisson argued that the consequence of Fresnel's theory was that the center of the shadow of a diffracting disk should be illuminated. This unexpected effect was subsequent observed, verifying Fresnel's theory. His is also known for his work on probability, including what have now come to be known as Poisson Distributions. He derived what is now known as the weak law of large numbers. His work on probability was published in Recherches sur la probabilité des jugements (1837).

Thus for all $V \subset U$
$$\int_V \operatorname{div} \mathbf{F} \, dx = 0$$
Thus $\operatorname{div} \mathbf{F} = 0$ in $U$.
If the flux is linear with respect to the concentration gradient:
$$\mathbf{F} = -aDu \tag{8}$$
with $a > 0$, then $u$ satisfies the Laplace equation.

If $u$ denotes the

1. chemical concentration
2. temperature
3. electrostatic potential

Then (8) is the

1. Fick's law of diffusion
2. Fourier's law of heat conduction
3. Ohm's law of electrical conduction

## 3.1 Fundamental solution

Write $\alpha(n)$ the volume of the unit ball of $\mathbb{R}^n$.

**Definition 3.2.** The function
$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \ln |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & (n \geq 3) \end{cases} \tag{9}$$
defined for $x \in \mathbb{R}^n$, $x \neq 0$ is the fundamental solution of Laplace's equation.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable, with compact support. Define $u$ by
$$u := \Phi * f \tag{10}$$
We remind the convolution formula $(u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy)$.

**Theorem 3.3. Solving Poisson's equation**. *Define $u$ by (10). Then*

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f$ *in* $\mathbb{R}^n$.

*Proof.* Direct computation by isolating the singularity of $\Phi$ at 0 with a ball $B(0, \epsilon)$ and taking the limit $\epsilon \downarrow 0$ □

**Exercise 3.4.** Show that the functions $u(x, y) = x^2 - y^2$ and $u(x, y) = e^x \sin(y)$ are solutions of the equation
$$\Delta u = 0$$

## 3.2 Mean-value formulas

**Theorem 3.5.** *If $u \in C^2(U)$ is harmonic then*

$$u(x) = \frac{1}{\int_{\partial B(x,r)} dS} \int_{\partial B(x,r)} u \, dS = \frac{1}{\int_{B(x,r)} dy} \int_{B(x,r)} u \, dy \tag{11}$$

*for each ball $B(x,r) \in U$.*

*Proof.* Write $\phi(r) := \int_{B(0,1)} u(x + rz) \, dS(z)$. Observe by computation and Green's formula that $\phi'(r) = 0$. □

**Theorem 3.6.** *If $u \in C^2(U)$ satisfies*

$$u(x) = \frac{1}{\int_{\partial B(x,r)} dS} \int_{\partial B(x,r)} u \, dS = \frac{1}{\int_{B(x,r)} dy} \int_{B(x,r)} u \, dy \tag{12}$$

*for each ball $B(x,r) \in U$, then $u$ is harmonic.*

*Proof.* Assume that there exists a ball $B(x,r) \subset U$ such that $\Delta u > 0$ on that ball, observe that $\phi'(r) > 0$ on that ball leading to a contradiction. □

**Exercise 3.7.** We say $v \in C^2(\bar{U})$ is subharmonic if $-\Delta v \leq 0$ in $U$.

1. Prove for subharmonic $v$ that
$$v(x) \leq \frac{1}{\int_{B(x,r)} dy} \int_{B(x,r)} dy \quad \forall B(x,r) \subset U$$
2. Prove therefore that $\max_{\bar{U}} v = \max_{\partial U} v$
3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that $u$ is harmonic and $v := \phi(u)$. Prove $v$ is subharmonic.
4. Prove $v := |Du|^2$ is subharmonic whenever $u$ is harmonic.

