Hyperbolic Systems and Characteristic Theory

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1 Lecture for Monday October 25, 2004

1.1 Motivation

Many physical systems lead to the formation of quasi-linear systems of first order partial differential equations. These equations can be thought as describing wave motion and the mathematical theory of characteristic for hyperbolic systems reveals the nature and structure of the wave system. Characteristic theory also helps in the understanding the nature of the conditions at the boundary.

1.2 Problem setup

We first begin with a system of two independent variables so that the system describes **plane waves**. The setup is as follows

- let x and t be the two independent variables;
- let $u_i(x,t)$ denote the dependent variables for $i = 1, 2, ..., n$;
- then the system can be written as

$$A_{ij}\frac{\partial u_j}{\partial t} + a_{ij}\frac{\partial u_j}{\partial x} + b_i = 0 \tag{1}$$

NOTE

1. The system is *quasi linear* if:
   $A_{ij} = A_{ij}(u_1, u_2, ..., u_n, x, t)$
   $a_{ij} = a_{ij}(u_1, u_2, ..., u_n, x, t)$
   $b_i = b_i(u_1, u_2, ..., u_3, x, t)$
2. The system is *linear* if:
   $A_{ij} = A_{ij}(x, t)$
   $a_{ij} = a_{ij}(x, t)$
   $b_i = b_i(x, t)$.
3. The quasi-linear or linear system is *homogeneous* if $b_i = 0$

1.3 Toward the characteristic form

We now wish to show under what conditions the system of Eq. (1) is hyperbolic.

Note that the system of Eq. (1) has information on the derivatives of u_j along different directions. It is desirable to obtain information on u_j along a single direction. To do this take linear combinations of the system in Eq. (1). This is equivalent to multiplying by the vector $\mathbf{l}$ where $l_i = l_i(u_1, u_2, ..., u_n, x, t)$ to obtain

$$l_i \left(A_{ij} \frac{\partial u_j}{\partial t} + a_{ij} \frac{\partial u_j}{\partial x} \right) + l_i b_i = 0 \tag{2}$$

This leads to

$$m_j \left(\alpha \frac{\partial u_j}{\partial t} + \beta \frac{\partial u_j}{\partial x} \right) + l_j b_j = 0. \tag{3}$$

Eq. (3) represents the derivative of u_j along the direction given by the vector field (α, β).

1.3.1 Derivative of a function along a curve

To understand this consider a function of two independent variables $p(x, y)$. Then $p(x, y)$ can be represented as a surface in (x, y, z)-space where the height of the surface is given by $p(x, y)$.

Now consider a curve on the surface parametrized by η and described by $(x(\eta), y(\eta)$. Then $p(x, y) = p(x(\eta), y(\eta)) = p(\eta)$. Now take the *total derivative* of $p(\eta)$ with respect to η to obtain

$$\frac{\mathrm{d}p}{\mathrm{d}\eta} = \frac{\partial p}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\eta} + \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\eta}.$$

Therefore Eq. (3) can be represented as the directional derivative of u_j along the direction given by the the vector field (α, β).

Let $x = X(\eta)$ and $t = T(\eta)$ so that $u_j(x, t) = u_j(X(\eta), T(\eta)) = u_j(\eta)$, take the total derivative of u_j with respect to η to obtain

$$\begin{aligned} \frac{\mathrm{d}u_j}{\mathrm{d}\eta} &= \frac{\partial u_j}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\eta} + \frac{\partial u_j}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\eta} \\ &= \frac{\partial u_j}{\partial x} X' + \frac{\partial u_j}{\partial t} T' \\ &= \frac{\partial u_j}{\partial x}\beta + \frac{\partial u_j}{\partial t}\alpha. \end{aligned}$$

Therefore the system of Eq. (3) now becomes

$$m_j \frac{\mathrm{d}u_j}{\mathrm{d}\eta} + l_j b_j = 0. \tag{4}$$

From Eq. (2) it is seen that

$$\begin{aligned} l_i A_{ij} &= m_j T' \\ l_i a_{ij} &= m_j X', \end{aligned}$$

the above can be solved for m_j to yield the linear system

$$l_i \left(A_{ij} X' - a_{ij} T'\right) = 0. \tag{5}$$

A necessary and sufficient condition for the linear system in Eq. (5) to admit non-trivial solutions is that is that

$$\det \left|A_{ij} X' - a_{ij} T'\right| = 0. \tag{6}$$

NOTE

1. It is OK for either the matrix A_{ij} or the matrix a_{ij} to be singular so that $\det |A_{ij}| = 0$ or $\det |a_{ij}| = 0$. However, they cannot both be singular.

