

Averaging Versus Chaos in Turbulent Transport?

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Abstract: In this paper we analyze the transport of passive tracers by deterministic stationary incompressible flows which can be decomposed over an infinite number of spatial scales without separation between them. It appears that a low order dynamical system related to local Peclet numbers can be extracted from these flows and it controls their transport properties. Its analysis shows that these flows are strongly self-averaging and super-diffusive: the delay $\tau(r)$ for any finite number of passive tracers initially close to separate till a distance r is almost surely anomalously fast ($\tau(r) \sim r^{2-\nu}$, with $\nu > 0$). This strong self-averaging property is such that the dissipative power of the flow compensates its convective power at every scale. However as the circulation increases in the eddies the transport behavior of the flow may (discontinuously) bifurcate and become ruled by deterministic chaos: the self-averaging property collapses and advection dominates dissipation. When the flow is anisotropic a new formula describing turbulent conductivity is identified.

1. Introduction

In this paper we study the passive transport in \mathbb{R}^d ($d \geq 2$) of a scalar T by a divergence free steady vector field v characterized by the following partial differential equation ($\kappa > 0$ being the molecular conductivity):

$$\partial_t T + v \nabla T = \kappa \Delta T \tag{1}$$

We will assume v to be given by an infinite (or large) number of spatial scales without any assumption of self-similarity [Ave96]. It will be shown that one can extract from the flow a low order dynamical system related to local Peclet tensors which controls the transport properties of the flow. Based on the analysis of this dynamical system we will show that the transport is almost surely super-diffusive, that is to say, the time of separation of any finite number of passive tracers driven by the same flow and independent thermal noise behave like $r^{2-\nu}$ with $\nu > 0$. Similar programs of investigations have shown that

the mean squared displacement of a single particle is anomalously fast when averaged with respect to space, time and the randomness of the flow ([Pit97, KO02, Fan02]). The point here is to show that the transport is strongly self-averaging: the diffusive properties are anomalously fast (before being averaged with respect to the thermal noise, or a probability law of the flow), moreover the pair separation is also anomalously fast. The fast behavior of the transport of a single particle can be created by long distance correlations in the structure of the velocity field but this is not sufficient to produce fast pair separation. In this paper non-asymptotic estimates will be given, showing that the transport is controlled by a never-ending averaging phenomenon ([Owh01a, Owh01b, BO02a, BO02b]). The analysis of the low order dynamical system allows to obtain a formula linking the minimal and maximal eigenvalues of the turbulent eddy diffusivity. It will be shown that the transport properties depend only on the power law in v and not on its particular geometry (which is not a priori obvious since we consider a quenched model). However, depending on the geometrical characteristics of the eddies at each scale, as the flow rate is increased in these eddies we observe that the super-diffusive behavior may bifurcate towards a Chaotic transport: the multi-scale averaging picture collapses and the flow becomes highly unstable, sensitive to the characteristics of the microstructure and dominated by convective terms.

2. The Model

We want to analyze the properties of the solutions of the following stochastic differential equation which is the Lagrangian formulation of the passive transport equation (1):

$$dy_t = \sqrt{2\kappa}d\omega_t + \nabla \cdot \Gamma(y_t) dt. \tag{2}$$

Here $\kappa > 0$ is the molecular conductivity of the flow, ω_t a standard Brownian Motion on \mathbb{R}^d related to the thermal noise, Γ is a skew-symmetric matrix on \mathbb{R}^d called the stream matrix of the flow and $\nabla \cdot \Gamma$ its divergence. Thus $\nabla \cdot \Gamma$ is the divergence free drift defined by $(\nabla \cdot \Gamma)_i = \sum_{j=1}^d \partial_j \Gamma_{ij}$. We assume that Γ is given by an infinite sum of periodic stream matrices with (geometrically) increasing periods and increasing amplitude,

$$\Gamma = \sum_{k=0}^{\infty} \gamma_k E^k \left(\frac{x}{R_k} \right). \tag{3}$$

In the formula (3) we have three important ingredients: the stream matrices E^k (also called eddies), the scale parameters R_k and the amplitude parameters γ_k (the stream matrices E^k are dimensionless and the parameters γ_k have the dimension of a conductivity). We will now describe the hypothesis we make on these three items of the model. Let us write $T^d := \mathbb{R}^d / \mathbb{Z}^d$ the torus of dimension d and side one and for $\alpha \in [0, 1]$, $\mathcal{S}^\alpha(T^d)$ the space of $d \times d$ skew-symmetric matrices on T^d with α -Holder continuous coefficients and $\|\cdot\|_\alpha$ the norm associated to that space. For $E \in \mathcal{S}^\alpha(T^d)$,

$$\|E\|_\alpha := \sup_{i,j \in \{1, \dots, d\}} \sup_{x \neq y} |E_{ij}(x) - E_{ij}(y)| / |x - y|^\alpha. \tag{4}$$

I Hypotheses on the stream matrices E^k .

There exists $0 < \alpha \leq 1$ such that for all $k \in \mathbb{N}$,

$$E^k \in \mathcal{S}^\alpha(\mathbb{T}^d). \tag{5}$$

The S^α -norm of the E^k are uniformly bounded, i.e.

$$K_\alpha := \sup_{k \in \mathbb{N}} \|E^k\|_\alpha < \infty. \tag{6}$$

Moreover for all k ,

$$E^k(0) = 0. \tag{7}$$

Observe that the S^0 -norms of the E^k are uniformly bounded and we will write

$$K_0 := \sup_{k \in \mathbb{N}} \sup_{i, j \in \{1, \dots, d\}} \|E^k\|_{0,ij}. \tag{8}$$

II Hypotheses on the scale parameters R_k .

R_k is a spatial scale parameter growing exponentially fast with k ; more precisely we will assume that $R_0 = r_0 = 1$ and that the ratios between scales defined by

$$r_k = R_k/R_{k-1} \in \mathbb{R}^* \tag{9}$$

for $k \geq 1$, are reals uniformly bounded away from 1 and ∞ : we will denote by

$$\rho_{\min} := \inf_{k \in \mathbb{N}^*} r_k \quad \text{and} \quad \rho_{\max} := \sup_{k \in \mathbb{N}^*} r_k, \tag{10}$$

and assume that

$$\rho_{\min} \geq 2 \quad \text{and} \quad \rho_{\max} < \infty. \tag{11}$$

III Hypotheses on the flow rates γ_k .

γ_k is an amplitude parameter (related to the local rate of the flow) growing exponentially fast with the scale k ; more precisely we will assume that $\gamma_0 = 1$ and that their ratios γ_k/γ_{k-1} for $k \geq 1$, are positive reals uniformly bounded away from 1 and ∞ : we will denote by

$$\gamma_{\min} := \inf_{k \in \mathbb{N}^*} (\gamma_k/\gamma_{k-1}) \quad \text{and} \quad \gamma_{\max} := \sup_{k \in \mathbb{N}^*} (\gamma_k/\gamma_{k-1}), \tag{12}$$

and assume that

$$\gamma_{\min} > 1 \quad \text{and} \quad \gamma_{\max} < \rho_{\min}^\alpha. \tag{13}$$

Remark 2.1. The uniform α -Holder continuity of the stream matrices E^k is sufficient to obtain a well defined α -Holder continuous stream matrix Γ , however Γ is not differentiable in general. In this case the stochastic differential equation (2) is formal. For the simplicity of the presentation and to start with, when referring to the SDE (2) we will assume that

$$\alpha = 1 \quad \text{and that the stream matrices } E^k \text{ are uniformly } C^1. \tag{14}$$

It follows from the Hypothesis **I**, **II** and **III** that Γ is a well defined uniformly C^1 skew-symmetric matrix on \mathbb{R}^d , thus the Stochastic Differential Equation 2 is well defined and admits a unique solution.

The differentiability hypothesis (14) though convenient in order to define the process y_t is in fact useless, the theorems are also meaningful and true for $0 < \alpha < 1$ (since they will refer to the diffusion associated to the weakly defined operator $\nabla \cdot (\kappa + \Gamma) \nabla$).

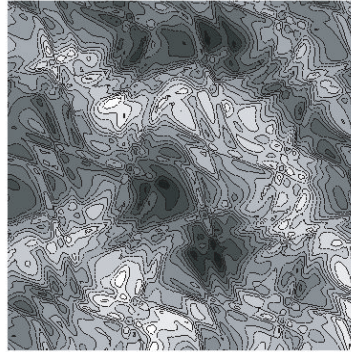


Fig. 1. A simple example of the multiscale flow

Remark 2.2. Observe that the power law of the flow in this paper is not Kolmogorov. Indeed if $v(l)$ is the velocity of the eddies of size l and $\mathcal{E}(k)$ the kinetic energy distribution in the Fourier modes then with the Kolmogorov law one should have

$$v(l) \sim l^{\frac{1}{3}} \quad \text{and} \quad \mathcal{E}(k) \sim k^{-\frac{5}{3}}.$$

In our Model we have

$$v(l) \sim l^{\frac{\ln \gamma}{\ln \rho} - 1} \quad \mathcal{E}(k) \sim k^{1 - 2 \frac{\ln \gamma}{\ln \rho}}.$$

Thus to be consistent with a Kolmogorov spectrum one should have $\gamma = \rho^{\frac{4}{3}}$; this case will be analyzed in a forthcoming paper.

As an example, we have illustrated in Fig. 1 the contour lines of a two scale flow with stream function $h_0^2(x, y) = \sum_{k=0}^2 \gamma^k h(\frac{x}{\rho^k}, \frac{y}{\rho^k})$, with $\rho = 3$, $\gamma = 1.1$ and $h(x, y) = 2 \sin(2\pi x + 3 \cos(2\pi y - 3 \sin(2\pi x + 1))) \sin(2\pi y + 3 \cos(2\pi x - 3 \sin(2\pi y + 1)))$.

3. A Reminder on the Eddy Conductivity

We write $\mathcal{M}_{d,\text{sym}}$ the space of $d \times d$ symmetric elliptic constant matrices and $\mathcal{S}_{L^\infty}(\mathbb{T}_R^d)$ the space of skew-symmetric matrices with coefficients in $L^\infty(\mathbb{T}_R^d)$ ($\mathbb{T}_R^d := \mathbb{R}\mathbb{T}^d$ stands for the torus of dimension d and side R). For $a \in \mathcal{M}_{d,\text{sym}}$ and E a skew symmetric matrix with bounded coefficients the heat kernel associated to the passive transport operator $\nabla \cdot (a + E) \nabla$ (defined in a weak sense) is Gaussian by Aronson estimates [Nor97]. We will now assume E to be periodic: $E \in \mathcal{S}_{L^\infty}(\mathbb{T}_R^d)$. In this case the process associated to the operator $L = \nabla \cdot (a + E) \nabla$ exhibits self-averaging properties and we will note $\sigma_{\text{sym}}(a, E)$ the effective conductivity associated to the homogenization of that operator [BLP78, JKO91]. Writing $p(t, x, y)$ the heat kernel associated to L , it is well known that $\sigma_{\text{sym}}(a, E)$ is a $d \times d$ elliptic symmetric matrix satisfying, for all $x, l \in \mathbb{R}^d$,

$$|l|_{\sigma_{\text{sym}}(a, E)}^2 = \frac{1}{2} \lim_{t \rightarrow \infty} t^{-1} \int_{\mathbb{R}^d} p(t, x, y) (y \cdot l)^2 dy. \tag{15}$$

We have used the notation $|l|_a^2 := {}^t l a l$. If z_t is the process generated by L , then as $\epsilon \downarrow 0$, $\epsilon z_{t/\epsilon^2}$ converges in law to a Brownian motion with covariance matrix $D(a, E)$ called effective diffusivity and proportional to the effective conductivity,

$$D(a, E) = 2\sigma_{\text{sym}}(a, E). \tag{16}$$

Let us recall that $\sigma_{\text{sym}}(a, E)$ is given by the following variational formula ([Nor97] Lemma 3.1): for $\xi \in \mathbb{R}^d$,

$$|\xi|_{\sigma_{\text{sym}}^{-1}(a, E)}^2 = \inf_{(f, H) \in C^\infty(\mathbb{T}_R^d) \times \mathcal{S}(\mathbb{T}_R^d)} R^{-d} \int_{\mathbb{T}_R^d} |\xi - \nabla \cdot H + (a + E)\nabla f|_{a^{-1}}^2 dx, \tag{17}$$

where we have written $\mathcal{S}(\mathbb{T}_R^d)$ the space of skew symmetric matrices with coefficients in $C^\infty(\mathbb{T}_R^d)$.

The symmetric tensor $\sigma_{\text{sym}}(a, E)$ is also called eddy conductivity: after averaging the information on particular geometry of the eddies associated to E is lost and the conductivity of the flow is replaced by an increased conductivity $\sigma_{\text{sym}}(a, E)$. Let us define for $P \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$ and $\rho \in \mathbb{R}^*$, $S_\rho P \in \mathcal{S}_{L^\infty}(\mathbb{T}_{\frac{1}{\rho}}^d)$ by

$$S_\rho P(x) := P(\rho x). \tag{18}$$

It is important to note that the effective conductivity is invariant by scaling, i.e. $\sigma_{\text{sym}}(a, S_\rho P) = \sigma_{\text{sym}}(a, P)$; thus we can assume for simplicity that $R = 1$ and $E \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$. When E is smooth $\sigma_{\text{sym}}(a, E)$ is given [BLP78] by solving the following cell problem:

$$\nabla \cdot (a + E)(l - \nabla \chi_l^{a, E}) = 0, \tag{19}$$

where $l \in \mathbb{R}^d$, $\chi_l^{a, E} \in C^\infty(\mathbb{T}^d)$ and $\int_{\mathbb{T}^d} \chi_l^{a, E}(x) dx = 0$. Write $F_l^{a, E} = l \cdot x - \chi_l^{a, E}(x)$, observe that $F_l^{a, E}$ is linear in l , thus we will write $F^{a, E}$ the vector field $(F^{a, E})_i := F_{e_i}^{a, E}$ and $\nabla F^{a, E}$ the matrix $(\nabla F^{a, E})_{ij} := \partial_i F_{e_j}^{a, E}$. The eddy conductivity is then given by

$$\sigma_{\text{sym}}(a, E) = \int_{\mathbb{T}^d} {}^t \nabla F^{a, E}(x) a \nabla F^{a, E}(x) dx. \tag{20}$$

Let us recall that the matrix $\sigma(a, E)$ defined by

$$\sigma(a, E) := \int_{\mathbb{T}^d} (a + E(x)) \nabla F^{a, E}(x) dx \tag{21}$$

is called the flow effective conductivity [FP94] and is also given by the following variational formula [Nor97]: for $\xi, l \in \mathbb{R}^d$,

$$\begin{aligned} & |\xi - \sigma(a, E)l|_{\sigma_{\text{sym}}^{-1}(a, E)}^2 \\ & := \inf_{(f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H - (a + E)(l - \nabla f)|_{a^{-1}}^2 dx. \end{aligned} \tag{22}$$

It is easy to check that $\sigma_{\text{sym}}(a, E)$ is the symmetric part of $\sigma(a, E)$ which implies the following variational formulation for the eddy conductivity:

$$|l|_{\sigma_{\text{sym}}}^2 = \inf_{\xi \perp l, (f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H - (a + E)(l - \nabla f)|_{a^{-1}}^2 dx, \tag{23}$$

where we have written $\xi \perp l$ is the subspace of $\xi \in \mathbb{R}^d$ orthogonal to the vector l : $\xi \cdot l = 0$.

4. Main Results

4.1. *Averaging with two scales.* Let $a \in \mathcal{M}_{d,\text{sym}}, P \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$ and $K \in \mathcal{S}^\alpha(\mathbb{T}^d)$. We will prove in Subsect. 5.1 the following estimate of $\sigma_{\text{sym}}(a, S_R P + K)$, the effective conductivity for a two-scale medium when R is an integer (and S_R is the scaling operator (18)).

Theorem 4.1. *There exists a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ increasing in each of its arguments such that for $a \in \mathcal{M}_{d,\text{sym}}, R \in \mathbb{N}^*, P \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$ and $K \in \mathcal{S}^\alpha(\mathbb{T}^d)$,*

$$(1 + \epsilon(R))^{-4} \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) \leq \sigma_{\text{sym}}(a, S_R P + K) \leq \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K)(1 + \epsilon(R))^4, \tag{24}$$

$$\text{with } \epsilon(R) = \left(\frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \right)^{\frac{1}{2}} f\left(d, \frac{\|a + P\|_\infty}{\lambda_{\min}(a)}\right). \tag{25}$$

Remark 4.2. Theorem 4.1 implies obviously that

$$\sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) = \lim_{R \rightarrow \infty} \sigma_{\text{sym}}(a, S_R P + K). \tag{26}$$

Thus $\sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K)$ should be interpreted as the effective conductivity of the two-scale flow with a complete separation of scales. So we will also write it $\sigma_{\text{sym}}(a, S_\infty P + K)$. Naturally $\sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K)$ is also computable from an explicit cell problem (see (20)).

