ACM 126a
Solutions for Homework Set 3

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Problem 1

1.1 Problem 4.13 page 123

[2] Let $\{g_n(t)\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathbf{L^2}(\mathbb{R})$. Prove that

$$\forall(t,\omega)\in\mathbb{R}^2\,,\ \sum_{n=0}^{+\infty} P_V g_n(t,\omega) = 1.$$

1.2 Solution

Let's start by proving several important facts that we are going to need to finish this exercise. The first result we will prove concerns the Delta function. For any orthobasis $\{g_n(x)\}_{n\in\mathbb{N}}$ of $\mathbf{L^2}(\mathbb{R})$ (in fact there is a more general result concerning the identity with respect to the scalar product in a Hilbert space) we can prove, in the distribution sense (tempered distributions), that

$$\delta(x-y) = \sum_{n=0}^{+\infty} g_n^*(y) g_n(x)$$

We can prove this in $\mathcal{C}_0^\infty$ (the space of infinitely differentiable functions with compact support), by just applying it for a function $f(y)$ and remembering that for these functions we can invert the sum and the integral.

$$f(x) = \int_{-\infty}^{+\infty} \delta(x-y) f(y)\,dy = \sum_{n=0}^{+\infty}\left(\int_{-\infty}^{+\infty} f(y) g_n^*(y)\,dy\right) g_n(x)$$

and we know this last result is true because if $f\in\mathcal{C}_0^\infty$ then $f\in\mathbf{L^2}(\mathbb{R})$ and $\{g_n(x)\}_{n\in\mathbb{N}}$ is an orthobasis of this last space. We can then enlarge the space of functions by using functions in the Schwarz class and then finally by density we can into $\mathbf{L^2}(\mathbb{R})$. Now if we change x into $t+\tau/2$ and y into $t-\tau/2$, we have that (notice that the Jacobian of this transformation is equal to 1 so the Delta function does not change)

$$\delta(\tau) = \delta((t+\frac{\tau}{2}) - (t-\frac{\tau}{2})) = \sum_{n=0}^{+\infty} g_n^*(t-\frac{\tau}{2}) g_n(t+\frac{\tau}{2})$$

Now, we calculate the Fourier transform of the Delta function and we have

$$1 = \int_{-\infty}^{+\infty} \delta(\tau) e^{-i\tau\omega}\,d\tau = \int_{-\infty}^{+\infty}\left(\sum_{n=0}^{+\infty} g_n^*(t-\frac{\tau}{2}) g_n(t+\frac{\tau}{2}) e^{-i\tau\omega}\right) d\tau$$

What we want to do now is to exchange the sum and the integral. In the space of distributions by the continuity of the Fourier transform we can exchange them (this will have to be proven with test functions in the Schwarz space and then by density we will go into $\mathbf{L^2}(\mathbb{R})$) and then we have that

$$\begin{aligned}
1 = \mathcal{F}(\delta(\tau)) &= \mathcal{F}(\sum_{n=0}^{+\infty} g_n^*(t - \frac{\tau}{2}) g_n(t + \frac{\tau}{2})) \\
&= \sum_{n=0}^{+\infty} \mathcal{F}(g_n^*(t - \frac{\tau}{2}) g_n(t + \frac{\tau}{2})) \\
&= \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} g_n^*(t - \frac{\tau}{2}) g_n(t + \frac{\tau}{2}) e^{-i\tau\omega} \, d\tau
\end{aligned}$$

Now that all this has been proven we can do the problem.

$$\begin{aligned}
\sum_{n=0}^{+\infty} P_V g_n(t,\omega) &= \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} g_n(t + \frac{\tau}{2}) g_n^*(t - \frac{\tau}{2}) e^{-i\tau\omega} \, d\tau \\
&= 1
\end{aligned}$$

because we recognize what we proved concerning the Delta function and any orthobasis of $\mathbf{L^2}(\mathbb{R})$.