2. If either a_{ij} or A_{ij} are singular, a rotation of the co-ordinate system can be introduced to remove the singularity. This is equivalent to taking linear combinations of A_{ij} and a_{ij}. Therefore a more general way to write Eq. (6) is that

$$\det \left|\lambda A_{ij} + \mu a_{ij}\right| \neq 0 \tag{7}$$

1.4 Definition of a hyperbolic system

The system

$$A_{ij} \frac{\partial u_j}{\partial t} + a_{ij} \frac{\partial u_j}{\partial x} + b_i = 0$$

such that

$$\det \left|\lambda A_{ij} + \mu a_{ij}\right| \neq 0$$

is *hyperbolic* if there exist $\underline{n \text{ linearly independent}}$ vectors $\mathbf{l^{(k)}}$ $k = 1, 2, ..., n$ such that

$$l_i^{(k)} \left(A_{ij} \alpha^{(k)} - a_{ij} \beta^{(k)}\right) = 0 \tag{8}$$

for each k and the directions $(\alpha^{(k)}, \beta^{(k)})$ are real with

$$\left(\alpha^{(k)}\right)^2 + \left(\beta^{(k)}\right)^2 \neq 0.$$

NOTE

1. If the system is hyperbolic then it can be written in *characteristic form*

$$m_j \frac{\mathrm{d} u_j}{\mathrm{d}\eta} + l_j b_j = 0$$

2. The curves described by the vector field $(\alpha^{(k)}, \beta^{(k)})$ are called *characteristics*.

3. When writing a system in characteristic form the equation for the characteristic must also be given

4. It is important to have n linearly independent vectors $\mathbf{l^{(k)}}$, it possible to have repeated values for $\alpha^{(k)}$ and $\beta^{(k)}$.

1.5 Special case $A_{ij} = \delta_{ij}$

For the case when $A_{ij} = \delta_{ij}$ the system of Eq. (1) can be written as

$$\frac{\partial u_i}{\partial t} + a_{ij}\frac{\partial u_j}{\partial x} + b_i = 0. \tag{9}$$

Use the parameter t itself to describe the variables $x = x(t)$ so that the total derivative of u_j with respect to t is given by

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = \frac{\partial u_j}{\partial t} + \frac{\partial u_j}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

The characteristic form is then given by

$$l_i\frac{\mathrm{d}u_i}{\mathrm{d}t} + l_i b_i = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t} = c \tag{10}$$

where c is found by Eq. (5) by the condition

$$\begin{aligned} l_i a_{ij} &= l_j c & (11)\\ \det|a_{ij} - c\delta_{ij}| &= 0 & (12) \end{aligned}$$

NOTE

1. c are the eigenvalues of the matrix a_{ij}
2. l are the left-eigenvectors of the matrix a_{ij}
3. If a_{ij} is real and symmetric, then the eigenvalues are real, there exist n linearly independent eigenvectors and the matrix can be diagonalized. Therefore the system is hyperbolic

1.6 Riemann invariants

Consider the system

$$l_i\frac{\mathrm{d}u_i}{\mathrm{d}t} + l_i b_i = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t} = c$$

and consider the following cases

1. Let l_i be constants , then change the order of the derivative to obtain

   $$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(l_i u_i) + l_i b_i &= 0\\ \frac{\mathrm{d}r}{\mathrm{d}t} + f(x, t, \mathbf{u}) &= 0 \end{aligned}$$

   where $r = l_i b_i$ is called the *Riemann invariant*

2. Let $l_i = l_i(x,t)$ then consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(l_i u_i\right) = l_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + u_i \frac{\mathrm{d}l_i}{\mathrm{d}t}$$

and the system reduces to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(l_i u_i\right) - u_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + l_i b_i &= 0 \\ \frac{\mathrm{d}r}{\mathrm{d}t} + f(x,t,\mathbf{u}) &= 0. \end{aligned}$$