Averaging versus chaotic coupling. Equation (24) basically says that when $\epsilon(R)$ is small, the mixing length of the smaller scale $P(Rx)$ is smaller than the scale at which the fluctuations of the larger scale $K(x)$ start to be felt. Now it is very important to observe that as $\lambda_{\min}(a) \downarrow 0$, $\epsilon(R)$ explode towards infinity and this collapse of the two-scale averaging is not an artefact, it is easy to see that the estimate (25) is sharp. What happens is a transition from averaging to a chaotic coupling between the two scales. More precisely as $\lambda_{\min}(a) \downarrow 0$, the mixing length of the smaller scale explode well above the visibility length of the larger scale, the two scales are no longer separated in the averaging and their particular geometry can no longer be ignored (collapse of the averaging paradigm). Moreover writing for $y \in [0, 1]^d$, Θ_y the translation operator acting on functions f of \mathbb{R}^d by $\Theta_y f(x) = f(x + y)$, observe that in the limit of complete separation between scales the two-scale averaging is invariant with respect to a relative translation of one scale with respect to another:

$$\lim_{R \rightarrow \infty} \sigma_{\text{sym}}(a, S_R \Theta_y P + K) = \lim_{R \rightarrow \infty} \sigma_{\text{sym}}(a, S_R P + K). \tag{27}$$

But the limit $\lambda_{\min}(a) \downarrow 0$ is singular and this invariance by translation is lost: for $l \in \mathbb{R}^d$,

$$\left({}^t l \sigma_{\text{sym}}(a, S_R \Theta_y P + K) l - {}^t l \sigma_{\text{sym}}(a, S_R P + K) l \right) \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l \right)^{-1} \tag{28}$$

may explode towards infinity. Indeed it is easy to see that for any $R \in \mathbb{N}^*$, there exist $P, K \in \mathcal{S}^1(\mathbb{T}^d)$ with $\|P\|_1 \leq C_d, \|K\|_1 \leq C_d$ such that there exists $y \in [0, 1]^d$ and $l \in \mathbb{R}^d$ with

$$\lim_{\zeta \downarrow 0} \left({}^t l \sigma_{\text{sym}}(\zeta I_d, S_R \Theta_y P + K) l \right) \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l \right)^{-1} = \infty. \tag{29}$$

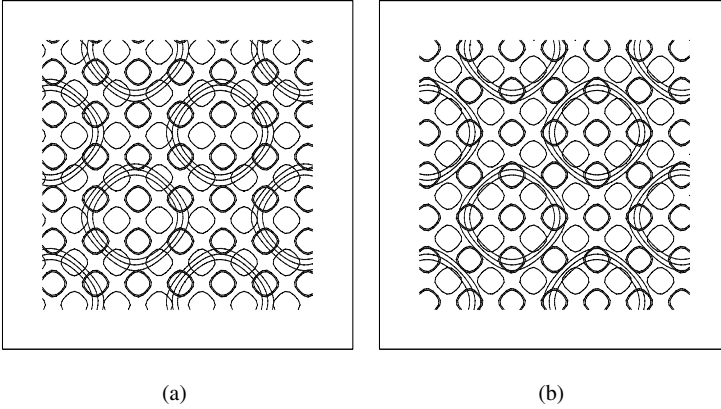


Fig. 2a,b. Two scales flow. (a) Stream lines of $S_R P$ and K ; (b) Stream lines of $S_R \Theta_y P$ and K

We have illustrated this symmetry breaking in Fig. 2 representing a two scale flow. In Fig. 2(a) the larger eddies are surrounded by a non-void region where the flow is null and asymptotic behavior of the effective conductivity at vanishing molecular conductivity is given by

$$\sigma_{\text{sym}}(\zeta I_d, S_R P + K) \sim C_1 \zeta I_d. \tag{30}$$

In Fig. 2(b) we have operated a small translation of the smaller scale with respect to the larger one. The result of this relative translation is the percolation of stream lines of the flow: a particle driven by the flow can go to infinity by following them. It follows after this small perturbation that asymptotic behavior of the effective conductivity of the two scale flow at vanishing molecular conductivity is given by (31).

$$\sigma_{\text{sym}}(\zeta I_d, S_R \Theta_y P + K) \sim C_2 \zeta^{\frac{1}{2}} I_d. \tag{31}$$

We call this sensibility with respect to the relative translation Θ_y , *chaotic coupling between scales*. The asymptotic (31) can be understood from a boundary layer analysis (see [Chi79] and [FP94]).

4.2. *Multiscale eddy conductivity and the renormalization core.* Let us write

$$\Gamma^{0,n} = \sum_{k=0}^n \gamma_k E^k \left(\frac{x}{R_k} \right). \tag{32}$$

For this subsection we will use the following hypothesis

IV. Hypothesis on the ratios between scales: For all $k \in \mathbb{N}$, $r_k \in \mathbb{N}^*$.

Our objective is to obtain quantitative estimates the multi-scale eddy viscosities $(\sigma(\Gamma^{0,n}))_{n \in \mathbb{N}}$; observe that under the hypothesis **IV**, $\Gamma^{0,n}$ is periodic, thus its effective conductivity is well defined by Eq. (17) (that is its only utility, we will not need this hypothesis to prove super-diffusion). These estimates (Theorem 4.4) will be proven by

induction on the number of scales. The basic step in this induction is the estimate (24) on the effective conductivity for a two scale periodic medium. We will need to introduce a dynamical system called the *renormalization core* which will play a central role in the transport properties of the stochastic differential equation (2).

Definition 4.3. We propose to call “renormalization core” the dynamical system $(A^n)_{n \in \mathbb{N}}$ of $d \times d$ symmetric strictly elliptic matrices defined by

$$A^0 = \frac{\kappa}{\gamma_0} I_d \quad \text{and} \quad A^{n+1} = \frac{\gamma_n}{\gamma_{n+1}} \sigma_{\text{sym}}(A^n, E^n). \tag{33}$$

For B a $d \times d$ symmetric coercive matrix let us define the function $g(B)$ by

$$g(B) := \left(\frac{K_\alpha}{\lambda_{\min}(B)(1 - \gamma_{\max}/\rho_{\min}^\alpha)} \right)^{\frac{1}{2}} f(d, (\lambda_{\max}(B) + K_0)/\lambda_{\min}(B)), \tag{34}$$

where f is the function appearing in Theorem 4.1.

We will prove the following theorem in Subsect. 5.2.

Theorem 4.4. Under Hypotheses I, II, III and IV for all $n \in \mathbb{N}^*$,

$$\gamma_{n+1} A^{n+1} \prod_{p=1}^n (1 + \epsilon_p)^{-4} \leq \sigma_{\text{sym}}(\kappa I_d, \Gamma^{0,n}) \leq \gamma_{n+1} A^{n+1} \prod_{p=1}^n (1 + \epsilon_p)^4 \tag{35}$$

with

$$\epsilon_p = \left(\frac{\gamma_p}{\gamma_{p-1} r_p^\alpha} \right)^{\frac{1}{2}} g(A^{p-1}), \tag{36}$$

A^n being the renormalization core (33).

Observe that $\gamma_{n+1} A^{n+1}$ is the estimate given by reiterated homogenization under the assumption of complete separation between scales, i.e. $\rho_{\min} \rightarrow \infty$ and the error term $\prod_{p=1}^n (1 + \epsilon_p)^4$ controlled by the *renormalization core* A^k which reflects the interaction between the scales k and $k + 1$. As $\lambda_{\min}(A_k) \downarrow 0$ one passes from a separation of the scales k and $k + 1$ to a chaotic coupling between these two scales. Moreover it is easy to obtain from Theorem 4.4 that

$$\lim_{r_1, \dots, r_{n-2} \rightarrow \infty} \sigma_{\text{sym}}(\kappa I_d, \Gamma^{0,n}) = \gamma_{n-1} \sigma_{\text{sym}}(A^{n-1}, S_{r_n} E^{n-1} + \frac{\gamma_n}{\gamma_{n-1}} E^n). \tag{37}$$

Assume that the multi-scale averaging scenario holds and $\rho_{\min} < \infty$. In that scenario, $\sigma_{\text{sym}}(\kappa I_d, \Gamma^{0,n})$ can be approximated by its limit at asymptotic separation between scales. We obtain a contradiction if $\liminf_{n \rightarrow \infty} \lambda_{\min}(A^n) = 0$ from (37) and the collapse of the two-scale averaging scenario given in Subsect. 4.1 and Fig. 2. In other words if $\liminf_{n \rightarrow \infty} \lambda_{\min}(A^n) = 0$, then the self-averaging property of the flow collapses towards a chaotic coupling between all the scales which is characterized by the breaking of the invariance by relative translation between the scales.

4.2.1. *What is the renormalization core?* First observe that it is a dimensionless tensor. At the limit of infinite separation between scales the eddy conductivity created by the scales $0, \dots, n - 1$ is $\lim_{\rho_{\min} \rightarrow \infty} \sigma_{\text{sym}}(\kappa I_d, \Gamma^{0,n-1})$. The typical scale length associated to the scale n is R_n and the velocity of the flow at this scale is of the order of $\gamma_n R_n^{-1}$ (we assume K_1 to be of order one). Thus at the scale n one can define a local renormalized Peclet tensor \mathbf{Pe}^n by:

$$\mathbf{Pe}^n := \mathbf{R}_n \times \frac{\gamma_n}{\mathbf{R}_n} \times \left(\lim_{\rho_{\min} \rightarrow \infty} \sigma_{\text{sym}}(\kappa \mathbf{I}_d, \Gamma^{0,n-1}) \right)^{-1}. \quad (38)$$

But at the limit of complete separation between scales $(A^n)^{-1}$ is equal to the ratio between the convective strength γ_n of the scale n and the local turbulent conductivity at the scale $n - 1$:

$$(A^n)^{-1} = \lim_{\rho_{\min} \rightarrow \infty} \gamma_n (\sigma_{\text{sym}}(\kappa I_d, \Gamma^{0,n-1}))^{-1}. \quad (39)$$

It follows that

$$(A^n)^{-1} = \mathbf{Pe}^n. \quad (40)$$

Thus one can interpret the *renormalization core* as the inverse of the Peclet tensor of the flow at the scale n assuming that all the smaller scales have been completely averaged.

Definition 4.5. We call the "local renormalized Peclet tensor" $(\mathbf{Pe}^n)_{n \in \mathbb{N}}$ the inverse of the renormalization core

$$\mathbf{Pe}^n = (\mathbf{A}^n)^{-1}. \quad (41)$$

4.2.2. *Pathologies of the renormalization core.*

Definition 4.6. We call stability of the renormalization core (33) the sequence

$$\lambda_n^- := \inf_{0 \leq p \leq n} \lambda_{\min}(A^p). \quad (42)$$

We write

$$\lambda_\infty^- := \lim_{n \rightarrow \infty} \lambda_n^- \quad \text{and} \quad \lambda^- := \liminf_{n \rightarrow \infty} \lambda_{\min}(A^n). \quad (43)$$

The renormalization core is said to be stable if and only if $\lambda^- > 0$.

Definition 4.7. We call anisotropic distortion of the renormalization core (33) the sequence

$$\mu_n = \sup_{0 \leq p \leq n} (\lambda_{\max}(A^p) / \lambda_{\min}(A^p)). \quad (44)$$

We write

$$\mu_\infty := \lim_{n \rightarrow \infty} \mu_n \quad \text{and} \quad \mu := \limsup_{n \rightarrow \infty} (\lambda_{\max}(A^n) / \lambda_{\min}(A^n)). \quad (45)$$

The renormalization core (33) is said to have unbounded (bounded) anisotropic distortion if and only if $\mu = \infty$ ($\mu < \infty$).

Definition 4.8. We call ubiety of the renormalization core (33) the sequence

$$\lambda_n^+ := \sup_{0 \leq p \leq n} \lambda_{\max}(A^p). \tag{46}$$

We write

$$\lambda_\infty^+ := \lim_{n \rightarrow \infty} \lambda_n^+ \quad \text{and} \quad \lambda^+ := \limsup_{n \rightarrow \infty} \lambda_{\max}(A^n). \tag{47}$$

The renormalization core is said to be vanishing if and only if $\lambda^+ = 0$.

Definition 4.9. The renormalization core (33) is said to be bounded if and only if $\lambda^+ < \infty$.

The *renormalization core* is gifted with remarkable properties which will be analyzed in detail in Subsect. 4.4. Before proceeding to super-diffusion we will give a first theorem stressing the role of the stability of the *renormalization core*, that is to say the fact that the local renormalized Peclet tensor stays bounded away from infinity. Indeed, it follows from Theorem 4.4 that the averaging paradigm for our model is valid if the *renormalization core* is stable, and has bounded anisotropic distortion. We may naturally wonder whether the fact that the local renormalized Peclet tensor stays bounded away from infinity is sufficient; the answer is positive as shown by the following theorem which will be proven in Subsect. 5.3.

Theorem 4.10. Writing $C = C_d K_0^2 (1 - 1/\gamma_{\min})^{-1}$ we have

1. If the renormalization core is not bounded ($\lambda^+ = \infty$) then it is not stable ($\lambda^- = 0$)
2. If the renormalization core is stable ($\lambda^- > 0$) then it is bounded and

$$\lambda^+ \leq \frac{C}{\lambda^-}. \tag{48}$$

3. The renormalization core has unbounded anisotropic distortion ($\mu = \infty$) if and only if it is not stable ($\lambda^- = 0$)
4. If the renormalization core is stable ($\lambda^- > 0$) then it has bounded anisotropic distortion ($\mu < \infty$) and

$$\mu \leq \frac{C}{(\lambda^-)^2}. \tag{49}$$

Combining Theorem 4.10 and 4.4 we obtain that if the *renormalization core* is stable then the local turbulent eddy conductivity diverges towards infinity like γ_n independently of the geometry of the eddies (if it is not stable the behavior of the local turbulent eddy conductivity depends on the geometry of the eddies). More precisely we have the following theorem.

Theorem 4.11. Under Hypotheses I, II, III and IV, if the renormalization core is stable then there exists C such that for $\rho_{\min}^\alpha > C\gamma_{\max}$ one has

$$\limsup_{n \rightarrow \infty} \frac{\ln \left(\lambda_{\max}(\sigma_{\text{sym}}(\kappa I_d, \Gamma^{0,n})) \right)}{\ln \gamma_n} \leq 1 + \epsilon \tag{50}$$

and

$$\liminf_{n \rightarrow \infty} \frac{\ln \left(\lambda_{\min}(\sigma_{sym}(\kappa I_d, \Gamma^{0,n})) \right)}{\ln \gamma_n} \geq 1 - \epsilon \tag{51}$$

with $\epsilon := 0.5(C\gamma_{\max}/(\rho_{\min}))^{\frac{1}{2}} < 0.5$ and $C := K_\alpha h(d, K_0/\lambda^-)$, h being a finite increasing positive function in each of its arguments.

Remark 4.12. For a real flow, call $\sigma(r)$ the local turbulent diffusivity of the flow at the scale r and $v(r)$ the magnitude of the vector velocity field at that scale. Then the key relation implying that the distortions created at the scale r are dissipated by the mixing power of the smaller scales is the relation

$$\sigma(r) \sim rv(r). \tag{52}$$

This relation is at the core of the Kolmogorov (K41) analysis and the analysis of fully developed turbulence by Landau-Lifschitz [LL84]. The result given in Theorem 4.11 corresponds to the relation (52) obtained and used from a heuristic point of view (dimension analysis) by physicists.

4.3. Super-diffusion.

Anomalous fast exit times. We write $\tau(r)$ the exit time of the process y_t (2) from the ball $B(0, r)$. We write \mathbb{E}_x the expectation associated to the process y_t started from the point x . We write $\text{Vol}(B(0, r))$ the Lebesgue measure of $B(0, r)$. We define $n(r)$ as the number of (smaller) scales which will be considered as averaged at the scale r ,

$$n(r) := \sup\{p \in \mathbb{N} : R_p \leq r\}. \tag{53}$$

Let m_r be the Lebesgue probability measure on the ball $B(0, r)$ defined by

$$m_r(dx) := \frac{dx}{\int_{B(0,r)} dx} 1_{B(0,r)}. \tag{54}$$

We will consider the mean exit time for the process started with initial distribution m_r , i.e.

$$\mathbb{E}_{m_r}[\tau(r)] = \frac{1}{\text{Vol}(B(0, r))} \int_{B(0,r)} \mathbb{E}_x[\tau(r)] dx. \tag{55}$$

We will prove in Subject. 5.4 the following theorem.

Theorem 4.13. *Under Hypotheses I, II, and III with $\alpha = 1$, if the renormalization core is stable ($\lambda^- > 0$) then there exists a constant Q such that for $\rho_{\min} > Q\gamma_{\max}$ one has*

$$\limsup_{r \rightarrow \infty} \frac{1}{\ln r} \ln \left(\mathbb{E}_{m_r}[\tau(r)] \right) < 2. \tag{56}$$

More precisely for $r > R_1$ one has

$$\mathbb{E}_{m_r}[\tau(r)] = r^{2-\nu(r)} \tag{57}$$

with

$$v(r) = \frac{\ln \gamma_n(r)}{\ln r} (1 + \epsilon(r)) + \frac{C(r)}{\ln r}, \tag{58}$$

and $|C(r)| \leq C(d, K_0, \gamma_{\max}) + |\ln \lambda_{\infty}^-|$,

$$|\epsilon(r)| < 0.5 \left(\frac{Q \gamma_{\max}}{\rho_{\min}} \right)^{\frac{1}{2}} \leq 0.5, \tag{59}$$

and there exists a finite increasing positive function in each of its arguments F such that

$$Q := \frac{1}{(\ln \gamma_{\min})^2} F \left(d, \frac{(1 + K_0)^2 (1 - 1/\gamma_{\min})^{-1} + \kappa}{\lambda_{\infty}^-} \right) (1 + K_1). \tag{60}$$

Remark 4.14. Equation (58) shows that the anomalous constant is directly related to the number of effective scales. Observe that

$$v(r) \leq \frac{\ln \gamma_{\max}}{\ln \rho_{\min}} \left(1 + 0.5 \left(\frac{Q \gamma_{\max}}{\rho_{\min}} \right)^{\frac{1}{2}} \right) + \frac{C(d, K_0, \gamma_{\max})}{\ln r} \tag{61}$$

and

$$v(r) \geq \frac{\ln \gamma_{\min}}{\ln \rho_{\max}} \left(1 - 0.5 \left(\frac{Q \gamma_{\max}}{\rho_{\min}} \right)^{\frac{1}{2}} \right) - \frac{C(d, K_0, \gamma_{\max})}{\ln r} \tag{62}$$

and $v(r) > 0.4(\ln \gamma_{\min} / \ln \rho_{\max})$ for r large enough. The anomalous parameter $v(r)$ is not a constant because the model is not self-similar; in a self similar case ($\gamma_{\min} = \gamma_{\max} = \gamma$ and $\rho_{\min} = \rho_{\max} = \rho$) one would have at a logarithmic approximation

$$\mathbb{E}[\tau(r)] \sim r^{2-v} \quad \text{with} \quad v \sim \frac{\ln \gamma}{\ln \rho}.$$

The error terms in $v(r)$ are explained by the interaction between the scales which are sensitive to the particular geometry of the eddies. We recall that we consider a quenched model and it is not a priori obvious that the transport should depend only on the power law in velocity field and not on its particular geometry.