## 3.3 Properties of harmonic functions

Assume $U \subset \mathbb{R}^n$ to be open and bounded

### 3.3.1 Strong maximum principle, uniqueness

**Theorem 3.8. Strong Maximum principle**. *Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U*

1. *then*
$$\max_{\bar{U}} u = \max_{\partial U} u$$

*2. Furthermore, if $U$ is connected and there exists a point $x_0 \in U$ such that*

$$u(x_0) = \max_{\bar{U}} u$$

*then $u$ is constant within $U$*

**Theorem 3.9. Uniqueness**. *Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary value problem*

$$\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases} \tag{13}$$

### 3.3.2 Regularity

**Theorem 3.10. Smoothness**. *If $u \in C(U)$ satisfies the mean value property (11) for each ball $B(x,r) \subset U$ then*

$$u \in C^\infty(U)$$

Note that $u$ may not be continuous up to $\partial U$.

### 3.3.3 Local estimates for harmonic functions

**Theorem 3.11. Estimates on derivatives**. *Assume $u$ is harmonic in $U$, then*

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

*for each ball $B(x_0,r) \subset U$ and each multi index $\alpha$ of order $|\alpha| = k$. Here*

$$\begin{cases} C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)} \quad (k = 1, \ldots) \end{cases} \tag{14}$$

### 3.3.4 Liouville's theorem

**Theorem 3.12. Liouville's theorem**. *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then $u$ is constant.*

**Theorem 3.13. Representation formula**. *Let $f \in C_c^2(\mathbb{R}^n)$, $n \geq 3$. Then any bounded solution of*

$$-\Delta u = f \quad in \quad \mathbb{R}^n$$

*has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C \quad x \in \mathbb{R}^n$$

*for some constant $C$.*

### 3.3.5 Analyticity

We say that a function $u$ is analytic in an open subset $U \subset \mathbb{R}^n$ if and only if for all $x_0 \in U$ there exists an open subset $V \subset U$ containing $x_0$ and a sequence of real numbers $c_n(x_0)$ such that for all $x \in V$

$$u(x) = \sum_{n=0}^{\infty} c_n(x_0)(x - x_0)^n$$

**Theorem 3.14. Analyticity**. *Assume $u$ is harmonic in $U$. Then $u$ is analytic in $U$*

### 3.3.6 Harnack's inequality

**Theorem 3.15. Harnack's inequality**. *For each connected open set $V \subset\subset U$, there exists a positive constant $C$, depending only on $V$, such that*

$$\sup_V u \leq C \inf_V u$$

*for all nonnegative harmonic functions in $U$*

Thus in particular

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

For all $x, y \in V$. These inequalities assert that the values of a non negative harmonic function within $V$ are comparable: $u$ cannot be very small (or very large) at any point of $V$ unless $u$ is very small (or very large) everywhere in $V$.

*Proof.* Let $r := \frac{1}{4} \operatorname{dist}(V, \partial U)$. Choose $x, y \in V$, $|x - y| \leq r$. Then

$$\begin{aligned} u(x) &= \frac{1}{\int_{B(x,2r)} dz} \int_{B(x,2r)} u \, dz \ geq \frac{1}{\alpha(n) 2^n r^n} \int_{B(y,r)} u \, dz \\ &= \frac{1}{2^n} \frac{1}{\int_{B(y,r)} dz} \int_{B(y,r)} u \, dz = \frac{1}{2^n} u(y) \end{aligned}$$

Thus $2^n u(y) \geq u(x) \geq \frac{1}{2^n} u(y)$ if $x, y \in V, \quad |x - y| \leq r$. Since $V$ is connected and $\bar{V}$ compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$ each of which has radius $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, \ldots, N$. Then

$$u(x) \geq \frac{1}{2^{nN}} u(y)$$

for all $x, y \in V$. ∎

#### 3.3.7 Green's function

Assume $U \subset \mathbb{R}^n$ is open, bounded, and $\partial U$ is $C^1$. Let $\Phi$ be defined by (9). Let $y \to \phi^x(y)$ be the solution of the following boundary value problem

$$\begin{cases} \Delta \phi^x = 0 \quad \text{in} \quad U \\ \phi^x = \Phi(x-y) \quad \text{on} \quad \partial U \end{cases} \tag{15}$$

**Definition 3.16.** Green's function for the region $U$ is

$$G(x,y) := \Phi(x-y) - \phi^x(y) \quad (x, y \in U,\, x \neq y) \tag{16}$$

Suppose now $u \in C^2(\bar{U})$ solves the boundary-value problem

$$\begin{cases} -\Delta u = f \quad \text{in} \quad U \\ u = g \quad \text{on} \quad \partial U \end{cases} \tag{17}$$

for given continuous functions $f, g$.