Remark: We have to realize that what we just proved concerns the class of functions equal to 1 almost everywhere. This means that on a set of points of measure zero our sum can differ from 1. For example, imagine we have a basis $\{e_n(t)\}_{n\in\mathbb{N}}$ of $\mathbf{L^2}(\mathbb{R}^+)$ then we can build a basis $\{g_n(t)\}_{n\in\mathbb{N}}$ of $\mathbf{L^2}(\mathbb{R})$ defined as

$$g_{2n}(t) = \begin{cases} e_n(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$
$$g_{2n+1}(t) = \begin{cases} e_n(-t) & \text{for } t \leq 0 \\ 0 & \text{for } t > 0 \end{cases}$$

This is evidently an orthobasis for $\mathbf{L^2}(\mathbb{R})$. Now if we write our sum at $t = 0$ we have

$$\begin{aligned}
\sum_{n=0}^{+\infty} P_V(0,\omega) &= \sum_{n=0}^{+\infty} \left(\int_{-\infty}^{+\infty} g_n(\frac{\tau}{2}) g_n^*(-\frac{\tau}{2}) e^{-i\tau\omega} \, d\tau \right) \\
&= 0 \qquad \text{because of the definition of } \{g_n(t)\}_{n\in\mathbb{N}}
\end{aligned}$$

Problem 2

Mallat chapter 5, problem 5.9 (a) and (b).

(a) Let us relate the Fourier series of $x[n]$ to the Fourier transform of $f(t)$.

$$\begin{aligned}\hat{x}(\omega) &= \sum_{n\in\mathbb{Z}} f(\frac{nT}{K})e^{-i\omega n} \\ &= \int c(t)f(t)e^{-i\omega tK/T}\,dt\end{aligned}$$

where $c(t)$ is the Dirac comb with sampling period T/K :

$$c(t) = \sum_n \delta(t - n\frac{T}{K}).$$

Its Fourier transform is

$$\hat{c}(\omega) = \frac{2\pi K}{T}\sum_k \delta(\omega - \frac{2\pi kK}{T}).$$

So, by the convolution theorem,

$$\begin{aligned}\hat{x}(\omega) &= \frac{1}{2\pi}\int \hat{c}(\frac{\omega K}{T} - \omega')\hat{f}(\omega')\,d\omega' \\ &= \frac{K}{T}\sum_k \hat{f}(\frac{\omega K}{T} - \frac{2\pi kK}{T}).\end{aligned}$$

If we call $\hat{f}_a(\omega) = \hat{f}(\frac{\omega}{a})$ the dilation of $\hat{f}$ by a factor a, then we have

$$\hat{x}(\omega) = \frac{K}{T}\sum_k \hat{f}_{T/K}(\omega - 2\pi kK).$$

Since by assumption supp $\hat{f} \subset [-\frac{\pi}{T}, \frac{\pi}{T}]$, it follows that supp $\hat{f}_{T/K} \subset [-\frac{\pi}{K}, \frac{\pi}{K}]$. When $K \geq 1$ no aliasing occurs in the sum over k so, for $-\pi \leq \omega \leq \pi$,

$$\hat{x}(\omega) = \frac{K}{T}\hat{f}_{T/K}(\omega).$$

As a result supp $\hat{x}$ = supp $\hat{f}_{T/K} \subset [-\frac{\pi}{K}, \frac{\pi}{K}]$. This shows that *oversampling* by a factor K in time reduces the bandwidth by the same factor K.

(b) If the wording is to be read as given in the book, one quickly concludes after a few lines of algebra that the MSE is infinite. This has to do with the fact that a comparable error is made at each sample $x[n]$ and these errors simply compound in the MSE. Since the sequence $x[n]$ is infinite, so is the MSE.

Instead, we are going to reformulate the problem in a fully discrete setting. Let us consider a finite sequence $x[n]$ of length N and its discrete Fourier transform $\hat{x}[k]$. To imitate the redundancy condition, assume that the DFT is nonzero only when $0 \leq k < N/K$, and that K divides N. We will replace regular convolutions by circular convolutions.

The mean-squared error is then

$$\begin{aligned}MSE &= E\{\sum_{n=0}^{N-1} |\sum_{n'=0}^{N-1}(\tilde{x}[n']h[n-n']) - x[n]|^2\} \\ &= E\{\sum_{n=0}^{N-1} |\sum_{n'=0}^{N-1}(x[n'](h[n-n'] - \delta_{nn'}) + W[n']h[n-n'])|^2\}.\end{aligned}$$

We can now use the discrete Plancherel's identity along with the discrete convolution theorem to obtain

$$\begin{aligned} MSE &= \frac{1}{N} E\{\sum_{k=0}^{N-1} |\hat{x}[k](\hat{h}[k]-1) + \hat{W}[k]\hat{h}[k]|^2\} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} E\{|\hat{x}[k]|^2\} |\hat{h}[k]-1|^2 \\ &\quad + \frac{1}{N} \sum_{k=0}^{N-1} 2\,\mathcal{R}e\, E\{\hat{x}[k]\overline{\hat{W}[k]}\}(\hat{h}[k]-1)\overline{\hat{h}[k]} \\ &\quad + \frac{1}{N} \sum_{k=0}^{N-1} E\{|\hat{W}[k]|^2\} |\hat{h}[k]|^2. \end{aligned}$$