Sufficient (and necessary) conditions for the stability of the *renormalization core* ($\lambda^- > 0$) will be given in Subsect. 4.4; we refer to Theorems 4.25, 4.26 and 4.30. In particular if $d = 2$ and if for all k , $E^k = E$, where E corresponds to the cellular flow ($E_{12}(x, y) := \sin(2\pi x) \cos(2\pi y)$) then the renormalization core is stable ($\lambda^- > 0$). We have illustrated the contour lines of the superposition of 4 scales of cellular flows in Fig. 3.



Fig. 3. Superposition of cellular flows

There exists an important literature on the fast transport phenomenon in turbulence addressed (from both the heuristic and rigorous point of view) by using the tools of homogenization or renormalization; we refer to [KS79, AM90, AM87, FGL+91, GLPP92, GZ92, Zha92, GK98, IK91, Gau98, Ave96, Bha99, FK01, BO02b, CP01, AC02] and this panorama is far from being complete; we refer to [MK99] and [Woy00] for a survey. For non-exactly solvable models (non-shear flows) asymptotic fast scaling in the transport behavior have been obtained in the framework of spectral averaging in turbulence. Along this axis L. Piterbarg has obtained [Pit97] fast asymptotic scaling after averaging the transport with respect to the law of the velocity field and the thermal noise and rescaling with respect to space and time. More recently S. Olla and T. Komorowski [KO02] have observed the asymptotic anomalously fast behavior of the mean squared displacement averaged with respect to the thermal noise, the law of the velocity field and time. A. Fannjiang [Fan02] has studied a model where the law of separation of two particles is postulated to be the transport law of a single one as studied in [KO02] and [Pit97].

Fast mixing. In order to show that the phenomenon presented in Theorem 4.13 is super-diffusion and not mere convection, we must compute the rate at which particles do separate and show that this rate follows the same fast behavior. More precisely we will consider $(y_t, z_t) \in \mathbb{R}^d \times \mathbb{R}^d$, where y_t is the solution of (2) and z_t follows the following stochastic differential equation:

$$dz_t = \sqrt{2\kappa}d\bar{\omega}_t + \nabla \cdot \Gamma(z_t) dt, \tag{63}$$

where $\bar{\omega}_t$ is a standard Brownian motion independent of ω_t . Thus y_t and z_t can be seen as two particles transported by the same drift but with independent identically distributed noise. Let us write $B(0, r, l)$ the following subset of $\mathbb{R}^d \times \mathbb{R}^d$:

$$B(0, r, l) := \{(y, z) \in \mathbb{R}^d \times \mathbb{R}^d : |y - z| < r \text{ and } y^2 + z^2 < l^2\}. \tag{64}$$

We write $\mathbb{E}_{y,z}[\tau(r, l)]$ the expectation of the exit time of the diffusion (y_t, z_t) from $B(0, r, l)$ with $(y_0, z_0) = (y, z)$. Let $m_{r,l}$ be the Lebesgue probability measure on the set $B(0, r, l)$ defined by

$$m_{r,l}(dy dz) := \frac{dy dz}{\int_{(y,z) \in B(0,r,l)} dy dz} 1_{B(0,r,l)}. \tag{65}$$

We will consider the mean exit time for the process (y_t, z_t) started with initial distribution $m_{r,l}$, i.e.

$$\mathbb{E}_{m_{r,l}}[\tau(r, l)] = \frac{1}{\text{Vol}(B(0, r, l))} \int_{B(0,r,l)} \mathbb{E}_{y,z}[\tau(r, l)] dx. \tag{66}$$

We have the following theorem proven in Subsect. 5.4.

Theorem 4.15. *Under Hypotheses I, II, and III with $\alpha = 1$, if the renormalization core is stable then there exists a constant Q such that for $\rho_{\min} > Q\gamma_{\max}$ one has*

$$\limsup_{r \rightarrow \infty} \lim_{l \rightarrow \infty} \frac{1}{\ln r} \ln \left(\mathbb{E}_{m_{r,l}}[\tau(r, l)] \right) < 2. \tag{67}$$

More precisely for $r > R_1$ one has

$$\lim_{l \rightarrow \infty} \mathbb{E}_{m_{r,l}}[\tau(r, l)] = r^{2-\nu(r)}, \tag{68}$$

where $\nu(r)$ is given by (58) and Q by (60).

Remark 4.16. It is easy to extend this theorem to any finite number of particles driven by the same flow but independent thermal noise.

Strong self-averaging property. A trivial consequence of Theorem 4.13 and 4.15 is the fact that fast mixing is an almost sure event. More precisely, let us write $H(r)$ and $H(r, l)$ the events

$$H(r) := \left\{ \tau(r) \leq r^{2-\delta} \right\} \quad \text{and} \quad H(r, l) := \left\{ \tau(r, l) \leq r^{2-\delta} \right\}$$

with $\delta = 0.9 \ln \gamma_{\min} / \ln \rho_{\max}$. Observe that $\delta > 0$ and we have the following theorem

Theorem 4.17. *Under Hypotheses I, II and III with $\alpha = 1$, if the renormalization core is stable then there exists a constant Q such that for $\rho_{\min} > Q\gamma_{\max}$ one has*

$$\lim_{r \rightarrow \infty} \mathbb{P}_{m_r} [H(r)] = 1 \quad \text{and} \quad \lim_{r \rightarrow \infty} \lim_{l \rightarrow \infty} \mathbb{P}_{m_{r,l}} [H(r, l)] = 1. \tag{69}$$

In this theorem Q is given by (60), by slightly modifying the constants.

4.4. Diagnosis of renormalization core’s pathologies. With Subsect. 4.3 we have seen that our model is super-diffusive if the *renormalization core* is stable. With this subsection we will give necessary and sufficient conditions for the stability of the *renormalization core* by analyzing in detail its dynamic. The results given here will be proven in Subsect. 5.3.

Diffusive properties of the eddies at vanishing molecular conductivity. We will need the following functions V and W describing the effective behavior of the eddies E^k of the *renormalization core* at vanishing molecular conductivity. For $E \in S_{L^\infty}(\mathbb{T}^d)$ and $\zeta > 0$ we write

$$V(\zeta, E) := \frac{\lambda_{\min}(\sigma_{\text{sym}}(\zeta I_d, E))}{\zeta}. \tag{70}$$

Observe that by the variational formulation (17) one has

$$\begin{aligned} 1/V(\zeta, E) = & \sup_{\xi \in \mathbb{S}^{d-1}} \inf_{(f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H + E \nabla f|^2 dx \\ & + \zeta^2 \int_{\mathbb{T}^d} |\nabla f(x)|^2 dx, \end{aligned} \tag{71}$$

where we have written \mathbb{S}^{d-1} the unit sphere of \mathbb{R}^d centered on 0. Observe that if for $\xi \in \mathbb{S}^{d-1}$, $\nabla \cdot E \cdot \xi \neq 0$ one has $\forall \zeta > 0$ $V(\zeta, E) > 1$. Moreover V is continuous and decreasing in ζ . Let us define

Definition 4.18.

$$V(\zeta) := \inf_{n \in \mathbb{N}} V(\zeta, E^n), \tag{72}$$

$$V(0) := \lim_{\zeta \downarrow 0} V(\zeta). \tag{73}$$

Observe that $V(\zeta)$ is a decreasing function in ζ thus the limit (73) is well defined and belongs to $[1, \infty]$. We define for $x \in (1, V(0))$, the inverse function V^{-1} as

$$V^{-1}(x) := \sup\{y > 0 : V(y) > x\}. \tag{74}$$

Observe that if $V(0) > 1$, $V^{-1}(x)$ is a decreasing function of x in $(1, V(0))$. Similarly we introduce

$$W(\zeta, E) := \frac{\lambda_{\max}(\sigma_{\text{sym}}(\zeta I_d, E))}{\zeta}. \tag{75}$$

Observe that by the variational formulation (23) one has

$$\begin{aligned} W(\zeta, E) = 1 + \sup_{l \in \mathcal{S}^{d-1}} \sup_{\xi \perp l, (f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \inf_{\zeta^{-2} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H - E(l - \nabla f)|^2 dx} \\ + \int_{\mathbb{T}^d} |\nabla f|^2 dx. \end{aligned} \tag{76}$$

Observe that if for $l \in \mathbb{S}^{d-1}$, $\nabla \cdot E \cdot l \neq 0$ one has $\forall \zeta > 0$ $W(\zeta, E) > 1$. Moreover W is continuous and decreasing in ζ . Let us define

Definition 4.19.

$$W(\zeta) := \sup_{n \in \mathbb{N}} W(\zeta, E^n), \tag{77}$$

$$W(0) := \lim_{\zeta \downarrow 0} W(\zeta). \tag{78}$$

Observe that $W(\zeta)$ is a decreasing function in ζ , thus the limit (78) is well defined and belongs to $[1, \infty]$.

We recall that for $E \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$ and $\zeta > 0$, one has

$$1 \leq V(\zeta, E) \leq 1 + C_d \zeta^{-2} \lambda_{\min} \left(\int_{\mathbb{T}^d} {}^t E(x) E(x) dx \right) \tag{79}$$

and

$$1 \leq W(\zeta, E) \leq 1 + C_d \zeta^{-2} \lambda_{\max} \left(\int_{\mathbb{T}^d} {}^t E(x) E(x) dx \right). \tag{80}$$

Moreover the behavior of $V(\zeta, E)$ and $W(\zeta, E)$ at vanishing molecular conductivity (as $\zeta \downarrow 0$) and their connections with the stream lines of the eddies has been widely studied in the literature (we refer to [IK91, FP94] and the references therein). Thus it has been obtained [FP94] that for any $\beta \in [-\frac{1}{2}, 0]$ there exist $E \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$ such that $V(\zeta, E) = W(\zeta, E)$ and as $\zeta \downarrow 0$,

$$V(\zeta, E) \sim c^* \zeta^\beta, \tag{81}$$

where c^* can be calculated explicitly in several cases. A particular example with $V(\zeta, E) \sim -c^* \ln \zeta$ is also given in [FP94]. For anisotropic cases, for any $\delta \in [0, 1/2]$ there exist $E \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$ such that [FP94]

$$W(\zeta, E) \sim c_1^* \zeta^{3\delta-2} \quad V(\zeta, E) \sim c_2^* \zeta^{-\delta}. \tag{82}$$

The stability of the renormalization core and its anisotropy. Theorem 4.20 shows that the anisotropy of the local turbulent conductivity is one of the causes of the instability of the *renormalization core*. It is natural to wonder whether the converse is true; the answer is positive at low flow rate as shown by the following theorem and corollary.

Theorem 4.20. *If the renormalization core has bounded anisotropic distortion ($\mu < \infty$) then*

1. *if $\gamma_{\max} < V(0)$ then the renormalization core is stable ($\lambda^- > 0$). Moreover if the monotony of V is strict then*

$$\lambda^- \geq \mu^{-\frac{1}{2}} \frac{V^{-1}(\gamma_{\max})}{\gamma_{\max}}. \tag{83}$$

2. *If $W(0) < \infty$ then the renormalization core is bounded from above and*

$$\lambda^+ \leq \mu^{\frac{1}{2}} C_d K_0 (\gamma_{\min} - 1)^{-\frac{1}{2}} (1 + W(0)). \tag{84}$$

Corollary 4.21. *If $\gamma_{\max} < V(0)$ and the renormalization core is not stable ($\lambda^- = 0$) then it has unbounded anisotropic distortion ($\mu = \infty$).*

Definition 4.22. *The flow is said to be isotropic if for all $\zeta > 0, k \in \mathbb{N}, \sigma_{\text{sym}}(\zeta I_d, E^k)$ is a multiple of the identity matrix.*

Definition 4.23. *The renormalization core is said to be isotropic if for all k, A^k is a multiple of the identity matrix. We then write $A^k = \lambda(A^k)I_d$.*

Observe that if the flow is isotropic then so is the *renormalization core*, $\mu = 1$ and from Theorem 4.20 we obtain the following corollary.

Corollary 4.24. *If the flow is isotropic then the renormalization core is stable for $\gamma_{\max} < V(0)$.*

Combining Theorem 4.20 and 4.10 we obtain that for $\gamma_{\max} < V(0)$ the *renormalization core* is not stable if and only if it has unbounded anisotropic distortion. Moreover we have the following theorem

Theorem 4.25. *If the renormalization core has bounded anisotropic distortion, the monotony of V is strict, $\gamma_{\max} < V(0)$ then the renormalization core is stable ($\lambda^- > 0$) and*

$$C_1 \leq \lambda^- \lambda^+ \leq C_2, \tag{85}$$

with $C_1 = (V^{-1}(\gamma_{\max})/\gamma_{\max})^2$ and $C_2 = C_d K_0^2 (1 - 1/\gamma_{\min})^{-1}$.

We believe that Eq. (85) could be at the origin of the isotropy of turbulence at small scales. Let us observe that if $V(0) = \infty$ then the stability of the *renormalization core* is equivalent to the fact that it has bounded anisotropic distortion. It is easy to build from Theorem 4.25 and the analysis of $V(\zeta, E)$ given above, examples of flows with stable *renormalization core* and thus a strongly-super-diffusive behavior. In particular we have the following theorem

Theorem 4.26. *If $V(0) = \infty$ and the renormalization core is isotropic then under hypotheses **I**, **II** and **III** with $\alpha = 1$ the flow is strongly super-diffusive for $\rho_{\min} > 11Q\gamma_{\max}$ (where Q is given by (60)) and Theorems 4.13, 4.15 and 4.17 are valid (with $\lambda^- > 0$).*

Observe that if $d = 2$ and if for all $k, E^k = E$, where E corresponds to the cellular flow ($E_{12}(x, y) := \sin(2\pi x) \cos(2\pi y)$), then $V(0) = \infty$ ([FP94]) and the renormalization core is stable ($\lambda^- > 0$).

Viscosity implosion. It is easy to obtain that if there exists $\delta > 0$ such that for all k the drift $\nabla \cdot E^k$ is null on $[0, 1]^d \setminus [\delta, 1 - \delta]^d$ then $W(0) < \infty$. Moreover we have the following theorem.

Theorem 4.27. *If $\gamma_{\min} > W(0)$ then the renormalization core is vanishing with exponential rate and*

$$\limsup_{n \rightarrow \infty} \frac{\ln(\lambda_{\max}(A^n))}{n} \leq \ln\left(\frac{W(0)}{\gamma_{\min}}\right). \tag{86}$$

It follows from Theorem 4.27 the *renormalization core* can be isotropic and not stable at the same time. Now it is natural to wonder whether a *renormalization core* (and thus the transport properties of the flow) may undergo a brutal alteration.

Definition 4.28. *We call viscosity implosion the bifurcation from a stable renormalization core to a vanishing renormalization core.*

We will now analyze this phenomenon.

Definition 4.29. *The flow is said to be self-similar if and only if $\gamma_{\max} = \gamma_{\min} = \gamma$, $\rho_{\max} = \rho_{\min} = \rho$ and for all k , $E^k = E^0 = E$.*

Let us recall that a real turbulent flow has a non self-similar multi-scale structure, we refer to [DC97]. Observe that if the flow is self-similar then $V(\zeta) = W(\zeta)$. In this case we will write

$$\gamma_c := V(0). \tag{87}$$

Theorem 4.30. *Assume the flow to be self-similar and isotropic.*

1. *If $\gamma < \gamma_c$ then the renormalization core is stable ($\lambda^- > 0$) and*

$$\lim_{n \rightarrow \infty} A^n = \zeta_0 I_d, \tag{88}$$

where ζ_0 is the unique solution of $V(\zeta_0) = \gamma$.

2. *If $\gamma = \gamma_c$ and $(V(0) - V(x))x^{-p}$ admits a non-null limit as $x \downarrow 0$ with $p > 0$ then the renormalization core is vanishing with polynomial rate (in particular $\lambda^+ = \lambda^- = 0$):*

$$\lim_{n \rightarrow \infty} \frac{\ln \lambda(A^n)}{\ln n} = -\frac{1}{p}. \tag{89}$$

3. *If $\gamma > \gamma_c$ then the renormalization core is vanishing with exponential rate (in particular $\lambda^+ = \lambda^- = 0$)*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda(A^n) = \ln\left(\frac{\gamma_c}{\gamma}\right). \tag{90}$$

It follows from Eq. (88) that if the flow is self-similar and the renormalization core isotropic and E non-constant then $V(0) > 1$ and for $1 < \gamma < V(0)$ the flow is strongly super-diffusive and Theorems 4.13, 4.15 and 4.17 are valid (with $\lambda^- > 0$).