**Theorem 3.17. Representation formula using Green's function**. *If $u \in C^2(\bar{U})$ solves the problem (17), then*

$$u(x) = -\int_{\partial U} g(y) \frac{\partial G(x,y)}{\partial \nu} dS(y) + \int_U f(y) G(x,y)\, dy \quad (x \in U) \tag{18}$$

*Remark* 3.18. The reverse statement is also true.

*Remark* 3.19. Fix $x \in U$. Regarding $G$ as a function of $y$, we may symbolically write

$$\begin{cases} -\Delta G = \delta_x \\ G = 0 \quad \text{on} \quad \partial U \end{cases} \tag{19}$$

where $\delta_x$ is the Dirac distribution with unit Lebesgue on $x$. Equation (19) can be justified within the framework of Distribution theory.

**Theorem 3.20. Symmetry of Green's function**. *For all $x, y \in U$, $x \neq y$, we have*

$$G(x,y) = G(y,x) \tag{20}$$

**Particular Cases** . When $U$ is the half space or a ball one can derive an explicit formula for $G$.

- If $U = \mathbb{R}^n_+$ (we note $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$) then

$$G(x,y) = \Phi(y-x) - \Phi(y - x + 2x_n e_n) \tag{21}$$

- If $U = B(0,1)$ then

$$G(x,y) = \Phi(y-x) - \Phi\big(|x|(y - \frac{x}{|x|^2})\big) \tag{22}$$

**Theorem 3.21. Poisson's formula for half-space**. *Assume* $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ *and define* $u$ *by*

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy \quad (x \in \mathbb{R}^n_+). \tag{23}$$

*Then*

1. $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$
2. $\Delta u = 0$ *in* $\mathbb{R}^n_+$ *and*
3. $\lim_{x\to x^0, x\in\mathbb{R}^n_+} u(x) = g(x^0)$ *for each* $x^0 \in \partial\mathbb{R}^n_+$

**Theorem 3.22. Poisson's formula for ball**. *Assume* $g \in C(\partial B(0,1)) \cap L^\infty(\mathbb{R}^{n-1})$ *and define* $u$ *by*

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \quad (x \in B^0(0,r). \tag{24}$$

*Then*

1. $u \in C^\infty(B^0(0,r))$
2. $\Delta u = 0$ *in* $B^0(0,r)$ *and*
3. $\lim_{x\to x^0, x\in B^0(0,r)} u(x) = g(x^0)$ *for each* $x^0 \in \partial B(0,r)$.

### 3.3.8 Energy methods

Our first analysis of Laplace's equation was done on the basis of rather explicit formulae. Let us observe that some of the results obtained above can be easily obtained when the Laplace's equation is seen as an operator on Sobolev spaces. We will investigate this point further on 201$b$. For instance if $U$ is bounded and $C^1$ then let us define the following energy functional

$$I[w] := \int_U \big(\frac{1}{2}|Dw|^2 - wf\big)\, dx \tag{25}$$

on the set

$$\mathcal{A} := \{w \in C^2(\bar{U}) |\, w = g \text{on} \partial U\}$$

Consider the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad U \\ u = g & \text{on} \quad \partial U \end{cases} \tag{26}$$

**Theorem 3.23. Dirichlet's principle**. $u \in C^2(\bar{U})$ *solves (26) if and only if*

$$I[u] = \min_{w\in\mathcal{A}} I[w] \tag{27}$$

Observe that if $v$ is in $C^2(\bar{U})$ and null on $\partial U$ then

$$I[u+v] = I[u] + \frac{1}{2}\int_U |Dv|^2\,dx$$

This also gives the uniqueness of $u$.