3. Let $l_i = l_i(x,t,\mathbf{u})$ then

2 Lecture for Wednesday October 27, 2004

2.1 Stokesian fluid

Recall that σ_{ij} is the stress tensor, S_{ij} is the rate-of-strain tensor defined as

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The Stokesian fluid defines the stress tensor as

$$\sigma_{ij} = (-p + \alpha)\,\delta_{ij} + \beta S_{ij} + \gamma S_{ik} S_{ki}.$$

Note

1. $\sigma_{ij} = \sigma_{ij}(S_{ij})$ and on the local thermodynamic state. It does not depend on other kinematical quantities such as Ω_{ij}.
2. The fluid is homogeneous so that σ_{ij} does not depend explicitly on the position $\mathbf{x}$.
3. The fluid is isotropic, so that there is no preferred direction. This implies that the principal direction of the stress tensor σ_{ij} and the principal directions of the rate-of-strain tensor S_{ij} coincide.
4. When there is no deformation so that $S_{ij} = 0$ then the stress is hydrostatic so that $\sigma_{ij} = -p\delta_{ij}$.
5. **Definition** The Newtonian fluid is a linear Stokesian fluid so that $\gamma = 0$ so that $\sigma_{ij} = (-p + \alpha)\delta_{ij} + \beta S_{ij}$ where $\alpha = \lambda S_{kk}$ and $\beta = 2\mu$

2.2 Riemann Invariants (continued)

Let $l_i = l_i(x, t, \mathbf{u})$ then the objective is to express the relation in Eq. (10) as

$$\begin{aligned} l_i \frac{\mathrm{d}u_i}{\mathrm{d}t} &= \lambda \frac{\mathrm{d}r}{\mathrm{d}t} \\ &= \lambda \frac{\partial r}{\partial u_i} \frac{\mathrm{d}u_i}{\mathrm{d}t} \end{aligned}$$

This yields

$$l_i = \lambda \frac{\partial r}{\partial u_i}.$$

For hyperbolic systems introduce the variable $r_k = l_i^{(k)} \mathrm{d}u_i$ and re-write the characteristic equations of Eq. (10) as

$$\frac{\mathrm{d}r_k}{\mathrm{d}t} + f_k(x, t, \mathbf{r}) = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t} = c_k(x, t, \mathbf{r}) \tag{13}$$

2.3 Examples of hyperbolic systems

2.3.1 The wave equation

Consider the wave equation $u_{tt} - \gamma u_{xx} = 0$, then let $u_x = v$ and $u_t = w$ to obtain the system

$$\begin{cases} v_t - w_x = 0 \\ w_t - \gamma v_x = 0. \end{cases}$$

Re-write in matrix form

$$\begin{bmatrix} v_t \\ w_t \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0.$$

This is the special case discussed above when $A_{ij} = \delta_{ij}$, therefore let A be the matrix above. The characteristic speeds are the eigenvalues of A. To find the eigenvalues solve $\det|A - \lambda I|$ and find that the characteristic equation is $\lambda^2 - \gamma = 0$ therefore obtain two eigenvalues $\lambda_1 = \sqrt{\gamma}$ and $\lambda_2 = -\sqrt{\gamma}$. Next to find the vectors $\mathbf{l}$ to put the system in characteristic form find the right eigenvectors of A. However, the right eigenvectors of A are the left eigenvectors of A^T. Therefore to find the first vector $\mathbf{l^{(1)}}$ solve the linear system $(A^T - \lambda_1 I)\mathbf{l^{(1)}} = 0$. Similarly to find $\mathbf{l^{(2)}}$ solve the linear system $(A^T - \lambda_2 I)\mathbf{l^{(2)}}$ and obtain

$$\mathbf{l^{(1)}} = \begin{bmatrix} -\sqrt{\gamma} \\ 1 \end{bmatrix} \qquad \mathbf{l^{(2)}} = \begin{bmatrix} \sqrt{\gamma} \\ 1 \end{bmatrix}$$