The viscosity implosion of the *renormalization core* implies that the strong self-averaging property of the flow collapses towards a chaotic coupling between the scales. Let us give a particular example to illustrate what we mean by such bifurcation. The flow is assumed to be self-similar and isotropic and the stream lines of the eddy E over a period $[0, 1]^3$ are given in Fig. 4(a). Since there exists $\delta > 0$ such that the drift $\nabla \cdot E$ is

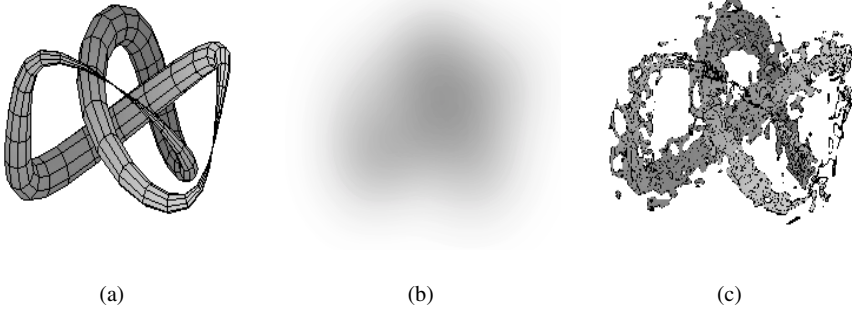


Fig. 4a–c. Viscosity implosion. (a) An implosive eddy geometry; (b) Stable *renormalization core*; (c) Vanishing *renormalization core*

null on $[0, 1]^d \setminus [\delta, 1 - \delta]^d$ we have $\gamma_c < \infty$ with the eddy illustrated in Fig. 4(a) Now imagine that one puts a drop of dye in such a flow and observe its transport at very large spatial scale. We have illustrated in Fig. 4 a metaphorical illustration of what one could see; it would be interesting to run numerical simulations to analyze the behavior of a drop of dye at the transition between a stable and vanishing renormalization core. For $\gamma < \gamma_c$ dye is transported by strong super-diffusion, and the density of its colorant in the flow is homogeneous (Fig. 4(b)). Moreover in the domain $(0, \gamma_c)$ an increase of the flow rate γ in the eddies is compensated by an increase of the diffusive (dissipative) power of the smaller eddies. The picture undergoes a brutal transformation at $\gamma \geq \gamma_c$; in this domain an increase of γ results in the growth of the advective power of the eddies but their diffusive power remains bounded and can no longer compensate convection. The diffusive power of the smaller scales becomes dominated by the convective power of the eddy at the observation scale (Fig. 4(c)). The drop dye is then transported by advection and presents high density gradients.

Variational formulae for γ_c . Assume the flow to be self-similar and isotropic. Thus from Eq. (71) it is easy to obtain that

$$\gamma_c^{-1} = \inf_{(f,H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\xi - \operatorname{div} H + E \nabla f|^2 dx; \tag{91}$$

from Eq. (76) it is also easy to obtain that for any unit vector l in \mathbb{R}^d ,

$$\begin{aligned} \gamma_c &= 1 + \lim_{\zeta \downarrow 0} \inf_{\xi \perp l, (f,H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \zeta^{-2} \int_{\mathbb{T}^d} |\xi - \operatorname{div} H - E(l - \nabla f)|^2 dx \\ &\quad + \int_{\mathbb{T}^d} |\nabla f|^2 dx. \end{aligned} \tag{92}$$

Write G the set of $f \in H^1(\mathbb{T}^d)$ such that there exists $\xi, l \in \mathbb{R}^d$ and H a skew symmetric matrix with coefficients in $H^1(\mathbb{T}^d)$ with $\xi \perp l$ and

$$\xi - \operatorname{div} H - E(l - \nabla f) = 0. \tag{93}$$

Then if $G = \emptyset$ it is easy to obtain from (92) that $\gamma_c = \infty$. If $G \neq \emptyset$ then one has

$$\gamma_c = 1 + \inf_{f \in G} \int_{\mathbb{T}^d} |\nabla f|^2 dx. \tag{94}$$

Equation (93) is degenerate, thus it is not easy to prove a solution for that equation in a general case and actually most of the time it has no solution which means that $\gamma_c = \infty$. It would be interesting to obtain non-trivial criteria ensuring the existence of a solution for (93). The most trivial example of a stream matrix E such that $\gamma_c < \infty$ is the following one. Take $d = 2$ and E a skew symmetric matrix with $E_{1,2} = h$, where h over the period $[0, 1]^2$ is equal to

$$h(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2) g\left(4((x_1 - 0.5)^2 + (x_2 - 0.5)^2)\right), \tag{95}$$

where g is any smooth function on $[0,1]$ such that $g = 1$ on $[0, 1/3]$ and $g = 0$ on $[2/3, 1]$. Then it is easy to check that $1 < \gamma_c < \infty$ and estimate it from the variational formulae given above. For instance write G' the set of smooth \mathbb{T}^d periodic function f such that $\nabla f = e_2$ on $\{x : (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 1/6\}$, then it is easy to check that

$$\gamma_c \leq 1 + \inf_{f \in G'} \int_{\mathbb{T}^d} |\nabla f|^2 dx. \tag{96}$$

The renormalization core with a finite number of scales. The results given above were related to the asymptotic behavior of the *renormalization core*. When the flow has only a finite number of scales we will give below quantitative estimates controlling the *renormalization core*.

Theorem 4.31. *The ubiety of the renormalization core is bounded from above by the inverse of its stability. Writing $C = C_d K_0^2 (1 - 1/\gamma_{\min})^{-1}$ we have*

$$\lambda_n^+ \leq \kappa + \frac{C}{\lambda_{n-1}^-} \quad \text{and} \quad \mu_n \leq \frac{\kappa}{\lambda_n^-} + \frac{C}{(\lambda_n^-)^2}.$$

Theorem 4.32. *We have*

$$\lambda_n^- \geq \min\left(\lambda_{\min}(\kappa), (\mu_n)^{-\frac{1}{2}} \frac{V^{-1}(\gamma_{\max})}{\gamma_{\max}}\right). \tag{97}$$

Theorem 4.33. *We have for $n \in \mathbb{N}$,*

$$\lambda_n^+ \leq \max\left(\lambda_{\max}(\kappa), C_d K_0 \mu_n^{\frac{1}{2}} (\gamma_{\min} - 1)^{-\frac{1}{2}}\right) (1 + W(0)). \tag{98}$$

In particular, observe that if $\gamma_{\max} < \infty$ and $\lambda^- = 0$ then the stability of the *renormalization core* should decrease according to the following relation $\ln \lambda_n^- \sim -0.5 \ln \mu_n$ and its ubiety should increase like $\ln \lambda_n^+ \sim 0.5 \ln \mu_n$.

5. Proofs

5.1. Averaging with two scales: Proof of Theorem 4.1. There are two strategies to prove Theorem 4.1; the first one is based on the relative translation method introduced in [Owh01a] and the variational formulations of the effective conductivity; this is the strategy used in [BO02a]. The second one is new and based only on the relative translation method. Although the first strategy in the case considered here would give (the proof is rather long) a sharper estimate of the error term: $(1 + \epsilon(R))^2$ with $\epsilon(R) = \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} f(d, \frac{\|a+P\|_\infty}{\lambda_{\min}(a)})$ instead of (24), we have preferred to write here the second one for its simplicity and the fact that it allows to obtain a lower and an upper bound at once without the need of any variational formulation. Let us now give this new alternative strategy.

By the variational formulation (23) the effective conductivity $\sigma_{\text{sym}}(a, S_R P + K)$ is continuous in $L^\infty(\mathbb{T}^d)$ norm with respect to the stream matrices P and K and by density it is sufficient to prove the estimate (24) assuming that P and K are smooth and belong to $\mathcal{S}(\mathbb{T}^d)$.

First we will prove the following proposition where we have used the notation introduced in Sect. 3 (we write $E := S_R P + K$).

Proposition 5.1. *Let $l \in \mathbb{R}^d$, $l \neq 0$,*

$$\left[\left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l \right)^{\frac{1}{2}} - \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l \right)^{\frac{1}{2}} \right]^2 \leq J_1 + J_2 + J_3 + J_4 \quad (99)$$

with

$$J_1 = \int_{(x,y) \in \mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(a - E \left(x + \frac{y}{R} \right) \right) \nabla \chi^{a,P} (Rx + y) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy, \quad (100)$$

$$J_2 = - \int_{\mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left((a + P(Rx + y)) \nabla F^{a,P} (Rx + y) - \sigma_{\text{sym}}(a, P) \right) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy, \quad (101)$$

$$J_3 = - \int_{\mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(K \left(x + \frac{y}{R} \right) - K(x) \right) \nabla F^{a,P} (Rx + y) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy, \quad (102)$$

$$J_4 = \int_{\mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) K(x) \nabla \chi^{a,P} (Rx + y) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy. \quad (103)$$

Proof. Let us write

$$I = \int_{(x,y) \in \mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) a \nabla F^{a,P} (Rx + y) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy. \quad (104)$$

Using the Cauchy-Schwartz inequality and formula (20) one obtains that

$$I \leq \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l \right)^{\frac{1}{2}} \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l \right)^{\frac{1}{2}}. \quad (105)$$

Now, writing $E = S_R P + K$ observe that

$$I = (I_1 + I_2)/2 \quad (106)$$

with

$$I_1 = \int_{(x,y) \in \mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(a + E \left(x + \frac{y}{R} \right) \right) \nabla F^{a,P} (Rx + y) \\ \times \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy$$

and

$$I_2 = \int_{(x,y) \in \mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(a - E \left(x + \frac{y}{R} \right) \right) \nabla F^{a,P} (Rx + y) \\ \times \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy.$$

Using

$$\int_{(x,y) \in \mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(a - E \left(x + \frac{y}{R} \right) \right) l = {}^t l \sigma_{\text{sym}}(a, S_R P + K) l$$

and the fact that $\nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(a - E \left(x + \frac{y}{R} \right) \right)$ is a divergence free vector field one obtains that

$$I_2 = {}^t l \sigma_{\text{sym}}(a, S_R P + K) l - J_1$$

with J_1 given by (100). Moreover

$$I_1 = G_0 - J_2 - J_3 - J_4 \quad (107)$$

with

$$G_0 = \int_{(x,y) \in \mathbb{T}^d \times [-\frac{1}{2}, \frac{1}{2}]^d} \nabla F_l^{a,E} \left(x + \frac{y}{R} \right) \left(\sigma_{\text{sym}}(a, P) + K(x) \right) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x) dx dy \\ = {}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l, \quad (108)$$

where we have used in the last equality the fact that $(\sigma_{\text{sym}}(a, P) + K(x)) \nabla F_l^{\sigma_{\text{sym}}(a,P),K} (x)$ is divergence free. And J_2, J_3, J_4 are given by (101), (102) and (103). Thus combining (105) and (106) we have obtained (99), which proves the proposition. \square

Now we will show that J_1, J_2, J_3 and J_4 act as error terms in the homogenization process. Using $\operatorname{div}((a + E)\nabla F_l^{a,E}) = 0$ and observing that, $\nabla \chi^{a,P}(Rx + y) = \nabla_y \chi^{a,P}(Rx + y)$ and integrating by parts in y one obtains (writing $\partial^i([-\frac{1}{2}, \frac{1}{2}]^d) = \{x \in [-\frac{1}{2}, \frac{1}{2}]^d : x_i = -\frac{1}{2}\}$)

$$J_1 = G_1 + G_2 \tag{109}$$

with (writing $(e_i)_{1 \leq i \leq d}$ the orthonormal basis of \mathbb{R}^d compatible with the axis of periodicity of \mathbb{T}^d)

$$G_1 = \sum_{i=1}^d \int_{(x,y^i) \in \mathbb{T}^d \times \partial^i([-\frac{1}{2}, \frac{1}{2}]^d)} ({}^t \nabla F_l^{a,E}(x + (y^i + e_i)/R) - \nabla F_l^{a,E}(x + y^i/R)) \times (a - P(Rx + y^i)) \cdot e_i \chi^{a,P}(Rx + y^i) \nabla F_l^{\sigma_{\text{sym}}(a,P),K}(x) dx dy^i \tag{110}$$

$$G_2 = - \sum_{i=1}^d \int_{(x,y^i) \in \mathbb{T}^d \times \partial^i([-\frac{1}{2}, \frac{1}{2}]^d)} ({}^t \nabla F_l^{a,E}(x + (y^i + e_i)/R)K(x + (y^i + e_i)/R) - {}^t \nabla F_l^{a,E}(x + y^i/R)K(x + y^i/R)) \cdot e_i \chi^{a,P}(Rx + y^i) \nabla F_l^{\sigma_{\text{sym}}(a,P),K}(x) dx dy^i, \tag{111}$$

Now we will need the following lemma which says that the solution of the two-scale cell problem keeps in its structure a signature of the fast period.

Lemma 5.2. *For $i \in \{1, \dots, d\}$ one has*

$$\int_{x \in \mathbb{T}^d} |\nabla F_l^{a,E}(x + e_i/R) - \nabla F_l^{a,E}(x)|_a^2 \leq |l|_{\sigma(a,E)}^2 C_d \left(\frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \right)^2. \tag{112}$$

Proof. Observe that

$$\begin{aligned} & \nabla \cdot (a + E(x)) \nabla (F_l^{a,E}(x + e_i/R) - F_l^{a,E}(x)) \\ &= \nabla \cdot ((K(x) - K(x + e_i/R)) \nabla F_l^{a,E}(x + e_i/R)). \end{aligned} \tag{113}$$

It follows that

$$\int_{x \in \mathbb{T}^d} |\nabla F_l^{a,E}(x + e_i/R) - \nabla F_l^{a,E}(x)|_a^2 = \int_{x \in \mathbb{T}^d} (\nabla F_l^{a,E}(x + e_i/R) - \nabla F_l^{a,E}(x)) (K(x) - K(x + e_i/R)) \nabla F_l^{a,E}(x + e_i/R), \tag{114}$$

thus using Cauchy-Schwartz inequality one obtains

$$\begin{aligned} & \int_{x \in \mathbb{T}^d} |\nabla F_l^{a,E}(x + e_i/R) - \nabla F_l^{a,E}(x)|_a^2 \\ & \leq \int_{x \in \mathbb{T}^d} |(K(x) - K(x + e_i/R)) \nabla F_l^{a,E}(x + e_i/R)|_{a^{-1}}^2, \end{aligned} \tag{115}$$

and Eq. 112 follows easily. \square

It follows from Lemma 5.2 Eq. 110 and Cauchy-Schwartz inequality that

$$G_1 \leq C_d \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \|\chi^{a,P}\|_\infty \frac{\|a - P\|_\infty}{\lambda_{\min}(a)} \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l\right)^{\frac{1}{2}} \times \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l\right)^{\frac{1}{2}}. \quad (116)$$

Now we will use the following lemma which is a consequence of Stampacchia estimates [Sta66, Sta65] for elliptic operators with discontinuous coefficients (see [Owh01a], Appendix B, Theorem B.1.1) (we recall that $\chi^{a,P}$ is uniquely defined by the cell problem and $\int_{\mathbb{T}^d} \chi^{a,P}(x) dx = 0$).

Lemma 5.3.

$$\|\chi^{a,P}\|_\infty \leq C_d \left(\frac{\|a + P\|_\infty}{\lambda_{\min}(a)}\right)^{3d+2}. \quad (117)$$

Thus one obtains from (117) and (116) that

$$G_1 \leq C_d \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \left(\frac{\|a + P\|_\infty}{\lambda_{\min}(a)}\right)^{3d+3} \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l\right)^{\frac{1}{2}} \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l\right)^{\frac{1}{2}} \quad (118)$$

similarly, observing that $\text{div}_y (K(x) \nabla F_l^{a,E}(x + \frac{y}{R})) = 0$ and integrating by part in y in Eq. (103) one obtains

$$J_4 = \sum_{i=1}^d \int_{(x,y^i) \in \mathbb{T}^d \times \partial^i([-\frac{1}{2}, \frac{1}{2}]^d)} \left({}^t \nabla F_l^{a,E}(x + (y^i + e_i)/R) - {}^t \nabla F_l^{a,E}(x + y^i/R)\right) K(x) \cdot e_i \chi^{a,P}(Rx + y^i) \nabla F_l^{\sigma_{\text{sym}}(a,P),K}(x) dx dy^i. \quad (119)$$

Adding Eq. (111) to Eq. (119) we obtain

$$J_4 + G_2 = \sum_{i=1}^d \int_{(x,y^i) \in \mathbb{T}^d \times \partial^i([-\frac{1}{2}, \frac{1}{2}]^d)} {}^t \nabla F_l^{a,E}(x + (y^i + e_i)/R) (K(x) - K(x + (y^i + e_i)/R)) \cdot e_i \times \chi^{a,P}(Rx + y^i) \nabla F_l^{\sigma_{\text{sym}}(a,P),K}(x) dx dy^i + \sum_{i=1}^d \int_{(x,y^i) \in \mathbb{T}^d \times \partial^i([-\frac{1}{2}, \frac{1}{2}]^d)} {}^t \nabla F_l^{a,E}(x + y^i/R) (K(x + y^i/R) - K(x)) \cdot e_i \times \chi^{a,P}(Rx + y^i) \nabla F_l^{\sigma_{\text{sym}}(a,P),K}(x) dx dy^i, \quad (120)$$

and by Cauchy Schwartz inequality and Lemma 117 one obtains that

$$J_4 + G_2 \leq C_d \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \left(\frac{\|a + P\|_\infty}{\lambda_{\min}(a)}\right)^{3d+3} \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l\right)^{\frac{1}{2}} \times \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l\right)^{\frac{1}{2}}. \quad (121)$$

Moreover from Eq. (102) and Cauchy Schwartz inequality one easily obtains

$$J_3 \leq C_d \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l \right)^{\frac{1}{2}} \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l \right)^{\frac{1}{2}}. \quad (122)$$

Now we will need the following lemma

Lemma 5.4. *If $V \in (C^\infty(\mathbb{T}^d))^d$ is such that $\text{div } V = 0$ and $\int_{\mathbb{T}^d} V(x) dx = 0$ then for $p > d$, there exists a skew symmetric \mathbb{T}^d -periodic matrix M such that $\|M\|_\infty \leq C_{d,p} \|V\|_{L^p(\mathbb{T}^d)}$ and $V = \nabla \cdot M$.*

Proof. From the proof of Lemma 4.7 of [BO02a] one obtains that there exists a \mathbb{T}^d -periodic smooth skew-symmetric matrix M such that

$$V_i = \sum_{j=1}^d M_{ij} \quad (123)$$

and M is given by

$$M_{ij} = B_i^j - B_j^i, \quad (124)$$

where B_j^i are the smooth \mathbb{T}^d periodic solutions of

$$\Delta B_j^i = \partial_i V_j \quad (125)$$

with 0 mean Lebesgue measure. Using Theorem 5.4 of [Sta66] one obtains that for $p > d$

$$\|B_j^i\|_\infty \leq C_{d,p} \|V_j\|_{L^p(\mathbb{T}^d)} \quad (126)$$

which proves the lemma.