**Exercise 3.24.** In $\mathbb{R}^3$, write the Laplacian in cylindrical coordinates and spherical coordinates.

**Exercise 3.25.** Prove that there exists a constant $C$, depending only on $n$, such that

$$\max_{B(0,1)} |u| \leq C\big(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|\big)$$

whenever $|u|$ is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in} \quad B(0,1) \\ u = g & \text{on} \quad \partial B(0,1) \end{cases} \tag{28}$$

# 4 Heat equation

We will study the heat equation

$$u_t - \Delta u = 0 \tag{29}$$

and the non homogeneous heat equation

$$u_t - \Delta u = f \tag{30}$$

Here $t > 0$, $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$ and the Laplacian is taken with respect to $x$. In (**??**) the function $f : U \times [0, \infty) \to \mathbb{R}$ is given.

**Physical interpretation** The Laplace equation describes the density $u$ of some quantity such as heat or chemical concentration. The heat equation describes the out of equilibrium dynamic of these quantities. If $V \subset U$ is any smooth subregion of $U$, the rate of change of the total quantity within $V$ equals the negative of the net flux through $\partial V$:

$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} \mathbf{F}.\nu dS. \tag{31}$$

where $\mathbf{F}$ is the flux density. Thus $u_t = -\operatorname{div}\mathbf{F}$. If the flux is linear with respect to the concentration gradient:

$$\mathbf{F} = -aDu \tag{32}$$

with $a > 0$, then $u$ satisfies "a heat" equation.

$$u_t = a\Delta u \tag{33}$$

Note that the heat equation is also called diffusion equation because it describes the dynamic of the probability density of the Brownian Motion.

## 4.1 The fundamental solution

**Definition 4.1.** The function

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t > 0) \\ 0 \quad (x \in \mathbb{R}^n, t < 0) \end{cases} \tag{34}$$

is called the fundamental solution of the heat equation.

**Lemma 4.2.** *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1$$

## 4.2 Initial-value problem

Let's consider the following initial-value (or Cauchy) problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{35}$$

We Define

$$\Phi(.,t) * g := \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy \tag{36}$$

**Theorem 4.3. Solution of initial value problem**. *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define $u$ by (36) ($u := \Phi(.,t) * g$). Then*

1. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,
2. $u_t(x,t) - \Delta u(x,t) = 0$ *($x \in \mathbb{R}^n, t > 0$) and*
3. $\lim_{(x,t)\to(x^0,0),t>0} u(x,t) = g(x^0)$ *for each point $x^0 \in \mathbb{R}^n$*

*Remark* 4.4. Thus the fundamental solution of the heat equation allows us to solve that equation for any initial condition. This fundamental solution can also be interpreted as the probability distribution of a Brownian Motion started from 0 at time $t$. Thus knowing the law of a stochastic process has allowed us to solve a PDE, this link between SDE and PDE is not restricted to the simple heat equation, it is a general phenomena (Ito Calculus) and also an active field of research (this link also exists with non linear PDE).

## 4.3 Non homogeneous problem

The nonhomogeneous initial-value problem can be written

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = 0 \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{37}$$

We assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and that $f$ has compact support. Let us define

$$u(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds \tag{38}$$

**Theorem 4.5. Solution of nonhomogeneous problem**. *Define $u$ by (38). Then*

1. $u \in C_1^2(\mathbb{R}^n \times (0, \infty))$,
2. $u_t(x,t) - \Delta u(x,t) = f(x,t)$ $(x \in \mathbb{R}^n, t > 0)$ *and*
3. $\lim_{(x,t)\to(x^0,0),t>0} u(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}^n$

*Remark* 4.6. By the linearity of the heat equation

$$u(x,t) := \Phi(.,t) * g + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds \tag{39}$$

solves

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{40}$$

## 4.4 Mean value formula

Assume $U \subset \mathbb{R}^n$ is open and bounded, fix a time $T > 0$.