Next test that by multiplying $\mathbf{l^{(1)}}$ by the system above the characteristic form emerges. Multiply and obtain

$$\begin{aligned} -\sqrt{\gamma}(v_t - w_x) + w_t - \gamma v_x &= 0 \\ -\sqrt{\gamma}(v_t + \sqrt{\gamma} v_x) + w_t + \sqrt{\gamma} w_x &= 0. \end{aligned}$$

Therefore we are done since the equation above is the characteristic form! Therefore write

$$-\sqrt{\gamma}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}w}{\mathrm{d}t} = 0 \quad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\gamma}.$$

Similarly for $\lambda_2 = -\sqrt{\gamma}$ obtain the characteristic form

$$\sqrt{\gamma}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}w}{\mathrm{d}t} = 0 \quad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{\gamma}.$$

2.4 1D Isentropic Flow

Consider one-dimensional isentropic flow

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + u u_x + \frac{1}{\rho} p_x = 0. \end{cases} \tag{14}$$

Use isentropic relations to express the pressure in terms of density and the speed of sound c and use the relation

$$\frac{1}{\rho}p_x = \frac{c^2}{\rho}\rho_x$$

where c is the speed of sound. Rewrite the system in matrix form to obtain

$$\begin{bmatrix} \rho_t \\ u_t \end{bmatrix} + \begin{bmatrix} u & \rho \\ \frac{c^2}{\rho} & u \end{bmatrix} \begin{bmatrix} \rho_x \\ u_x \end{bmatrix} = 0.$$

Again we are very luck at $A_{ij} = \delta_{ij}$. Therefore to find the characteristic speeds find the eigenvalues of the matrix above.

$$\begin{aligned} \det \begin{bmatrix} u-\lambda & \rho \\ \frac{c^2}{\rho} & u-\lambda \end{bmatrix} &= 0 \\ (u-\lambda)^2 - c^2 &= 0 \\ u - \lambda &= \pm c \end{aligned}$$

Therefore find the characteristic speeds $\lambda_1 = u - c$ and $\lambda_2 = u + c$. Next, find the left-eigenvectors $\mathbf{l}$ of the transposed matrix by solving

$$(A^T - \lambda I)\mathbf{l} = \begin{bmatrix} u-\lambda & \frac{c^2}{\rho} \\ \rho & u-\lambda \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = 0.$$

Now for $\lambda_1 = u - c$ find $l_1^{(1)} = 1$ and $l_2^{(1)} = -c/\rho$ so obtain

$$\rho_t + u\rho_x + \rho u_x - \frac{\rho}{c}\left(u_t + uu_x + \frac{c^2}{\rho}\rho_x\right) = 0,$$

so obtain for the characteristic equation

$$\rho_t + (u-c)\rho_x - \frac{\rho}{c}\left[u_t + (u-c)u_x\right] = 0,$$

giving the characteristic form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\rho}{c}\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c. \tag{15}$$

For $\lambda_2 = u + c$ obtain $l_1^{(2)} = 1$ and $l_2^{(2)} = \rho/c$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho}{c}\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t} = u + c. \tag{16}$$

2.5 Solving problems with the method of characteristics

2.5.1 Finding the Riemann invariants for the 1D isentropic equations

Next, find the Riemann invariants for the characteristic Equations (15) and (16). Rewrite the equations as

$$\begin{aligned}\frac{c}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}-\frac{\mathrm{d}u}{\mathrm{d}t} &= 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \frac{c}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\frac{\mathrm{d}u}{\mathrm{d}t} &= 0, \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{aligned}$$

and integrate along the characteristic to obtain

$$\begin{aligned}\int\frac{c\mathrm{d}\rho}{\rho}-u &= const \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \int\frac{c\mathrm{d}\rho}{\rho}+u &= const \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{aligned}$$