Let us now prove the following lemma.

Lemma 5.5.

$$\left((a + P(Rx + y)) \nabla F^{a,P}(Rx + y) - \sigma(P) \right)_{kj} = \sum_{i=1}^d \partial_i M_{kij}^P(Rx + y), \quad (127)$$

where M^P is a \mathbb{T}^d periodic $d \times d \times d$ tensor such that $M_{ikj}^P = -M_{kij}^P$ and

$$\|M^P\|_\infty \leq C_{d, \|a+P\|_\infty / \lambda_{\min}(a)} \|a + P\|_\infty. \quad (128)$$

Proof. From Lemma 5.4 one obtains that for $p > d$,

$$\|M^P\|_\infty \leq C_{d,p} \|(a + P(\cdot)) \nabla F^{a,P}(\cdot) - \sigma(P)\|_{L^p(\mathbb{T}^d)}. \quad (129)$$

Using Meyers argument [Mey63] one obtains that there exists $\rho(\|a+E\|_\infty / \lambda_{\min}(a)) > d$ such that

$$\|\nabla \chi^{a,P}(\cdot)\|_{L^p(\mathbb{T}^d)} \leq C_{d, \|a+P\|_\infty / \lambda_{\min}(a)}, \quad (130)$$

which implies Eq. (128). \square

Using Eq. (127) and integrating by part in y in (101) one obtains

$$J_2 = - \sum_{i,j,k=1}^d \int_{(x,y^i) \in \mathbb{T}^d \times \partial^i([-1/2, 1/2]^d)} ({}^t \nabla F_l^{a,E}(x+(y^i+e_i)/R) - {}^t \nabla F_l^{a,E}(x+y^i/R)) \cdot e_k M_{kij}^P(Rx+y^i)e_j \cdot \nabla F_l^{\sigma_{\text{sym}}(a,P),K}(x) dx dy^i. \tag{131}$$

Combining this with (128) and (112) one obtains from Cauchy-Schwartz inequality that

$$J_2 \leq C_{d,\|a+P\|_\infty/\lambda_{\min}(a)} \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l\right)^{\frac{1}{2}} \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l\right)^{\frac{1}{2}}. \tag{132}$$

In conclusion we have obtained from Eq. (99), (109), (118), (121), (122) and (132) that

$$\begin{aligned} & \left[\left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l\right)^{\frac{1}{2}} - \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l\right)^{\frac{1}{2}} \right]^2 \\ & \leq C_{d,\|a+P\|_\infty/\lambda_{\min}(a)} \frac{\|K\|_\alpha}{R^\alpha \lambda_{\min}(a)} \left({}^t l \sigma_{\text{sym}}(a, S_R P + K) l\right)^{\frac{1}{2}} \left({}^t l \sigma_{\text{sym}}(\sigma_{\text{sym}}(a, P), K) l\right)^{\frac{1}{2}}. \end{aligned} \tag{133}$$

Now we will use the following lemma whose proof is trivial

Lemma 5.6. *If $(X - Y)^2 \leq \delta XY$ then $X/Y \leq (1 + 8\sqrt{\delta})^2$.*

And the estimate (24) is a simple consequence of (133) and Lemma 5.6 which proves the theorem.

5.2. Averaging with n scales: Proof of Theorem 4.4. The proof of Theorem 4.4 is based on Theorem 4.1 and a reverse induction. It is important to note that contrary to reiterated homogenization, here the larger scales are homogenized first; this reversion in the inductive process is essential to obtain sharp estimates. Observe that by the variational formula 17 one has for $\gamma > 0$, $B \in \mathcal{M}_{d,\text{sym}}$ and $K \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$,

$$\sigma_{\text{sym}}(B, \gamma K) = \gamma \sigma_{\text{sym}}\left(\frac{B}{\gamma}, K\right). \tag{134}$$

From this we deduce that for $p \in \{0, \dots, n - 1\}$,

$$\sigma_{\text{sym}}\left(\sigma_{\text{sym}}(B, E^p), \frac{1}{\gamma_p} \Gamma^{p+1,n}\right) = \frac{\gamma_{p+1}}{\gamma_p} \sigma_{\text{sym}}\left(\frac{\gamma_p}{\gamma_{p+1}} \sigma_{\text{sym}}(B, E^p), \frac{1}{\gamma_{p+1}} \Gamma^{p+1,n}\right). \tag{135}$$

Combining this with Theorem 4.1 one obtains that for $p \in \{0, \dots, n - 1\}$,

$$\sigma_{\text{sym}}\left(B, \frac{1}{\gamma_p} \Gamma^{p,n}\right) \leq \frac{\gamma_{p+1}}{\gamma_p} \sigma_{\text{sym}}\left(\frac{\gamma_p}{\gamma_{p+1}} \sigma_{\text{sym}}(B, E^p), \frac{1}{\gamma_{p+1}} \Gamma^{p+1,n}\right) (1 + \epsilon_{p+1}(B))^4, \tag{136}$$

$$\sigma_{sym}\left(B, \frac{1}{\gamma_p} \Gamma^{p,n}\right) \geq \frac{\gamma_{p+1}}{\gamma_p} \sigma_{sym}\left(\frac{\gamma_p}{\gamma_{p+1}} \sigma_{sym}(B, E^p), \frac{1}{\gamma_{p+1}} \Gamma^{p+1,n}\right) (1 + \epsilon_{p+1}(B))^{-4} \tag{137}$$

with

$$\epsilon_p(B) = \left(\frac{\gamma_p K_\alpha}{\gamma_{p-1} r_p^\alpha \lambda_{\min}(B) (1 - \gamma_{\max}/\rho_{\min}^\alpha)}\right)^{\frac{1}{2}} f((d, (\lambda_{\max}(B) + K_0)/\lambda_{\min}(B))). \tag{138}$$

Then one obtains by a simple induction that

$$\gamma_{n+1} A^{n+1} \prod_{p=0}^{n-1} (1 + \epsilon_{p+1}(A^p))^{-4} \leq \sigma_{sym}(a, \Gamma^{0,n}) \leq \gamma_{n+1} A^{n+1} \prod_{p=0}^{n-1} (1 + \epsilon_{p+1}(A^p))^4, \tag{139}$$

where A^k , is the *renormalization coreization sequence* given in Definition 4.3 which proves Theorem 4.4.

5.3. Diagnosis of renormalization core’s pathologies: Proofs. Let $a \in \mathcal{M}_{d,\text{sym}}$ and $E \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$; it is well known ([AM91]) and a simple consequence of (17) and (23) that

$$a \leq \sigma_{sym}(a, E) \leq a + \int_{\mathbb{T}^d} {}^t E(x) a^{-1} E(x) dx. \tag{140}$$

Then the following proposition follows from (33), (140) and a simple induction on the number of scales.

Proposition 5.7. *For all $n \in \mathbb{N}$,*

$$\gamma_n^{-1} I_d \leq A^n \leq (\kappa/\gamma_n) I_d + \sum_{p=0}^{n-1} (\gamma_p/\gamma_n) \int_{\mathbb{T}_1^d} {}^t E^p(x) (A^p)^{-1} E^p(x) dx. \tag{141}$$

Theorems 4.20 and 4.31 are straightforward consequences of Proposition 5.7. We will need the following proposition giving isotropic estimates on anisotropic viscosities.

Proposition 5.8. *For $a \in \mathcal{M}_{d,\text{sym}}$ and $E \in \mathcal{S}_{L^\infty}(\mathbb{T}^d)$, one has for all $l \in (\mathbb{R}^d)^*$,*

$$\left(\frac{\lambda_{\max}(a)}{\lambda_{\min}(a)}\right)^{-\frac{1}{2}} \leq \frac{{}^t l \sigma_{sym}(a, E) l}{{}^t l \sigma_{sym}\left((\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} I_d, E\right) l} \leq \left(\frac{\lambda_{\max}(a)}{\lambda_{\min}(a)}\right)^{\frac{1}{2}}. \tag{142}$$

Proof. By the variational formula 17 one has for $\xi \in \mathbb{R}^d$,

$$\begin{aligned} |\xi|_{\sigma_{sym}^{-1}(a, E)}^2 &= \inf_{(f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H + E \nabla f|_{a^{-1}}^2 dx + \int_{\mathbb{T}^d} |\nabla f(x)|_a^2 dx \\ &\leq (\lambda_{\max}(a)/\lambda_{\min}(a))^{\frac{1}{2}} \inf_{(f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} (\lambda_{\max}(a)\lambda_{\min}(a))^{-\frac{1}{2}} \\ &\quad \int_{\mathbb{T}^d} |\xi - \nabla \cdot H + E \nabla f|^2 dx + (\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} \int_{\mathbb{T}^d} |\nabla f(x)|^2 dx. \end{aligned} \tag{143}$$

It follows that

$$\sigma_{\text{sym}}(a, E) \geq (\lambda_{\max}(a)/\lambda_{\min}(a))^{-\frac{1}{2}} \sigma_{\text{sym}}\left((\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} I_d, E\right). \quad (144)$$

Similarly from the variational formula 23 one obtains that for $l \in \mathbb{R}^d$,

$$\begin{aligned} |l|_{\sigma_{\text{sym}}}^2 &= \inf_{\xi \perp l, (f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H - E(l - \nabla f)|_{a^{-1}}^2 dx \\ &\quad + \int_{\mathbb{T}^d} |l - \nabla f|_a^2 dx \\ &\leq (\lambda_{\max}(a)/\lambda_{\min}(a))^{\frac{1}{2}} \inf_{\xi \perp l, (f, H) \in C^\infty(\mathbb{T}^d) \times \mathcal{S}(\mathbb{T}^d)} \\ &\quad (\lambda_{\max}(a)\lambda_{\min}(a))^{-\frac{1}{2}} \int_{\mathbb{T}^d} |\xi - \nabla \cdot H - E(l - \nabla f)|^2 dx + (\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} \\ &\quad \int_{\mathbb{T}^d} |l - \nabla f|_a^2 dx, \end{aligned} \quad (145)$$

which leads us to

$$\sigma_{\text{sym}}(a, E) \leq (\lambda_{\max}(a)/\lambda_{\min}(a))^{\frac{1}{2}} \sigma_{\text{sym}}\left((\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} I_d, E\right). \quad (146)$$

□

A direct consequence of Proposition 5.8 is the following corollary which controls the minimal and maximal enhancement of the conductivity in the flow associated to the stream matrix E by the geometric mean $(\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}}$ of the maximal and minimal eigenvalues of a .

Corollary 5.9.

$$\frac{\lambda_{\min}(\sigma_{\text{sym}}(a, E))}{\lambda_{\min}(a)} \geq \frac{\lambda_{\min}\left(\sigma_{\text{sym}}\left((\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} I_d, E\right)\right)}{(\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}}}, \quad (147)$$

$$\frac{\lambda_{\max}(\sigma_{\text{sym}}(a, E))}{\lambda_{\max}(a)} \leq \frac{\lambda_{\max}\left(\sigma_{\text{sym}}\left((\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}} I_d, E\right)\right)}{(\lambda_{\max}(a)\lambda_{\min}(a))^{\frac{1}{2}}}. \quad (148)$$

It is then a simple consequence of Corollary 5.9 that

Proposition 5.10.

$$\frac{\lambda_{\min}(\sigma_{\text{sym}}(a, E))}{\lambda_{\min}(a)} \geq V\left((\lambda_{\min}(a)\lambda_{\max}(a))^{\frac{1}{2}}, E\right) \quad (149)$$

and

$$\frac{\lambda_{\max}(\sigma_{\text{sym}}(a, E))}{\lambda_{\max}(a)} \leq W\left((\lambda_{\min}(a)\lambda_{\max}(a))^{\frac{1}{2}}, E\right). \quad (150)$$

From Proposition 5.10 one obtains that for $n \in \mathbb{N}$,

$$\frac{\lambda_{\min}(A^{n+1})}{\lambda_{\min}(A^n)} \geq \frac{\gamma_n}{\gamma_{n+1}} V\left(\left(\lambda_{\min}(A^n)\lambda_{\max}(A^n)\right)^{\frac{1}{2}}, E^n\right). \tag{151}$$

It follows from Eq. (151) and the monotony of V that

$$\frac{\lambda_{\min}(A^{n+1})}{\lambda_{\min}(A^n)} \geq \frac{1}{\gamma_{\max}} V\left(\lambda_{\min}(A^n)\left(\frac{\lambda_{\max}(A^n)}{\lambda_{\min}(A^n)}\right)^{\frac{1}{2}}\right) \tag{152}$$

it follows from (152) that $\lambda_{\min}(A^n)$ is increasing if it belongs to $\left(0, \left(\frac{\lambda_{\max}(A^n)}{\lambda_{\min}(A^n)}\right)^{-\frac{1}{2}} V^{-1}(\gamma_{\max})\right)$; which implies Eq. (83) of Theorem 4.20 and Eq. (97) of Theorem 4.32. Now, observe that from the variational formulation (76) one obtains that

$$W(\zeta, E) \leq 1 + \zeta^{-2} \lambda_{\max}\left(\int_{\mathbb{T}^d} {}^t E(x) E(x) dx\right). \tag{153}$$

It follows from Proposition 5.10 that

$$\frac{\lambda_{\max}(\sigma_{\text{sym}}(a, E))}{\lambda_{\max}(a)} \leq 1 + C_d K_0^2 (\lambda_{\min}(a)\lambda_{\max}(a))^{-1}. \tag{154}$$

Thus one obtains for all $n \in \mathbb{N}$,

$$\frac{\lambda_{\max}(A^{n+1})}{\lambda_{\max}(A^n)} \leq \gamma_{\min}^{-1} \left(1 + C_d K_0^2 \frac{\lambda_{\max}(A^n)}{\lambda_{\min}(A^n)} (\lambda_{\max}(A^n))^{-2}\right). \tag{155}$$

It follows from (155) that $\lambda_{\max}(A^n)$ is decreasing if it belongs to $\left(\left(C_d K_0^2 \frac{\lambda_{\max}(A^n)}{\lambda_{\min}(A^n)} (\gamma_{\min} - 1)^{-1}\right)^{\frac{1}{2}}, \infty\right)$; which implies Eq. (84) of Theorem 4.20 and Eq. (98) of Theorem 4.33.

Now observe that by Proposition 5.10 one has

$$\frac{\lambda_{\max}(A^{n+1})}{\lambda_{\max}(A^n)} \leq \frac{1}{\gamma_{\min}} W\left(\lambda_{\min}(A^n)\left(\frac{\lambda_{\max}(A^n)}{\lambda_{\min}(A^n)}\right)^{\frac{1}{2}}\right), \tag{156}$$

which proves Theorem 4.27 since W is decreasing.

Now if the flow is self-similar and isotropic, Theorem 4.30 is a simple consequence of the following recursive relation:

$$\frac{\lambda(A^{n+1})}{\lambda(A^n)} = \frac{1}{\gamma} V(\lambda(A^n)). \tag{157}$$

5.4. Super diffusion: Proofs.

5.4.1. A variational formula for the exit times. Let Ω be a smooth subset of \mathbb{R}^d , we write for $a \in \mathcal{M}_{d,\text{sym}}$ and E a skew symmetric matrix with coefficients in $L^\infty(\bar{\Omega})$,

$$\psi^{a,E} = \mathbb{E}_x[\tau^{a,E}(\Omega)], \tag{158}$$

the expectation of the exit time from Ω of the diffusion associated to the generator $\nabla \cdot (a + E)\nabla$ started from x . Observe that $\psi^{a,E}$ can be defined as the weak solution of the following equation with null Dirichlet boundary condition on $\partial\Omega$,

$$\nabla \cdot \left((a + E(x))\nabla \psi^{a,E}(x) \right) = -1. \tag{159}$$

We will need the following variational formulation for the mean exit times.