**Definition 4.7.** 1. We define the parabolic cylinder

$$U_t := U \times (0, T]$$

2. The parabolic boundary of $U_T$ is

$$\Gamma_T := \bar{U}_T - U_T$$

**Definition 4.8.** For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$ we define

$$E(x,t,r) := \left\{(y,s) \in \mathbb{R}^{n+1} \mid s \leq t, \Phi(x-y, t-s) \geq \frac{1}{r^n}\right\}$$

Observe that the boundary of that region is a level set of $\Phi(x-y, t-s)$ ($(x,t)$ is located on the boundary because of time direction).

**Theorem 4.9. A Mean value property for the heat equation**. *Let $u \in C_1^2(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds \tag{41}$$

*For each $E(x,t,r) \subset U_T$*

*Proof.* Direct computation (shift and rescale the problem so that $x = 0$, $t = 0$. Observe that the derivative of the right hand side of (41) with respect to 0 is null. Take the limit $r \downarrow 0$. □

## 4.5 Properties of solutions

### 4.5.1 Strong maximum principle, uniqueness

**Theorem 4.10. Strong maximum principle for the heat equation**. *Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation in $U_T$*

1. *Then*
$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u$$

2. *Furthermore if $U$ is connected and there exists a point $(x_0, t_0) \in U_T$ such that*
$$u(x_0, t_0) = \max_{\bar{U}_T} u$$
*the $u$ is constant in $\bar{U}_{t_0}$*

*Remark* 4.11. So if $u$ attains it maximum (or minimum) at an interior point, then $u$ is constant at all earlier times.

**Theorem 4.12. Uniqueness on bounded domains**. *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem.*

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad U_T \\ u = g & \text{on} \quad \Gamma_T \end{cases} \tag{42}$$

**Theorem 4.13. Maximum principle for the Cauchy problem**. *Assume $u \in C_1^2\big(\mathbb{R}^n \times (0,T]\big) \cap C\big(\mathbb{R}^n \times [0,T]\big)$ solves*

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,T) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{43}$$

*and satisfies the growth estimate*

$$u(x,t) \le Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \le t \le T)$$

*for some constants $A, a > 0$. Then*

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$$

**Theorem 4.14. Uniqueness for Cauchy problem**. *Let $g \in C(\mathbb{R}^n), f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C_1^2\big(\mathbb{R}^n \times (0,T]\big) \cap C\big(\mathbb{R}^n \times [0,T]\big)$ of the initial value problem*

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,T) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{44}$$

*satisfying the growth estimate*

$$u(x,t) \le Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \le t \le T)$$

*for constants $A, a > 0$.*

*Remark* 4.15. Without the growth estimate one can build infinitely many solutions to (44) with $f = 0$ and $g = 0$, by "putting an infinite source of heat at infinity", these solutions (besides $u \equiv 0$) grows very rapidly as $|x| \to \infty$.

#### 4.5.2 Regularity.

**Theorem 4.16. Smoothness** *Suppose $u \in C_1^2(U_T)$ solves the heat equation in $U_T$. Then*

$$u \in C^\infty(U_T).$$

#### 4.5.3 Local estimates for solutions of the heat equation.

**Theorem 4.17. Estimates on derivatives** *There exists for each pair of integers $k, l = 0, 1, \ldots$ a constant $C_{k,l}$ such that*

$$\max_{C(x,t,r/2)} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t,r)}$$

*for all cylinders $C(x,t,r/2) \subset C(x,t,r) \subset U_T$, and all solutions $u$ of the heat equation in $U_T$*

### 4.6 Energy Methods.

#### 4.6.1 Uniqueness.

We consider the initional boundary value problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad U_T \\ u = g & \text{on} \quad \Gamma_T \end{cases} \tag{45}$$

We assume $U$ to be open and $\partial U$ to be $C^1$. The terminal time $T$ is given.

**Theorem 4.18. Uniqueness** *There exists at most one solution $u \in C_1^2(\bar{U}_T)$ of (45).*

#### 4.6.2 Backward uniqueness.

Suppose $u$ and $\tilde{u}$ are both smooth solutions of the heat equation in $U_T$, with the same boundary condition on $\partial U$:

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad U_T \\ u = g & \text{on} \quad \Gamma_T \end{cases} \tag{46}$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = f & \text{in} \quad U_T \\ \tilde{u} = g & \text{on} \quad \Gamma_T \end{cases} \tag{47}$$

for some function $g$. We are not supposing $u = \tilde{u}$ at time $t = 0$.