We easily deduce from the white noise properties that

$$\begin{aligned} E\{\hat{x}[k]\overline{\hat{W}[k]}\} &= E\{\hat{x}[k]\}E\{\overline{\hat{W}[k]}\} \qquad \text{by independence} \\ &= E\{\hat{x}[k]\} \sum_{n=0}^{N-1} E\{W[n]\} e^{2\pi i k n/N} = 0, \end{aligned}$$

and that

$$\begin{aligned} E\{|\hat{W}[k]|^2\} &= E\{\sum_{n}\sum_{n'} W[n]W[n'] e^{-2\pi i k(n-n')/N}\} \\ &= \sigma^2 \sum_{n}\sum_{n'} \delta_{nn'} e^{-2\pi i k(n-n')/N} \\ &= \sigma^2 \sum_{n=0}^{N-1} 1 = N\sigma^2. \end{aligned}$$

No assumption is given at this stage on the stochastic process generating the samples $x[n]$, so we leave $E\{|\hat{x}[k]|^2\}$ as is, and call it β_k.

$$MSE = \frac{1}{N} \sum_{k=0}^{N-1} (\beta_k |\hat{h}[k]-1|^2 + N\sigma^2 |\hat{h}[k]|^2).$$

The minimization can be carried out term-by-term, and yields

$$\hat{h}[k] = \frac{\beta_k}{\beta_k + N\sigma^2}.$$

This specifies the *ideal* filter $h[n]$ in terms of its transfer function $\hat{h}[k]$. After substitution, the MSE becomes

$$MSE(\text{ideal}) = \sigma^2 \sum_{k=0}^{N-1} \frac{\beta_k}{\beta_k + N\sigma^2}.$$

We cannot expect this sum to be convergent as $N \to \infty$, for the reason explained above. Instead, it makes more sense to introduce the *averaged* MSE,

$$aMSE(\text{ideal}) = \frac{MSE}{N} = \frac{\sigma^2}{N} \sum_{k=0}^{N-1} \frac{\beta_k}{\beta_k + N\sigma^2}$$

If we assume that, due to oversampling, $\hat{x}[k] = 0$ for $\frac{N}{K} \leq k \leq N-1$, we can bound the fraction $\frac{\beta_k}{\beta_k + N\sigma^2}$ by zero if $\frac{N}{K} \leq k \leq N-1$, and one otherwise. Thus we obtain the bound

$$aMSE(\text{ideal}) \leq \frac{\sigma^2}{N}\frac{N}{K} = \frac{\sigma^2}{K}.$$

As explained in p.137 in Mallat, the transfer function of the filter corresponding to frame projection is given by the indicator function of the interval $[-\frac{\pi}{K}, \frac{\pi}{K}]$. In the discrete setting this becomes $\hat{h}_0[k] = 1$ if $0 \leq k < N/K$, and zero otherwise. In that case, it is easy to see that the aMSE reduces to

$$aMSE(\text{indicator}) = \frac{\sigma^2}{K}.$$

This result is in accordance with equation (5.43). We have therefore checked that the averaged ideal MSE is always smaller than the averaged MSE corresponding to the "indicator" filter, as expected.

Note that the ideal filter is so called because in practice we might not have access to the β_k's. At best the spectral information of the unknown signal $x[n]$ has to be *estimated* or *modeled*. This will introduce an secondary source of error in the MSE.

Problem 3

Please be aware that $\delta = \sup_n |t_{n+1} - t_n|$ in the header, and δ in part (a) of the wording do not refer to the same quantity.

(a) We are implementing the formula

$$f(t_n) = \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{t_n - nT}{T}) f(nT).$$

In practice the sum is truncated to $-M/2 \leq n < M/2$ for some large M. This is a system of linear equations which can be inverted to obtain $f(nT)$ as a function of $f(t_n)$. If N, the number of t_n's, is greater than M, the number of nT's, then the system is invertible. We choose to solve it using least-squares, i.e. $Ax = b$ is solved by $x = (A^T A)^{-1} A^T b$. This allows the use of conjugate gradients for the inversion of $A^T A$. If $N < M$ the system is underdetermined and we still endeavor to find an approximate solution using least-squares, $x = A^T (AA^T) b$. See the attached matlab code.