Theorem 5.11.

$$\begin{aligned} & \int_{\Omega} \mathbb{E}_x[\tau^{a,E}(\Omega)] dx \\ &= \sup_{f \in C_0^\infty(\Omega), H \in \mathcal{S}(\bar{\Omega})} \left[2 \int_{\Omega} f(x) dx - \int_{\Omega} |\nabla f|_a^2 dx - \int_{\Omega} |\nabla \cdot H + E \nabla f|_{a^{-1}}^2 dx \right], \end{aligned} \tag{160}$$

where the minimization (160) is done over smooth functions f on Ω , null on $\partial\Omega$ and smooth skew symmetric matrices H on $\bar{\Omega}$. From Theorem 5.11 we deduce the following corollary

Corollary 5.12.

$$\int_{\Omega} \mathbb{E}_x[\tau^{a+\lambda I_d,0}(\Omega)] dx \leq \int_{\Omega} \mathbb{E}_x[\tau^{a,E}(\Omega)] dx \leq \int_{\Omega} \mathbb{E}_x[\tau^{a,0}(\Omega)] dx \tag{161}$$

with

$$\lambda := \sup_{x \in \Omega} \lambda_{\max}({}^t E(x) a^{-1} E(x)). \tag{162}$$

Let us now prove Theorem 5.11. By density we can first assume E to be smooth. Our purpose is to show that

$$\begin{aligned} & \int_{\Omega} \mathbb{E}_x[\tau^{a,E}(\Omega)] dx \\ &= -2 \inf_{f \in C_0^\infty(\Omega), H \in \mathcal{S}(\bar{\Omega})} \left[\frac{1}{2} \int_{\Omega} |\nabla \cdot H + (a - E)\nabla f|_{a^{-1}}^2 dx - \int_{\Omega} f(x) dx \right]. \end{aligned} \tag{163}$$

By considering variations around the minimum one obtains that

$$\nabla \cdot H_0 + (a - E)\nabla f_0 = a \nabla \psi(x) \tag{164}$$

with $\psi = 0$ on $\partial\Omega$ and

$$\nabla \cdot (a + E)\nabla \psi(x) = -1. \tag{165}$$

From which one obtains that $\psi(x) = \mathbb{E}_x[\tau^E(\Omega)]$ and $f_0(x) = (\mathbb{E}_x[\tau^E(\Omega)] + \mathbb{E}_x[\tau^{-E}(\Omega)])/2$. Thus at the minimum

$$\begin{aligned}
 & - \inf_{H,f} \left[\frac{1}{2} \int_{\Omega} |\nabla \cdot H + (a - E)\nabla f|_{a^{-1}}^2 dx - \int_{\Omega} f(x) dx \right] \\
 & = - \left[\frac{1}{2} \int_{\Omega} {}^t \nabla f_0(a + E)\nabla \psi dx - \int_{\Omega} f_0(x) dx \right] = -1/2 \int_{\Omega} f_0(x) dx, \tag{166}
 \end{aligned}$$

since

$$\int_{\Omega} {}^t \nabla f_0(a + E)\nabla \psi(x) dx = \int_{\Omega} |a\nabla \psi(x)|_{a^{-1}}^2 dx \tag{167}$$

but also

$$\int_{\Omega} |a\nabla \psi(x)|_{a^{-1}}^2 dx = \int_{\Omega} {}^t \nabla \psi(x)(a + E)\nabla \psi(x) dx = \int_{\Omega} \psi(x) dx \tag{168}$$

which leads to the result, which can be written as (160).

5.4.2. Averaging with two scales the exit times. We will use the notation of Subject. 5.4.1 and assume that

$$E = P(Rx) + K(x), \tag{169}$$

where $x \in \Omega$, $R \in [2, \infty)$, P belongs to $S_{L^\infty}(\mathbb{T}^d)$ and K is a Lipschitz-continuous skew symmetric matrix on \mathbb{R}^d ($\alpha = 1$). Our purpose is to obtain sharp quantitative estimates on the mean exit time.

$$\int_{\Omega} \psi^{a,E}(x) dx. \tag{170}$$

It follows from Theorem 5.11 that the mean exit time (170) is continuous in L^∞ norm with respect to E , thus we can by density assume E , P and K to be smooth and $\psi^{a,E}$ shall be a strong solution of (159).

To estimate (170) we will need to introduce a relative translation with respect to the fast scale associated to the medium E , i.e. we introduce for $x, y \in \Omega \times [0, 1]^d$, $E(x, y)$ as

$$E(x, y) := P(Rx + y) + K(x). \tag{171}$$

We will write for $y \in [0, 1]^d$, $\psi^{a,E}(x, y)$ the strong solution of the following equation with null Dirichlet boundary condition on $\partial\Omega$,

$$\nabla_x \left((a + E(x, y))\nabla_x \psi^{a,E}(x, y) \right) = -1. \tag{172}$$

Let us define

$$\begin{aligned}
 J := & - \int_{x \in \Omega, y \in [0, 1]^d} {}^t \nabla \psi^{a,E}(x, y) \left((a + P(Rx + y))\nabla F^{a,P}(Rx + y) - \sigma(a, P) \right) \\
 & \times \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy + \int_{x \in \Omega, y \in [0, 1]^d} {}^t \nabla \psi^{a,E}(x, y) (a - P(Rx + y)) \\
 & \times \nabla \chi^{a,P}(Rx + y)\nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy. \tag{173}
 \end{aligned}$$

Now we will show that J controls the multi-scale homogenization associated to $\psi(x, y)$

Proposition 5.13. *One has*

$$\left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \right)^{\frac{1}{2}} - \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}} \leq J. \quad (174)$$

Proof. Let us write

$$I = \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{a,E}(x, y) a \nabla F^{a,P}(Rx + y) \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy. \quad (175)$$

Observing that

$$\begin{aligned} & \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{a,E}(x, y) a \nabla \psi^{a,E} dx dy \\ &= \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{a,E}(x, y) (a + E(x, y)) \nabla \psi^{a,E} dx dy \\ &= \int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \end{aligned} \quad (176)$$

and

$$\begin{aligned} & \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) {}^t \nabla F^{a,P}(Rx + y) a \nabla F^{a,P}(Rx + y) \\ & \times \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy = \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx, \end{aligned} \quad (177)$$

one obtains by Cauchy-Schwartz inequality from (175) that

$$I \leq \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}}. \quad (178)$$

Now let us polarize I as

$$I = (I_1 + I_2)/2 \quad (179)$$

with

$$I_1 = \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{a,E}(x, y) (a - E(x, y)) \nabla F^{a,P}(Rx + y) \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy \quad (180)$$

and

$$\begin{aligned} I_2 &= \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{a,E}(x, y) (a + E(x, y)) \nabla F^{a,P}(Rx + y) \\ & \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy. \end{aligned} \quad (181)$$

Using $\nabla \cdot (a + E(x + y/R)) \nabla \psi^{a,E}(x, y) = -1$ one obtains that

$$\begin{aligned} I_1 &= \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx - \int_{x \in \Omega, y \in [0,1]^d} {}^t \nabla \psi^{a,E}(x, y) (a - E(x, y)) \\ & \times \nabla \chi^{a,P}(Rx + y) \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy. \end{aligned} \quad (182)$$

Moreover

$$\begin{aligned}
 I_2 &= \int_{x \in \Omega, y \in [0, 1]^d} {}^t \nabla \psi^{a, E}(x, y) (\sigma(a, P) + K(x)) \nabla \psi^{\sigma_{\text{sym}}(a, P), K}(x) dx dy \\
 &\quad + \int_{x \in \Omega, y \in [0, 1]^d} {}^t \nabla \psi^{a, E}(x, y) ((a + P(Rx + y)) \nabla F^{a, P}(Rx + y) - \sigma(a, P)) \\
 &\quad \nabla \psi^{\sigma_{\text{sym}}(a, P), K}(x) dx dy \\
 &\quad - \int_{x \in \Omega, y \in [0, 1]^d} {}^t \nabla \psi^{a, E}(x, y) K(x) \nabla \chi^{a, P}(Rx + y) \nabla \psi^{\sigma_{\text{sym}}(a, P), K}(x) dx dy,
 \end{aligned} \tag{183}$$

and observing that

$$\begin{aligned}
 &\int_{x \in \Omega, y \in [0, 1]^d} {}^t \nabla \psi^{a, E}(x, y) (\sigma(a, P) + K(x)) \nabla \psi^{\sigma_{\text{sym}}(a, P), K}(x) dx dy \\
 &= \int_{x \in \Omega, y \in [0, 1]^d} \psi^{a, E}(x, y) dx dy,
 \end{aligned} \tag{184}$$

one obtains from the combination of (179), (182) and (183) that

$$2I = \int_{x \in \Omega, y \in [0, 1]^d} \psi^{a, E}(x, y) dx dy + \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a, P), K}(x) dx - J \tag{185}$$

with J given by Eq. (173). Next one easily obtains (174) from (185) and (178). \square

We will now show that J acts as an error term. We will need the following lemmas.

Lemma 5.14. *Let σ be a positive definite symmetric constant matrix. There exists a constant C_d depending only on the dimension d such that for any function $f \in C_0^\infty(\Omega)$ one has*

$$\sum_{i, j=1}^d \int_{\Omega} (\partial_i \partial_j f(x))^2 dx \leq C_d (\lambda_{\min}(\sigma))^{-2} \int_{\Omega} (\nabla \sigma \nabla f(x))^2 dx. \tag{186}$$

Proof. When $\Omega = \mathbb{R}^d$ and $f \in C_0^\infty(\mathbb{R}^2)$, the inequality (186) is standard, we refer to Theorem 1.7 of [Sim72]. When Ω is a bounded open subset of \mathbb{R}^d with smooth boundary the proof follows trivially from the density of $C_0^\infty(\mathbb{R}^2)$ in $C_0^2(\Omega)$. \square

We write $T(\Omega)$ the set of smooth d -dimensional vector field on $\bar{\Omega}$, $\xi \in (C^\infty(\bar{\Omega}))^d$ such that

$$\forall z \in \partial\Omega, \quad \xi(z) \cdot n(z) = 1, \tag{187}$$

where $\partial\Omega$ is the boundary of Ω and $n(z)$ the exterior orthonormal vector at the point z of the boundary. For Ω a bounded open subset of \mathbb{R}^d with smooth boundary we write $\Theta(\Omega)$ the following isoperimetric constant associated to Ω :

$$\Theta(\Omega) := \inf_{\xi \in T(\Omega)} \max (\|\xi\|_\infty, \|\nabla \xi\|_\infty). \tag{188}$$

Lemma 5.15. *We have*

$$\begin{aligned} \int_{z \in \partial\Omega, y \in [0,1]^d} ({}^t \nabla \psi^{a,E}(z, y) a \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(z)) dz dy &\leq C_{d, \|a+P\|_\infty / \lambda_{\min}(a)} \Theta(\Omega) \\ &\left(\left(1 + \frac{\|K\|_1}{\lambda_{\min}(a)} \right) \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}} \right. \\ &\left. + \left(\frac{\text{Vol}(\Omega)}{\lambda_{\min}(a)} \right)^{\frac{1}{2}} \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx + \int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \right)^{\frac{1}{2}} \right). \end{aligned} \tag{189}$$

Proof. Let f and v be a smooth function and a smooth vector field on $\bar{\Omega}$ we will use the following Green formula:

$$\int_{\Omega} f(x) \text{div } v(x) dx = - \int_{\Omega} \nabla f(x) v(x) dx + \int_{\partial\Omega} f(z) (v(z) \cdot n(z)) dz, \tag{190}$$

where dz is the measure surface at the boundary. Let $\xi \in T(\Omega)$. Let us write

$$\begin{aligned} G &= \sum_{i,j,k=1}^d \int_{x \in \Omega, y \in [0,1]^d} \partial_i \psi^{a,E}(x, y) \\ &\quad \times (a - P(Rx + y))_{ij} \partial_j \xi_k(x) \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy. \end{aligned} \tag{191}$$

Applying formula (190) to Eq. (191) with $\nabla f = \nabla \xi_k(x)$ we obtain that

$$G = G_1 + G_2 + G_3 \tag{192}$$

with (using the skew symmetry of P_{ij} in ij)

$$\begin{aligned} G_1 &= - \sum_{i,j,k=1}^d \int_{x \in \Omega, y \in [0,1]^d} \partial_j \left((a + P(Rx + y))_{ji} \partial_i \psi^{a,E}(x, y) \right) \\ &\quad \times \xi_k(x) \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy, \end{aligned} \tag{193}$$

$$\begin{aligned} G_2 &= - \sum_{i,j,k=1}^d \int_{x \in \Omega, y \in [0,1]^d} \partial_i \psi^{a,E}(x, y) (a - P(Rx + y))_{ij} \xi_k(x) \\ &\quad \partial_j \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy, \end{aligned} \tag{194}$$

$$\begin{aligned} G_3 &= \sum_{i,j,k=1}^d \int_{z \in \partial\Omega, y \in [0,1]^d} \partial_i \psi^{a,E}(z, y) (a - P(Rz + y))_{ij} \xi_k(z) \\ &\quad \times n_j(z) \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(z) dz dy, \end{aligned} \tag{195}$$

where n_j are the coordinates of the exterior orthonormal vector $n(z)$. Using the fact that $\nabla \psi^{a,E}(z, y)$ and $\nabla \psi^{\sigma_{\text{sym}}(a,P),K}(z)$ are parallel to $n(z)$ at the boundary of Ω and both

heading towards the opposite direction of n , we obtain that (using the skew symmetry of P_{ij} in ij)

$$G_3 = \int_{z \in \partial\Omega, y \in [0,1]^d} ({}^t \nabla \psi^{a,E}(z, y) a \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(z)) (\xi(z) \cdot n(z)) dz dy. \tag{196}$$

Thus by Eq. (187),

$$G_3 = \int_{z \in \partial\Omega, y \in [0,1]^d} ({}^t \nabla \psi^{a,E}(z, y) a \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(z)) dz dy. \tag{197}$$

Now, by Cauchy-Schwartz inequality we obtain from (191)

$$|G| \leq C_d \|a + P\|_\infty (\lambda_{\min}(a))^{-1} \|\nabla \xi\|_\infty \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}}. \tag{198}$$

Using Cauchy-Schwartz inequality and $\nabla \cdot (a + P(Rx + y)) \nabla \psi^{a,E}(x, y) = -1 - \nabla \cdot K(x) \nabla \psi^{a,E}(x, y)$ we obtain from Eq. (193) that

$$|G_1| \leq C_d (\lambda_{\min}(\sigma_{\text{sym}}(a, P)))^{-\frac{1}{2}} \|\xi\|_\infty \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}} \left(\text{Vol}(\Omega) + \|K\|_1^2 (\lambda_{\min}(a))^{-1} \int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \right)^{\frac{1}{2}}. \tag{199}$$

Using Cauchy-Schwartz inequality, Lemma 5.14 and $\nabla \cdot \sigma_{\text{sym}}(a, P) \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x) = -1 - \nabla \cdot K(x) \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(x)$, we obtain from Eq. (194) that

$$|G_2| \leq C_d \|a + P\|_\infty (\lambda_{\min}(a))^{-\frac{1}{2}} (\lambda_{\min}(\sigma_{\text{sym}}(a, P)))^{-1} \|\xi\|_\infty \times \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \right)^{\frac{1}{2}} \times \left(\text{Vol}(\Omega) + \|K\|_1^2 (\lambda_{\min}(\sigma_{\text{sym}}(a, P)))^{-1} \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}}. \tag{200}$$

Combining (192), (197), (198), (199) and (200) we obtain that

$$\begin{aligned} & \int_{z \in \partial\Omega, y \in [0,1]^d} ({}^t \nabla \psi^{a,E}(z, y) a \nabla \psi^{\sigma_{\text{sym}}(a,P),K}(z)) dz dy \\ & \leq C_{d, \|a+P\|_\infty / \lambda_{\min}(a)} \left(1 + \frac{\|K\|_1}{\lambda_{\min}(a)} \right) \\ & \quad (\|\xi\|_\infty + \|\nabla \xi\|_\infty) \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}} \\ & \quad + C_{d, \|a+P\|_\infty / \lambda_{\min}(a)} \|\xi\|_\infty \left(\frac{\text{Vol}(\Omega)}{\lambda_{\min}(a)} \right)^{\frac{1}{2}} \\ & \quad \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx + \int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \right)^{\frac{1}{2}}, \end{aligned} \tag{201}$$

which proves the lemma by optimization on the vector field ξ . \square

Proposition 5.16. *We have*

$$\begin{aligned}
 |J| &\leq R^{-1} C_{d, \|a+P\|_\infty/\lambda_{\min}(a)} (\Theta(\Omega) + 1) \\
 &\left(\left(1 + \frac{\|K\|_1}{\lambda_{\min}(a)} \right) \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}} \right. \\
 &+ \left(\frac{\text{Vol}(\Omega)}{\lambda_{\min}(a)} \right)^{\frac{1}{2}} \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx \right)^{\frac{1}{2}} \\
 &\left. + \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \right)^{\frac{1}{2}} \right). \tag{202}
 \end{aligned}$$

Proof. Using formulae (173) and (127) one obtains that

$$J = \sum_{i,j,k=1}^d \int_{x \in \Omega, y \in [0,1]^d} \partial_i \psi^{a,E}(x, y) B_{i,j,k}(x, y) \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy \tag{203}$$

with

$$B_{i,j,k}(x, y) = -\partial_j M_{ijk}^P(Rx + y) + (a - P(Rx + y))_{ij} \partial_j \chi_k^{a,P}(Rx + y). \tag{204}$$