**Theorem 4.19. Backwards uniqueness** *Suppose* $u, \tilde{u} \in C^2(\bar{U}_T)$ *solve (46), (47). If*

$$u(x,T) = \tilde{u}(x,T) \quad (x \in U)$$

*then*

$$u \equiv \tilde{u} \quad \textit{within} \quad U_T$$

# 5 Wave Equation

In this section we investigate the *wave equation*

$$u_{tt} - \Delta u = 0$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta u = f,$$

subject to appropriate initial and boundary conditions. Here $u(x,t) : \overline{U} \times [0,\infty) \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open.

**Physical interpretation.** The wave equation is a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In these physical interpretations $u(x,t)$ represents the displacement in some direction of the point $x$ at time $t \geq 0$.

Let $V$ represent any smooth subregion of $U$. The acceleration within $V$ is the

$$\frac{d^2}{dt^2}\int_V u \, dx = \int_V u_{tt} \, dx$$

and the net contact force is

$$-\int_{\partial V} \mathbf{F} \cdot \nu \, dS.$$

Newton's law

$$\int_V u_{tt} \, dx = -\int_{\partial V} \mathbf{F} \cdot \nu \, dS.$$

## 5.1 Solution by spherical means

### 5.1.1 Solution for $n = 1$, d'Alembert's formula.

The initial-value problem for the one-dimensional wave equation in all of $\mathbb{R}$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, \; u_t = h & \text{in } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{48}$$

where $g$, $h$ are give. Let us first note the PDE in (48) can be "factored", to read

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0. \tag{49}$$

Write

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t). \tag{50}$$

Then (49) says

$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R},\ t > 0).$$

We find

$$v(x,t) = a(x-t)$$

for $a(x) := v(x,0)$. Repeating the same procedure with $u$ in (50), taking into account the initial conditions, we are able to reach

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy \quad (x \in \mathbb{R},\ t \geq 0). \tag{51}$$

This is *d'Alember's formula.*

**Theorem 5.1.** ***Solution of wave equation,*** $n = 1$. *Assume* $g \in C^2(\mathbb{R})$, $h \in C^2(\mathbb{R})$, *and define* $u$ *by d'Alembert's formula (51). Then*

*(i)* $u \in C^2(\mathbb{R} \times [0,\infty)$,

*(ii)* $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times [0,\infty)$,

*and (iii)* $\lim\limits_{\substack{(x,\,t)\,\to\,(x^0,\,0)\\ t\,>\,0}} u(x,t) = g(x^0)$, $\lim\limits_{\substack{(x,\,t)\,\to\,(x^0,\,0)\\ t\,>\,0}} u_t(x,t) = h(x^0)$ *for each point* $x^0 \in \mathbb{R}$.

**A reflection method.** To illustrate a further application of d'Alembert's formula, let us consider this initial/boundary-value problem on the half-line $\mathbb{R}_+ = \{x > 0\}$ :

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g,\ u = h_t & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{in } \{x = 0\} \times (0,\infty), \end{cases}$$

where $g$, $h$ are given, with $g(0) = h(0) = 0$.

### 5.1.2 Spherical means.

Now suppose $n \geq 2$, $m \geq 2$, and $u \in C^m(\mathbb{R}^n \times [0,\infty))$ solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{52}$$

**Notation.** (i) Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. Define

$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\ dS(y),$$

the average of $u(\cdot,t)$over the sphere $\partial B(x,r)$.