Now to do the integration consider that for a polytropic gas $c^2=\kappa\gamma\rho^{\gamma-1}$ and therefore obtain

$$\begin{aligned}\frac{2c}{\gamma-1}-u &= r_1 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \frac{2c}{\gamma-1}+u &= r_2 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{aligned}$$

Where r_1 and r_2 are the Riemann invariants. Therefore the characteristic equations become

$$\frac{\mathrm{d}r_1}{\mathrm{d}t} = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c \tag{17}$$

$$\frac{\mathrm{d}r_2}{\mathrm{d}t} = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c \tag{18}$$

showing that the Riemann invariants are constant along characteristics.

2.5.2 Using Riemann invariants to solve problems

Therefore consider a graph of the equations in the (x,t)-plane. In this plane the two characteristics are straight lines with slope given by $1/(u-c)$ and $1/(u+c)$ respectively. Because $u-c$ is negative and $u+c$ is positive it is expected that the family of characteristics corresponding to r_1 are going to the left and the family of characteristics corresponding to r_2 are going to the right.

Therefore given the values of r along the x-axis, for any point P above the x-axis it is possible to determine the velocity u and the speed of sound c based on the 1-characteristic coming from the points Q_2 and the 2 characteristic coming from the point

Q_1. In particular note that

$$\begin{aligned} c(P) &= \frac{\gamma-1}{4}\left(r_2(Q_1)+r_1(Q_2)\right) \\ u(P) &= \frac{1}{2}\left(r_2(Q_1)-r_1(Q_2)\right). \end{aligned}$$

Riemann invariants also help in determining how to impose boundary conditions. For the flow above, it is necessary to impose r_1 and r_2 along the x-axis at $t = 0$ but at the boundary condition along the t-axis corresponding to $x = 0$ one can only impose r_2 since r_1 is obtained from the x-axis.

NOTE

1. Initial conditions cannot be specified along a characteristic
2. The number of boundary condition needed is equal to the number of characteristics pointing inside the domain

3 Lecture for Friday October 29, 2004

3.1 Using Riemann invariants to solve problems (continued)

Definitions:

- *Domain of dependence* Solution P depends on the region between Q_1 and Q_2. It is bounded between the first and the second characteristic.
- *Domain of influence* All points where the solution is affected by the point P. Also the set of points whose domain of dependence includes P.

3.2 Weak solutions

3.2.1 One dimensional wave equation

Consider the following one dimensional wave equation

$$\begin{cases} \rho_t + c_0 \rho_x = 0 \\ \rho(x, t = 0) = f(x) \end{cases}$$

Then the solution is given by $\rho(x,t) = f(x - c_0 t)$.

Consider now the following quasi-linear system

$$\begin{cases} \rho_t + c(\rho) \rho_x = 0 \\ \rho(x, t = 0) = f(x) \end{cases}$$

Solve using the method of characteristics. The characteristic speed is given by $c(\rho)$. Consider $\rho(x(t), t)$ then

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t}$$

Therefore confronting this with the equation above it is found that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \qquad \text{on } \frac{\mathrm{d}x}{\mathrm{d}t} = c(\rho). \tag{19}$$

Therefore ρ is constant on characteristics. Furthermore the characteristics are straight lines with slope given by $1/c(\rho)$. Since the characteristics originate on the x-axis the slope is given by $1/c(f(x_0))$ for a characteristic starting at the point $x = x_0$.

Consider the following example

$$\begin{cases} \rho_t + c(\rho)\rho_x = 0 \\ \rho(x,0) = \begin{cases} \rho_1 & x > 0 \\ \rho_2 & x < 0 \end{cases} \\ c(\rho) = \begin{cases} c_1 & \rho = \rho_1 \\ c_2 & \rho = \rho_2 \end{cases} \end{cases}$$

Now suppose that $c_2 > c_1$ this constitutes a breakdown in the solution as the characteristics intersect. We can overcome this by allowing discontinuities in the solution.

Q: How can we differentiate a discontinuity and in what sense is this a "solution" to the PDE?