Applying formula (190) to Eq. (203) first with $\text{div } \xi = \sum_{j=1}^d -\partial_j M_{ijk}^P(Rx + y)$, next with $\nabla f = \nabla \chi_k^{a,P}(Rx + y)$, we obtain that

$$J = J_1 + J_2 + J_3 \tag{205}$$

with (using the skew symmetry of M_{ijk}^P in ij)

$$\begin{aligned}
 J_1 &= -R^{-1} \sum_{i,j,k=1}^d \int_{x \in \Omega, y \in [0,1]^d} \partial_j (a + P(Rx + y))_{ji} \partial_i \psi^{a,E}(x, y) \\
 &\quad \times \chi_k^{a,P}(Rx + y) \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy, \tag{206}
 \end{aligned}$$

$$\begin{aligned}
 J_2 &= R^{-1} \sum_{i,j,k=1}^d \int_{x \in \Omega, y \in [0,1]^d} \partial_i \psi^{a,E}(x, y) \\
 &\quad \times \left(M_{ijk}^P(Rx + y) - (a - P(Rx + y))_{ij} \chi_k^{a,P}(Rx + y) \right) \\
 &\quad \partial_j \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx dy, \tag{207}
 \end{aligned}$$

$$\begin{aligned}
 J_3 &= R^{-1} \sum_{i,j,k=1}^d \int_{z \in \partial\Omega, y \in [0,1]^d} \partial_i \psi^{a,E}(z, y) \\
 &\quad \times \left(-M_{ijk}^P(Rz + y) + (a - P(Rz + y))_{ij} \chi_k^{a,P}(Rz + y) \right) \\
 &\quad n_j(z) \partial_k \psi^{\sigma_{\text{sym}}(a,P),K}(z) dz dy. \tag{208}
 \end{aligned}$$

Using the fact that $\nabla\psi^{a,E}(z, y)$ and $\nabla\psi^{\sigma_{\text{sym}}(a,P),K}(z)$ are parallel to $n(z)$ at the boundary of Ω and both heading towards the opposite direction of n , we obtain that (using the skew symmetry of M_{ijk}^P and P_{ij} in ij)

$$J_3 = R^{-1} \int_{z \in \partial\Omega, y \in [0,1]^d} ({}^t\nabla\psi^{a,E}(z, y)a\nabla\psi^{\sigma_{\text{sym}}(a,P),K}(z))(\chi^{a,P}(Rz + y).n(z)) dz dy. \tag{209}$$

Thus by Lemma 5.15 and Eq. (117)

$$\begin{aligned} |J_3| &\leq R^{-1} \|\chi^{a,P}\|_\infty \int_{z \in \partial\Omega, y \in [0,1]^d} ({}^t\nabla\psi^{a,E}(z, y)a\nabla\psi^{\sigma_{\text{sym}}(a,P),K}(z)) dz dy \\ &\leq R^{-1} C_{d, \|a+P\|_\infty/\lambda_{\min}(a)} \Theta(\Omega) \\ &\quad \left(\left(1 + \frac{\|K\|_1}{\lambda_{\min}(a)}\right) \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx\right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\frac{\text{Vol}(\Omega)}{\lambda_{\min}(a)}\right)^{\frac{1}{2}} \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx + \int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy\right)^{\frac{1}{2}} \right). \end{aligned} \tag{210}$$

Using Cauchy-Schwartz inequality and $\nabla.(a + P(Rx + y))\nabla\psi^{a,E}(x, y) = -1 - \nabla.K(x)\nabla\psi^{a,E}(x, y)$ we obtain from Eq. (206) and (117) that

$$\begin{aligned} |J_1| &\leq C_{d, \|a+P\|_\infty/\lambda_{\min}(a)} R^{-1} \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx\right)^{\frac{1}{2}} \left(\frac{\text{Vol}(\Omega)}{\lambda_{\min}(a)}\right)^{\frac{1}{2}} \\ &\quad + \frac{\|K\|_1}{\lambda_{\min}(a)} \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy\right)^{\frac{1}{2}}. \end{aligned} \tag{211}$$

Using Cauchy-Schwartz inequality, Lemma 5.14 and $\nabla.\sigma_{\text{sym}}(a, P)\nabla\psi^{\sigma_{\text{sym}}(a,P),K}(x) = -1 - \nabla.K(x)\nabla\psi^{\sigma_{\text{sym}}(a,P),K}(x)$, we obtain from Eq. (207), (117) and (128) that

$$\begin{aligned} |J_2| &\leq C_d R^{-1} C_{d, \|a+P\|_\infty/\lambda_{\min}(a)} \left(\int_{x \in \Omega, y \in [0,1]^d} \psi^{a,E}(x, y) dx dy\right)^{\frac{1}{2}} \left(\frac{\text{Vol}(\Omega)}{\lambda_{\min}(a)}\right)^{\frac{1}{2}} \\ &\quad + \frac{\|K\|_1}{\lambda_{\min}(a)} \left(\int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a,P),K}(x) dx\right)^{\frac{1}{2}}. \end{aligned} \tag{212}$$

Proposition (5.16) is then a straightforward combination of (205), (210), (211) and (212).

We will now need the following lemma whose proof is trivial algebra

Lemma 5.17. *Assume $X, Y, \delta, \eta > 0$ and*

$$(X - Y)^2 \leq \delta XY + \eta(X + Y), \tag{213}$$

then

$$X^{\frac{1}{2}} \leq Y^{\frac{1}{2}}(1 + \sqrt{\delta}) + \sqrt{\eta} \tag{214}$$

and

$$X^{\frac{1}{2}} \geq (Y^{\frac{1}{2}} - \sqrt{\eta})(1 + \sqrt{\delta})^{-1}. \tag{215}$$

Proof. The upper root of Eq. (213) is $X_0 = Y(1 + \frac{\delta}{2}) + \frac{\eta}{2} + \frac{\sqrt{\Delta}}{2}$ with $\Delta = Y^2\delta(\delta + 4) + Y\eta(8 + 2\delta) + \eta^2$. Then by applying the Minkowski inequality to $\sqrt{\Delta}$ we obtain

$$X \leq Y(1 + \sqrt{\delta})^2 + 2\sqrt{Y}\sqrt{\eta}(1 + \sqrt{\delta}) + \eta \tag{216}$$

which leads to (214). Equation (215) is then obtained by the symmetry of (213) in X and Y . \square

Combining Proposition 5.13 and 5.16 with Lemma 5.17 we obtain Theorem 5.18:

Theorem 5.18. *There exists a finite function $h : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ increasing in each argument such that the following inequalities are valid:*

$$X \leq Y(1 + \delta) + \eta \quad \text{and} \quad X \geq (Y - \eta)(1 + \delta)^{-1} \tag{217}$$

with

$$X := \left(\frac{1}{\text{Vol}(\Omega)} \int_{x \in \Omega, y \in [0, 1]^d} \psi^{a, E}(x, y) dx dy \right)^{\frac{1}{4}}, \tag{218}$$

$$Y := \left(\frac{1}{\text{Vol}(\Omega)} \int_{x \in \Omega} \psi^{\sigma_{\text{sym}}(a, P), K}(x) dx \right)^{\frac{1}{4}}, \tag{219}$$

$$\delta := R^{-\frac{1}{2}} h \left(d, \frac{1 + \|a + P\|_{\infty}}{\lambda_{\min}(a)} \right) (\Theta(\Omega) + 1)^{\frac{1}{2}} \left(1 + \|K\|_1 \right)^{\frac{1}{2}}, \tag{220}$$

and

$$\eta := R^{-\frac{1}{2}} h \left(d, \frac{1 + \|a + P\|_{\infty}}{\lambda_{\min}(a)} \right) (\Theta(\Omega) + 1)^{\frac{1}{2}}. \tag{221}$$

5.4.3. Effect of relative translation on averaging. For Ω a bounded open subset of \mathbb{R}^d with smooth boundary and E a skew symmetric matrix with smooth coefficients in $L_{loc}^{\infty}(\mathbb{R}^d)$ and $a \in \mathcal{M}_{d, \text{sym}}$, let $\psi_{\Omega}^{a, E}(x)$ be the solution of $\nabla \cdot (a + E)\nabla \psi_{\Omega}^{a, E} = -1$ in Ω . For $y \in [0, 1]^d$ let us introduce the operator θ_y such that for any function f on \mathbb{R}^d , $\theta_y f(x) = f(x + y)$. Using the notation (172), let us observe that for $y \in [0, 1]^d$,

$$\psi_{\Omega}^{a, \theta_{y/R}(S_R P) + K}(x) = \psi_{\Omega}(x, y). \tag{222}$$

Lemma 5.19. *For $y \in \mathbb{R}^d$ one has*

$$\int_{x \in \Omega} |\nabla \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) - \nabla \psi_{\Omega}(x, y)|_a^2 \leq C_d \left(\frac{\|K\|_1}{R\lambda_{\min}(a)} \right)^2 \int_{x \in \Omega} \psi_{\Omega}^{a, E}(x, y) dx. \tag{223}$$

Proof. Observe that

$$\begin{aligned} & \nabla \cdot (a + E(x + y/R)) \nabla (\psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) - \psi_{\Omega}^{a, E}(x, y)) \\ &= \nabla \cdot \left((K(x + y/R) - K(x)) \nabla \psi_{\Omega}^{a, E}(x, y) \right). \end{aligned} \tag{224}$$

It follows that

$$\begin{aligned} & \int_{x \in \Omega} |\nabla \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) - \nabla \psi_{\Omega}^{a, E}(x, y)|_a^2 \\ &= \int_{x \in \Omega} (\nabla \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) - \nabla \psi_{\Omega}^{a, E}(x, y)) (K(x + y/R) - K(x)) \nabla \psi_{\Omega}^{a, E}(x, y), \end{aligned} \tag{225}$$

thus by Cauchy-Schwartz inequality

$$\begin{aligned} & \int_{x \in \Omega} |\nabla \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) - \nabla \psi_{\Omega}^{a, E}(x, y)|_a^2 \\ & \leq \int_{x \in \Omega} \left| (K(x + y/R) - K(x)) \nabla \psi_{\Omega}^{a, E}(x, y) \right|_{a^{-1}}^2, \end{aligned} \tag{226}$$

and Eq. (223) follows easily. \square

Now we will need the following lemma

Lemma 5.20. For $y \in [0, 1]^d$,

$$\int_{x \in \Omega} \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) dx \leq \int_{x \in \Omega} \psi_{\Omega}(x, y) dx \left(1 + C_d \frac{\|K\|_1}{R \lambda_{\min}(a)} \right)^2 \tag{227}$$

and

$$\int_{x \in \Omega} \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) dx \geq \int_{x \in \Omega} \psi_{\Omega}(x, y) dx \left(1 + C_d \frac{\|K\|_1}{R \lambda_{\min}(a)} \right)^{-2}. \tag{228}$$

Proof. Combining the identity

$$\int_{x \in \Omega} \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) dx = \int_{x \in \Omega} |\nabla \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x)|_a^2 dx \tag{229}$$

with Minkowski inequality we obtain that

$$\begin{aligned} \left(\int_{x \in \Omega} \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) dx \right)^{\frac{1}{2}} & \leq \left(\int_{x \in \Omega} |\nabla \psi_{\Omega}^{a, E}(x, y)|_a^2 dx \right)^{\frac{1}{2}} \\ & \quad + \left(\int_{x \in \Omega} |\nabla \psi_{\Omega}^{a, \theta_{y/R}(S_R P + K)}(x) - \nabla \psi_{\Omega}(x, y)|_a^2 dx \right)^{\frac{1}{2}}, \end{aligned} \tag{230}$$

and Eq. (227) follows from Lemma 5.19. The proof of inequality (228) is similar. \square

We write

$$X(\Omega, a, E) := (\text{Vol}(\Omega))^{-1} \int_{x \in \Omega} \psi_{\Omega}^{a, E}(x) dx. \tag{231}$$

For $y \in \mathbb{R}^d$ we write $\theta_y \Omega := \{x + y : x \in \Omega\}$. From Lemma 5.20 we obtain the following proposition

Proposition 5.21.

$$X(\theta_{\frac{\gamma}{R}}\Omega, a, S_R P + K) \leq X(\Omega, a, \theta_{\frac{\gamma}{R}}(S_R P) + K) \left(1 + C_d \frac{\|K\|_1}{R\lambda_{\min}(a)}\right)^2 \quad (232)$$

and

$$X(\theta_{\frac{\gamma}{R}}\Omega, a, S_R P + K) \geq X(\Omega, a, \theta_{\frac{\gamma}{R}}(S_R P) + K) \left(1 + C_d \frac{\|K\|_1}{R\lambda_{\min}(a)}\right)^{-2}. \quad (233)$$

5.4.4. *Reverse iteration to obtain supper-diffusion.* It is easy to obtain from Theorem (5.11), that for any $\gamma > 0$,

$$X(\Omega, a, \gamma E) = \gamma^{-1} X(\Omega, \gamma^{-1} a, E). \quad (234)$$

Moreover for $R > 0$, writing $S_R \Omega := \{x \in \mathbb{R}^d : R^{-1}x \in \Omega\}$ it is easy to obtain by scaling that

$$X(S_R \Omega, a, E) = R^2 X(\Omega, a, S_R E). \quad (235)$$

Let us write for $0 \leq p \leq n - 1$,

$$Z(p, B) := \left(\int_{(y_p, \dots, y_{n-1}) \in [0,1]^{d \times (n-p)}} X\left(S_{R_n} \prod_{k=p}^{n-1} \theta_{\frac{y_k R_k}{R_n}} \Omega, B, \frac{1}{\gamma_p} \Gamma^{p, \infty}\right) dy_p \dots dy_{n-1} \right)^{\frac{1}{4}}. \quad (236)$$

We will need the following proposition

Proposition 5.22. *There exists a finite increasing function $F : (R^+)^2 \rightarrow \mathbb{R}^+$ such that for*

$$\rho_{\min} > \gamma_{\max} Q_n \left(1 + \frac{R_n}{(Z(n, A^n))^2}\right) \quad (237)$$

one has

$$Z(0, A_0) \geq 0.5 \left(\frac{R_n^2}{\gamma_n}\right)^{\frac{1}{4}} \frac{Z(n, A^n)}{R_n^{\frac{1}{2}}} \left(1 + \left(\frac{Q_n \gamma_{\max}}{\rho_{\min}}\right)^{\frac{1}{2}}\right)^{-n} \quad (238)$$

and

$$Z(0, A_0) \leq 2 \left(\frac{R_n^2}{\gamma_n}\right)^{\frac{1}{4}} \frac{Z(n, A^n)}{R_n^{\frac{1}{2}}} \left(1 + \left(\frac{Q_n \gamma_{\max}}{\rho_{\min}}\right)^{\frac{1}{2}}\right)^n \quad (239)$$

with

$$Q_n := F(d, \frac{1 + K_0}{\lambda_n^-} + \mu_n) (\Theta(\Omega) + 1)(1 + K_1), \quad (240)$$

where λ_n^- is the stability of the renormalization core $(A^n)_{n \in \mathbb{N}}$ and μ_n its anisotropic distortion.

Proof. From Proposition 5.21 and Eq. (235) we obtain that

$$\begin{aligned}
 Z(p, B) \leq & \left(\int_{(y_p, \dots, y_{n-1}) \in [0, 1]^{d \times (n-p)}} X \left(\prod_{k=p+1}^{n-1} \theta_{\frac{y_k R_k}{R_n}} \Omega, B, S_{\frac{R_n}{R_p}} \theta_{y_p} E^p \right. \right. \\
 & \left. \left. + \frac{1}{\gamma_p} S_{R_n} \Gamma^{p+1, \infty} \right) dy_p \dots dy_{n-1} \right)^{\frac{1}{4}} R_n^{\frac{1}{2}} \left(1 + C_d \frac{\| \frac{1}{\gamma_p} S_{R_n} \Gamma^{p+1, \infty} \|_1 R_p}{R_n \lambda_{\min}(B)} \right)^{\frac{1}{2}}
 \end{aligned} \tag{241}$$

and

$$\begin{aligned}
 Z(p, B) \geq & \left(\int_{(y_p, \dots, y_{n-1}) \in [0, 1]^{d \times (n-p)}} X \left(\prod_{k=p+1}^{n-1} \theta_{\frac{y_k R_k}{R_n}} \Omega, B, S_{\frac{R_n}{R_p}} \theta_{y_p} E^p \right. \right. \\
 & \left. \left. + \frac{1}{\gamma_p} S_{R_n} \Gamma^{p+1, \infty} \right) dy_p \dots dy_{n-1} \right)^{\frac{1}{4}} R_n^{\frac{1}{2}} \left(1 + C_d \frac{\| \frac{1}{\gamma_p} S_{R_n} \Gamma^{p+1, \infty} \|_1 R_p}{R_n \lambda_{\min}(B)} \right)^{-\frac{1}{2}}.
 \end{aligned} \tag{242}$$

Now let us observe that

$$\begin{aligned}
 \| S_{R_n} \frac{1}{\gamma_p} \Gamma^{p+1, \infty} \|_1 & \leq K_1 \frac{R_n}{R_{p+1}} \sum_{k=p+1}^{\infty} (\gamma_k / \gamma_p) (R_{p+1} / R_k) \\
 & \leq K_1 \frac{R_n}{R_{p+1}} \frac{\gamma_{p+1}}{\gamma_p} (1 - \gamma_{\max} / \rho_{\min})^{-1}.
 \end{aligned} \tag{243}$$