(ii) Similarly,

$$G(x;r) := \oint_{\partial B(x,r)} g(y)\ dS(y),$$

$$H(x;r) := \oint_{\partial B(x,r)} h(y)\ dS(y).$$

**Lemma 5.2.** ***Euler-Poisson-Darboux equation***. *Fix $x \in \mathbb{R}^n$,and let $u$ satisfy (52). Then $U \in C^m(\bar{\mathbb{R}}_+ \times [0,\infty))$ and*

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G,\ U_t = H & \text{on } \mathbb{R}_+ \times \{t=0\}. \end{cases} \tag{53}$$

### 5.1.3 Solution for $n = 3$, 2, Kirchhoff's and Poisson's formulas.

**Solution for $n = 3$.** Set

$$\tilde{U} := rU, \tag{54}$$

$$\tilde{G} := rG,\ \tilde{H} := rH.$$

We now assert that $\tilde{U}$ solves

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G},\ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{U} = 0 & \text{on } \{r=0\} \times (0,\infty). \end{cases} \tag{55}$$

We therefore conclude

$$u(x,t) = \oint_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\ dS(y) \quad (x \in \mathbb{R}^3,\ t > 0).$$

This is *Kirchhoff's formula* for the solution of the initial-value problem (53) in three dimensions.

**Solution for $n = 2$.** No transformation like (54) works to convert the Euler-Poisson-Darboux equation into the one-dimensional wave equation when $n = 2$. Instead we will take the initial-value problem (53) for $n = 2$ and simply regard it as a problem for $n = 3$, in which the third spatial variable $x_3$ does not appear. The solution is

$$u(x,t) = \frac{1}{2} \oint_{B(x,t)} \frac{tg(y) + t^2 h(y) + tDg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\ dy.$$

This is *Poisson's formula* for the solution of the initial-value problem (53) in two dimensions.

#### 5.1.4 Solution for odd n.

**Lemma 5.3.** ***Some useful identities.*** *Let $\phi$: $\mathbb{R} \to \mathbb{R}$ be $C^{k+1}$. Then for $k = 1, 2, \ldots$:*

*(i)* $\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right)$,

*(ii)* $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r)$, *where the constants* $\beta_j^k (j = 0, \ldots, k-1)$ *are independent of* $\phi$.

*Furthermore,*

*(iii)* $\beta_0^k = 1 \cdot 3 \cdot 5 \cdots (2k-1)$.

Now assume $n = 2k+1 \quad (k \geq 1)$.

**Notation.** We write

$$\begin{cases} \tilde{U}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right) \\ \tilde{G}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}G(x;r)\right) \\ \tilde{H}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}H(x;r)\right). \end{cases} \qquad (r > 0, t \geq 0) \tag{56}$$

**Lemma 5.4.** *$\tilde{U}$solves the one-dimensional wave equation). We have*

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G},\ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ U \tilde{=} 0 & \text{on } \{r = 0\} \times (0,\infty). \end{cases} \tag{57}$$

**Theorem 5.5.** ***Solution of wave equation in odd dimensions***. *Assume $n$ is an odd integer, $n \geq 3$, and suppose also $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$, for $m = \frac{n+1}{2}$. Define $u$ by*

$$\begin{cases} u(x,t) = \frac{1}{\gamma_n}\Bigg[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\oint_{\partial B(x,t)} g\ dS\right) \\ \qquad + \left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-3}{2}}\left(t^{n-2}\oint_{\partial B(x,t)} h\ dS\right)\Bigg] \\ \qquad \text{where } n \text{ is odd and } \gamma = 1 \cdot 3 \cdot 5 \cdots (n-2), \end{cases} \tag{58}$$

*for $x \in \mathbb{R}^n$, $t > 0$. Then*

*(i)* $u \in C^2(\mathbb{R}^n \times [0,\infty)$,

*(ii)* $u_{tt} - \Delta u = 0$ *in* $\mathbb{R}^n \times [0,\infty)$,

*and*

*(iii)* $\lim\limits_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0), \qquad \lim\limits_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u_t(x,t) = h(x^0)$

*for each point $x^0 \in \mathbb{R}$.*

#### 5.1.5 Solution for even n.

Assume now $n$ is an even integer. We want to fashion a representation formula like (58) for $u$. The trick, as above for $n = 2$, is to note

$$\bar{u}(x_1, \ldots, x_{n+1}, t) := u(x_1, \ldots, x_n, t)$$

solves the wave equation in $\mathbb{R}^{n+1} \times (0,\infty)$, with the initial conditions

$$\bar{u} = \bar{g},\ \bar{u}_t = \bar{h} \quad \text{on } \mathbb{R}^{n+1} \times \{t=0\},$$

where

$$\begin{cases} \bar{g}(x_1,\dots,x_{n+1}) := g(x_1,\dots,x_n) \\ \bar{h}(x_1,\dots,x_{n+1}) := h(x_1,\dots,x_n). \end{cases}$$