A: Use the concept of *weak solutions*

3.2.2 Theory of weak solutions

Consider now the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) = 0 \tag{20}$$

where $Q(\rho)$ represents the flux. Consider now a rectangle R in (x, t)-space and let ρ be a continuous, smooth function such that $\phi = 0$ on the boundary of R. Now integrate over the entire rectangular region

$$\iint_R \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) \right] \phi \mathrm{d}x\mathrm{d}t = 0.$$

Next integrate by parts and since $\phi = 0$ on the boundary Eq. (20) becomes

$$-\iint_R \left[\rho \frac{\partial \phi}{\partial t} + Q(\rho) \frac{\partial \phi}{\partial x} \right] \mathrm{d}x\mathrm{d}t \tag{21}$$

Definition $\rho(x, t)$ is a *weak solution* of the conservation Law (20) if relation (21) is satisfied for all test functions ϕ.

NOTE

1. If ρ and $Q(\rho)$ are differentiable then Eq. (20) and Eq. (21) are equivalent
2. However Eq. (21) allows more general solutions as ρ does not have to be differentiable, therefore allowing discontinuities

3.2.3 Velocity of the shock

Returning back to the conservation law of Eq. (20) and now consider a surface of discontinuity parametrized by the curve $(s(e), e)$ in (x, t)-space which divides the region R into R_1 and R_2. Let ρ_1 and $Q(\rho_1)$ be the parameters on one side and ρ_2 and $Q(\rho_2)$ be the parameters on the other side. Then consider the normal to the surface given by $N(l, m)$ where

$$l = -\frac{\mathrm{d}s}{\mathrm{d}e} \qquad m = \frac{\mathrm{d}t}{\mathrm{d}e}.$$

Apply the weak solution formulation of the PDE to find

$$\iint_{R_1} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) \right] \phi \mathrm{d}x\mathrm{d}t + \iint_{R_2} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) \right] \phi \mathrm{d}x\mathrm{d}t$$
$$+ \int_S \left\{ [\rho] l + [Q(\rho)]\, m \right\} \phi \mathrm{d}e = 0,$$

where $[\rho] = \rho_1 - \rho_2$ and $[Q(\rho)] = Q(\rho_1) - Q(\rho_2)$. Since the first two terms in the integral are satisfied by the continuity of ρ and $Q(\rho)$ it follows that

$$\begin{aligned} [\rho]l + [Q(\rho)]\,m &= 0 \\ -\frac{l}{m} &= \frac{[Q(\rho)]}{[\rho]} \end{aligned}$$

But

$$\frac{\frac{\mathrm{d}s}{\mathrm{d}e}}{\frac{\mathrm{d}t}{\mathrm{d}e}} = \frac{\mathrm{d}s}{\mathrm{d}t} = U,$$

where U is the velocity of propagation of the shock. Therefore it follows that

$$U = \frac{[Q(\rho)]}{[\rho]}. \tag{22}$$

This procedure also shows that a solution containing a discontinuity can be a weak solution to the original conservation law Eq. (20).

3.2.4 Returning to the one dimensional example

Therefore returning to the example above with

$$\begin{cases} \rho_t + c(\rho)\rho_x = 0 \\ \rho(x,0) = \begin{cases} \rho_1 & x > 0 \\ \rho_2 & x < 0 \end{cases} \\ c(\rho) = \begin{cases} c_1 & \rho = \rho_1 \\ c_2 & \rho = \rho_2 \end{cases} \end{cases}$$

and $c_2 > c_1$. Then the velocity of the shock is given by Eq. (22). However, it is necessary to express the system in conservation form. Consider

$$Q(\rho) = \begin{cases} c_1\rho_1 & \rho = \rho_1 \\ c_2\rho_2 & \rho = \rho_2 \end{cases}$$

Therefore the velocity of the shock is given by

$$U = \frac{[Q(\rho)]}{[\rho]} = \frac{c_1\rho_1 - c_2\rho_2}{\rho_1 - \rho_2}$$

Thus the final solution is given by

$$\rho(x,t) = \begin{cases} \rho_1 & x > Ut \\ \rho_2 & x < Ut \end{cases}$$