Combining (241) and (242) with Theorem 5.18 (with $R = R_n / R_p$, $P = E^p$, $K = S_{R_n} \Gamma^{p+1, \infty} / \gamma_p$) and (234) one obtains that

$$Z(p, B) \leq \left(\frac{\gamma_p}{\gamma_{p+1}} \right)^{\frac{1}{4}} Z(p+1, \frac{\gamma_p}{\gamma_{p+1}} \sigma_{sym}(B, E^p)) (1 + \delta_p(B)) + \eta_p(B) \tag{244}$$

and

$$Z(p, B) \geq \left(\left(\frac{\gamma_p}{\gamma_{p+1}} \right)^{\frac{1}{4}} Z(p+1, \frac{\gamma_p}{\gamma_{p+1}} \sigma_{sym}(B, E^p)) - \eta_p(B) \right) (1 + \delta_p(B))^{-1} \tag{245}$$

with

$$\begin{aligned}
 \delta_p(B) := & \left(\frac{R_p}{R_{p+1}} \frac{\gamma_{p+1}}{\gamma_p} \right)^{\frac{1}{2}} h(d, \frac{1 + \lambda_{\max}(B) + K_0}{\lambda_{\min}(B)}) (\Theta(\Omega) + 1)^{\frac{1}{2}} (1 + K_1)^{\frac{1}{2}} \\
 & \times (1 - \gamma_{\max} / \rho_{\min})^{-1}
 \end{aligned} \tag{246}$$

and

$$\eta_p(B) := R_p^{\frac{1}{2}} h(d, \frac{1 + \lambda_{\max}(B) + K_0}{\lambda_{\min}(B)}) (\Theta(\Omega) + 1)^{\frac{1}{2}} (1 - \gamma_{\max} / \rho_{\min})^{-\frac{1}{2}}, \tag{247}$$

where, in (246) we have used the inequality (243) and we have integrated the error terms involving B appearing in (241) and (242) in the function h and used the assumption $K_1\gamma_{\max} \leq \rho_{\min}$. Then one obtains from (244) by a simple induction that for $n \geq 2$,

$$\begin{aligned} Z(0, A_0) &\leq \left(\frac{\gamma_0}{\gamma_n}\right)^{\frac{1}{4}} Z(n, A^n) \prod_{k=0}^{n-1} (1 + \delta_k(A_k)) \\ &\quad + \sum_{p=0}^{n-2} \eta_{p+1}(A^{p+1}) \left(\frac{\gamma_0}{\gamma_{p+1}}\right)^{\frac{1}{4}} \prod_{k=0}^p (1 + \delta_k(A_k)) \\ &\quad + \eta_0(A_0). \end{aligned} \tag{248}$$

Similarly one obtains from (245) by a simple induction that for $n \geq 2$,

$$\begin{aligned} Z(0, A_0) &\geq \left(\frac{\gamma_0}{\gamma_n}\right)^{\frac{1}{4}} Z(n, A^n) \prod_{k=0}^{n-1} (1 + \delta_k(A_k))^{-1} \\ &\quad - \sum_{p=0}^{n-2} \eta_{p+1}(A^{p+1}) \left(\frac{\gamma_0}{\gamma_{p+1}}\right)^{\frac{1}{4}} \prod_{k=0}^p (1 + \delta_k(A_k))^{-1} \\ &\quad - \eta_0(A_0)(1 + \delta_0(A_0))^{-1}, \end{aligned} \tag{249}$$

where $(A^k)_{k \in \mathbb{N}}$, is the *renormalization core* (33). Now combining (249) and (235) we obtain that

$$\begin{aligned} Z(0, A_0) &\geq \left(\frac{R_n^2}{\gamma_n}\right)^{\frac{1}{4}} \frac{Z(n, A^n)}{R_n^{\frac{1}{2}}} \prod_{k=0}^{n-1} \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1}\gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right)^{-1} \\ &\quad - \sum_{p=0}^{n-2} R_{p+1}^{\frac{1}{2}} \delta(A^{p+1}) \left(\frac{\gamma_0}{\gamma_{p+1}}\right)^{\frac{1}{4}} \prod_{k=0}^p \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1}\gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right)^{-1} \\ &\quad - \delta(A_0)(1 + \delta(A_0))^{-1} \end{aligned} \tag{250}$$

with

$$\delta(B) := h\left(d, \frac{1 + \lambda_{\max}(B) + K_0}{\lambda_{\min}(B)}\right) (\Theta(\Omega) + 1)^{\frac{1}{2}} (1 + K_1)^{\frac{1}{2}} (1 - \gamma_{\max}/\rho_{\min})^{-1}. \tag{251}$$

Thus

$$Z(0, A_0) \geq (1 - \zeta_i) \left(\frac{R_n^2}{\gamma_n}\right)^{\frac{1}{4}} \frac{Z(n, A^n)}{R_n^{\frac{1}{2}}} \prod_{k=0}^{n-1} \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1}\gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right)^{-1} \tag{252}$$

with

$$\begin{aligned} \zeta_i &\leq \frac{R_n^{\frac{1}{2}}}{Z(n, A^n)} \left(\sum_{p=0}^{n-2} \left(\frac{R_{p+1}^2 \gamma_n}{R_n^2 \gamma_{p+1}}\right)^{\frac{1}{4}} \delta(A^{p+1}) \prod_{k=p+1}^{n-1} \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1}\gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right)\right. \\ &\quad \left. + \left(\frac{\gamma_n}{R_n^2}\right)^{\frac{1}{4}} \prod_{k=0}^{n-1} \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1}\gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right)\right). \end{aligned} \tag{253}$$

Thus writing

$$W_n := h\left(d, \frac{1 + K_0}{\lambda_n^-} + \mu_n\right) (\Theta(\Omega) + 1)^{\frac{1}{2}} (1 + K_1)^{\frac{1}{2}} (1 - \gamma_{\max}/\rho_{\min})^{-1}, \tag{254}$$

we obtain from (253) under the assumption $\rho_{\min}^2 > \gamma_{\max}(16 + W_n^2)$ that

$$\zeta_i \leq \frac{R_n^{\frac{1}{2}}}{Z(n, A^n)} 4W_n \left(\frac{\gamma_{\max}}{\rho_{\min}^2}\right)^{\frac{1}{4}}. \tag{255}$$

Thus for

$$\rho_{\min} > \gamma_{\max}^{\frac{1}{2}} 64 \left(1 + W_n \left(1 + \frac{R_n^{\frac{1}{2}}}{Z(n, A^n)}\right)\right)^2, \tag{256}$$

$\zeta_i < 0.5$ and ζ_i acts as an error term in the inequality (252). Then combining (252), (256) and (254) we obtain the control (238). Moreover, we obtain from (235) and (248) that

$$\begin{aligned} Z(0, A_0) &\leq \left(\frac{R_n^2 \gamma_0}{\gamma_n}\right)^{\frac{1}{4}} \frac{Z(n, A^n)}{R_n^{\frac{1}{2}}} \prod_{k=0}^{n-1} \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1} \gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right) \\ &\quad + \sum_{p=0}^{n-2} R_{p+1}^{\frac{1}{2}} \delta(A^{p+1}) \left(\frac{\gamma_0}{\gamma_{p+1}}\right)^{\frac{1}{4}} \prod_{k=0}^p \left(1 + \left(\frac{\gamma_{k+1}}{r_{k+1} \gamma_k}\right)^{\frac{1}{2}} \delta(A_k)\right) + \delta(A_0). \end{aligned} \tag{257}$$

From this point the proof of Eq. (239) is similar to the one of Eq. (238) \square

We will need the following lemma

Lemma 5.23. *We have*

$$Z(n, A^n) \leq R_n^{\frac{1}{2}} (X(\Omega, A_n, 0))^{\frac{1}{4}} \tag{258}$$

and

$$\begin{aligned} Z(n, A^n) &\geq R_n^{\frac{1}{2}} (X(\Omega, I_d, 0))^{\frac{1}{4}} (\lambda_{\max}(A^n))^{-\frac{1}{4}} \\ &\quad \times \left(1 + \gamma_n^{-2} \sup_{x \in S_{R_n} \Omega} \lambda_{\max}({}^t \Gamma^{n, \infty}(x) \Gamma^{n, \infty}(x))\right)^{-\frac{1}{4}}. \end{aligned} \tag{259}$$

Proof. Equations (258) and (259) are an easy application of Theorem 5.11, Corollary 5.12 and Eq. (235). \square

For $r > 0$ we write $T[r]\Omega$ the set of $x \in \mathbb{R}^d$ such that there exists $y \in \Omega$ with $|x - y| \leq r\sqrt{d}$. We write $T[-r]\Omega$ the set of $x \in \Omega$ such that there exists $y \notin \Omega$ with $|x - y| > r\sqrt{d}$. From Eq. (236), using $\prod_{k=0}^{n-1} \theta_{\frac{y_k R_k}{K_n}} \Omega \subset T[(\rho_{\min} - 1)^{-1}]\Omega$ we obtain that

$$X(S_{R_n} T[(\rho_{\min} - 1)^{-1}]\Omega, A_0, \Gamma^{0, \infty}) \geq (Z(0, A_0))^{\frac{1}{4}}. \tag{260}$$

Similarly we obtain that

$$X(S_{R_n} T[-(\rho_{\min} - 1)^{-1}] \Omega, A_0, \Gamma^{0,\infty}) \leq (Z(0, A_0))^{\frac{1}{4}}. \tag{261}$$

We will need the following proposition

Proposition 5.24. *There exists a finite increasing function $F : (R^+)^2 \rightarrow \mathbb{R}^+$ such that for*

$$\rho_{\min} > \gamma_{\max} Q_n \quad \text{and} \quad r > R_1 \tag{262}$$

one has

$$X(B(0, r), A_0, \Gamma^{0,\infty}) \geq \frac{C_{d,K_0}}{\gamma_{\max}^2} \frac{r^2}{\gamma_n \lambda_{\max}(A_n)} \left(1 + \left(\frac{Q_n \gamma_{\max}}{\rho_{\min}}\right)^{\frac{1}{2}}\right)^{-4n} \tag{263}$$

and

$$X(B(0, r), A_0, \Gamma^{0,\infty}) \leq \frac{r^2}{\gamma_n \lambda_{\max}(A_n)} \left(1 + \left(\frac{Q_n \gamma_{\max}}{\rho_{\min}}\right)^{\frac{1}{2}}\right)^{4n} \tag{264}$$

with

$$n = \sup\{p \in \mathbb{N} : R_p \leq r\}, \tag{265}$$

$$Q_n := F\left(d, \frac{1 + K_0}{\lambda_n^-} + \mu_n\right)(1 + K_1), \tag{266}$$

where λ_n^- is the stability of the renormalization core $(A_n)_{n \in \mathbb{N}}$ and μ_n its anisotropic distortion.

Proof. Taking $\Omega := B(0, (r/R_n) - \sqrt{d}(\rho_{\min} - 1)^{-1})$ in Eq. (260) we obtain from Eq. (238) of Proposition 5.22 and (259) that

$$\begin{aligned} X(B(0, r), A_0, \Gamma^{0,\infty}) &\geq (0.5)^4 \frac{R_n^2}{\gamma_n \lambda_{\max}(A_n)} X(B(0, r/R_n), I_d, 0) \left(1 + \left(\frac{Q_n \gamma_{\max}}{\rho_{\min}}\right)^{\frac{1}{2}}\right)^{-4n} \\ &\quad \times \left(1 + \gamma_n^{-2} \sup_{x \in B(0,r)} \lambda_{\max}({}^t \Gamma^{n,\infty}(x) \Gamma^{n,\infty}(x))\right)^{-1}, \end{aligned} \tag{267}$$

which leads to (263) by (262), incorporating the new constants in Q_n (observing that for $r \geq 1$, $\Theta(B(0, r))$ is uniformly bounded away from infinity by a constant depending only on the dimension) and using

$$\sup_{x \in B(0,r)} \gamma_n^{-2} \lambda_{\max}({}^t \Gamma^{n,\infty}(x) \Gamma^{n,\infty}(x)) \leq C_d \left(K_0 + \gamma_{\max} \left(1 - \frac{\gamma_{\max}}{\rho_{\min}}\right)^{-1} \frac{r}{R_{n+1}}\right)^2. \tag{268}$$

The proof of (264) follows similarly by taking $\Omega := B(0, (r/R_n) + \sqrt{d}(\rho_{\min} - 1)^{-1})$ in Eq. (261) and using Eq. (239) of Proposition 5.22 and (258).

Let us write

$$n(r) := \sup\{p \in \mathbb{N} : R_p \leq r\}. \tag{269}$$

From Proposition (5.24) we easily deduce the following theorem

Theorem 5.25. *There exists a finite increasing function $F : (R^+)^2 \rightarrow \mathbb{R}^+$ and a function $|C(r)| \leq C(d, K_0, \gamma_{\max})$ such that for*

$$\frac{\rho_{\min}}{\gamma_{\max}} > Q(r) \quad \text{and} \quad r > R_1 \tag{270}$$

one has

$$\frac{1}{\text{Vol}(B(0, r))} \int_{B(0, r)} \mathbb{E}_x[\tau(r)] dx = \frac{r^{2-v(r)}}{\lambda_{\max}(A^n(r))} \tag{271}$$

with

$$v(r) = \frac{\ln \gamma_n(r)}{\ln r} (1 + \epsilon(r)) + \frac{C(r)}{\ln r} \tag{272}$$

and

$$|\epsilon(r)| \leq \left(\frac{Q(r)\gamma_{\max}}{\rho_{\min}} \right)^{\frac{1}{2}} \tag{273}$$

with

$$Q(r) := \frac{1}{(\ln \gamma_{\min})^2} F\left(d, \frac{1 + K_0}{\lambda_{n(r)}^-} + \mu_{n(r)}\right) (1 + K_1). \tag{274}$$

Then Theorem 4.13 is a simplified version of Theorem 5.25 (using Theorem 4.31). Now we will show that the anomalous fast behavior of the exit times from $B(0, r)$ is a super-diffusive phenomenon and not a convective phenomenon. We will consider $\mathbb{E}_{m,r,l}[\tau(r, l)]$ defined by Eq. (66). The following theorem implies Theorem 4.15.

Theorem 5.26. *There exists a finite increasing function $F : (R^+)^2 \rightarrow \mathbb{R}^+$ such that for*

$$\rho_{\min} > \gamma_{\max} 10Q(r) \quad \text{and} \quad r > R_1 \tag{275}$$

one has

$$\lim_{l \rightarrow \infty} \left(\text{Vol}(B(0, r, l)) \right)^{-1} \int_{(y,z) \in B(0,r,l)} \mathbb{E}_{y,z}[\tau(B(0, r, l))] = \frac{r^{2-v(r)}}{\lambda_{\max}(A^n(r))} \tag{276}$$

where $v(r)$ is given by (272) and $Q(r)$ by (274).

Proof. Let us observe that

$$B(0, r, l) \subset \hat{B}(0, r, l) \quad \text{and} \quad \hat{B}(0, r, l) \subset B(0, r, l + r). \tag{277}$$

Thus, it is sufficient to control exit times from $\hat{B}(0, r, l)$ in order to prove Theorem 5.26. Now let us observe that the diffusion (y_t, z_t) is associated to the following generator L acting on $f \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$:

$$Lf(y, z) := \nabla_y \cdot (\kappa I_d + \Gamma(y) - \Gamma(z)) \nabla_y + \nabla_z \cdot (\kappa I_d + \Gamma(z) - \Gamma(y)) \nabla_z. \tag{278}$$

Thus one can apply Proposition 5.22 with $\Omega = \hat{B}(0, r, l)$. Let us observe that the *renormalization core* associated to (y_t, z_t) is

$$\begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}. \tag{279}$$

Moreover it is easy to observe that $\Theta(\hat{B}(0, r, l))$ is bounded uniformly away from infinity on $r \leq l$ and that

$$\|\Gamma^{n,\infty}(y) - \Gamma^{n,\infty}(z)\| \leq C_d \sum_{k=n}^{\infty} C_d K_1 \gamma_k R_k^{-1} |y - z|. \tag{280}$$

From this point the proof of Theorem 5.26 is trivially similar to the one of Theorem 5.25. \square

Super-diffusion as a common event. Let us write $G(r)$ as the set of points of $B(0, r)$ such that if y_t starts from those points, its exit time from $B(0, r)$ is anomalously fast with probability asymptotically close to one. We also write $G(r, l)$ as the set of points of $B(0, r, l)$ such that if (y_t, z_t) starts for those points, their separation time is anomalously fast with probability asymptotically close to one. More precisely let us write

$$\delta(r) = \frac{\ln \gamma_n(r)}{\ln r} (1 - 3C(r)) - \frac{C_{d, K_0, \gamma_{\max}}}{\ln r} \tag{281}$$

with

$$C(r) = \left(\frac{Q(r)\gamma_{\max}}{\rho_{\min}} \right)^{\frac{1}{2}}, \tag{282}$$

where $Q(r)$ is given by (274). Let us write

$$\epsilon_2(r) = \exp(-\ln \gamma_n(r) C(r)). \tag{283}$$

We will consider

$$G(r) := \left\{ x \in B(0, r) : \mathbb{P}_x \left[\tau(r) \leq \frac{r^{2-\delta(r)}}{\lambda_{\max}(A^{n(r)})} \right] \geq 1 - \epsilon_2(r) \right\}$$

and

$$G(r, l) := \left\{ (y, z) \in B(0, r, l) : \mathbb{P}_{y,z} \left[\tau(B(0, r, l)) \leq \frac{r^{2-\delta(r)}}{\lambda_{\max}(A^{n(r)})} \right] \geq 1 - \epsilon_2(r) \right\}.$$

Let us write $m_r, m_{r,l}$ the Lebesgue probability measure defined on $B(0, r)$ and $B(0, r, l)$ by

$$m_r(G(r)) := \frac{\int_{G(r)} dx}{\int_{B(0,r)} dx}, \tag{284}$$

$$m_{r,l}(G(r, l)) := \frac{\int_{(y,z) \in G(r,l)} dy dz}{\int_{(y,z) \in B(0,r,l)} dy dz}. \tag{285}$$

A trivial consequence of Theorems 5.25 and 5.26 is the following theorem.

Theorem 5.27. *There exists a finite increasing function $F : (R^+)^2 \rightarrow \mathbb{R}^+$ such that for*

$$\rho_{\min} > 10\gamma_{\max} Q(r) \quad \text{and} \quad r > R_1, \quad (286)$$

$$m_r(G(r)) \geq 1 - \epsilon_2(r), \quad (287)$$

$$m_{r,l}(G(r, l)) \geq 1 - \epsilon_2(r). \quad (288)$$

Theorem 4.17 is a particular case of Theorem 5.27.

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