The resulting representation formula for even $n$ is:

$$\begin{cases} u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy\right) \right. \\ \qquad \left. + \left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{h(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy\right)\right] \\ \qquad \text{where } n \text{ is even and } \gamma = 2\cdot 4\cdot 6\cdots(n-2)\cdot n, \end{cases} \tag{59}$$

for $x \in \mathbb{R}^n$, $t > 0$.

**Theorem 5.6.** ***(Solution of wave equation in even dimensions****. Assume $n$ is an even integer, $n \geq 2$, and suppose also $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$, for $m = \frac{n+2}{2}$. Define $u$ by (59). Then*

*(i)* $u \in C^2(\mathbb{R}^n \times [0,\infty)$,

*(ii)* $u_{tt} - \Delta u = 0$ *in* $\mathbb{R}^n \times [0,\infty)$,

*and*

*(iii)* $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0), \quad \lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u_t(x,t) = h(x^0)$

*for each point* $x^0 \in \mathbb{R}$.

## 5.2 Nonhomogeneous problem.

We next investigate the initial-value problem for the nonhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0,\ u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{60}$$

Motivated by Duhamel's principle, we define $u = u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot;s) = 0,\ u_t(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{61}$$

Now set

$$u(x,t) := \int_0^t u(x,t;s)\, ds \quad (x \in \mathbb{R}^n,\ t \geq 0). \tag{62}$$

**Theorem 5.7.** ***Solution of nonhomogeneous wave equation****. Assume $n \geq 2$ and $f \in C^{[n/2]+1}(\mathbb{R}^n \times [0,\infty))$. Define $u$ by (62). Then*

*(i)* $u \in C^2(\mathbb{R}^n \times [0,\infty)$,

*(ii)* $u_{tt} - \Delta u = f$ *in* $\mathbb{R}^n \times [0,\infty)$,

*and*

*(iii)* $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0, \quad \lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u_t(x,t) = 0$

*for each point* $x^0 \in \mathbb{R}$.

**2.4.3. Energy methods.**

### 5.2.1 Uniqueness.

Let $U \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary $\partial U$, and as usual set $U_T = U \times (0,T]$, $\Gamma_T = \bar{U}_T - U_T$, where $T > 0$. We are interested in the initial/boundary-value problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } U_T \\ \qquad u = g, & \text{on } \Gamma_T \\ \qquad u_t = h, & \text{on } \ U \times \{t = 0\}. \end{cases} \tag{63}$$

**Theorem 5.8.** ***Uniqueness for wave equation****. There exists at most one function $u \in C^2(\bar{U}_T)$ solving (63).*

### 5.2.2 Domain of dependence.

Suppose $u \in C^2$solves

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$and consider the cone

$$C = \{(x,t)\,|0 \le t \le t_0, |x - x_0| \le t_0 - t\}\,.$$

**Theorem 5.9.** ***Finite propagation speed.*** *If $u \equiv u_t \equiv 0$ on $B(x_0, t_0)$, then $u \equiv 0$ within the cone $C$.*

## 5.3 PROBLEMS

1. (a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions $F$, $G$.
(b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.
(c) Use (a) and (b) to re-derive d'Alembert's formula.

2. (Equipartition of energy). Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \ u_t = h & \text{in } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose $g$, $h$ have compact support. The *kinetic energy* is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the *potential energy* is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$. Prove

(i) $k(t) + p(t)$ is constant in $t$,

(ii)$k(t) = p(t)$ for all large enough times $t$.

3. Let $u$ solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, \ u_t = h & \text{in } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where $g$, $h$ are smooth and have compact support. Show there exists a constant $C$ such that

$$|u(x,t)| \leq C/t \quad (x \in \mathbb{R}^3, \ t > 0).$$