(b),(c) We choose $f(t) = (\text{sinc}(\frac{t}{5}))^4$ because we need a fast decay in space in order to avoid errors due to truncation of the sum to M terms. The Fourier transform of f is supported in the interval $[-\frac{4\pi}{5}, \frac{4\pi}{5}]$. We can therefore choose $T = 1$. The dilation factor 5 is chosen to avoid sampling at the roots of sinc (because this will allow us to compute a relative error.)

See attached code and plots. We draw the following conclusions. Note that all plots are averaged over 200 random realizations of the t_n's in order to best render the main trends.

- There is a dramatic loss of accuracy (a 'phase transition') as $\delta \geq T$, because the samples $f(t_n)$ do not uniquely determine $f(nT)$. The accuracy is roughly identical for all values of $\delta < T$. As a case study would reveal, the main source of error is *spatial truncation*. Taking M very large it is easy to check the empirical convergence of the error to machine precision for $\delta < T$. Large errors should be expected for most choices of slow-decaying f.

- The condition number of $A^T A$ increases steadily but moderately as δ approaches T from below. For $\delta > T$ the matrix $A^T A$ is rank-deficient. The condition number of AA^T does not exhibit any interesting behavior for $\delta > T$.

- The number of iterations needed to solve the linear system (up to some tolerance accuracy) correlates well with the condition number for $\delta < T$. For $\delta > T$, the smaller the number of samples $f(t_n)$, the faster the convergence. In that case speed of convergence is surprisingly not correlated to the condition number.

Note: Matlab's backslash command, or 'mldivide', automatically chooses the correct least-squares formulation. However I don't know if it can be used to keep track of the number of iterations (of whichever preconditioned iterative solver is being used).

```
----------------------------- file us2es.m ---------------------------------

% ACM126 Hw3 q3(a) 2005 Laurent Demanet

function [x,K,niter] = us2es(f,t_us,t_es)

% us2es.m - Turns unequispaced samples of a function into equispaced samples
% In:   values f = f(t_n) of a function at the sample points t = t_n.
%       T is the sampling period for the equispaced grid.
%       N is the number of equispaced samples to be recovered.
% Out:  x = f(nT) for n = 0..N-1
%       K is the condition number of the system Ax = f
%       niter is the number of CG iterations needed to accurately invert A
% All vectors are column vectors

M = length(t_es);
T = t_es(2) - t_es(1);
N = length(t_us);
overdetermined = (N >= M);
arg1 = t_us * ones(1,M);
arg2 = ones(N,1) * t_es';
A = sinc((arg1 - arg2)/T);
if overdetermined, B = A' * A; else B = A * A'; end;
K = cond(B);
cheat = 0;
if cheat,
    x = A \ f;
    niter = [];
else
    tol = 1e-12; niter = 0;
    if overdetermined, init = A'*f; else init = f; end;
    x = 0; r = init; p = init; prevp = zeros(size(p));
    Bp = B*p; Bprevp = B*prevp;
    while norm(r) > tol,
        niter = niter + 1;
        lambda = (r'*p) / (p'*Bp);
        x = x + lambda * p;
        r = r - lambda * Bp;
        if niter == 1,
            nextp = Bp - (Bp'*Bp)/(p'*Bp)*p;
        else
            nextp = B*p - (Bp'*Bp)/(p'*Bp)*p - (Bp'*Bprevp)/(prevp'*Bprevp)*prevp;
        end
        prevp = p; p = nextp;
        Bprevp = Bp; Bp = B*p;
    end
    if ~overdetermined, x = A'*x; end;
end
```

```
---------------------------- file hw3q3b.m ---------------------------------

% ACM126 Hw3 q3(b) 2005 Laurent Demanet

T = 1;                          % sampling period
L = 40;                         % essential support of f is [-L/2,L/2]
t_es = ((-L/2):T:(L/2-T))';     % equispaced times
M = length(t_es);               % number of equispaced samples
f_es = sinc(t_es/5).^4;         % equispaced samples of f

nrand = 200;
dataplots = 0;
for a = 1:nrand                 % average over a few noise realizations
    dataplot = [];
    for N = M+(-15:40);             % number of unequispaced samples
        delta = L/N;                % underlying sampling step
        t_us = delta*rand(N,1) + ((-L/2):delta:(L/2-delta))';
                                    % unequispaced times
        f_us = sinc(t_us/5).^4;     % unequispaced samples of f
        [x,K,niter] = us2es(f_us,t_us,t_es);
        dataplot = [dataplot; N, delta, norm(x-f_es)/norm(f_es), K, niter];
    end
    dataplots = dataplots + dataplot;
end
dataplot = dataplots/nrand;

figure(1); clf; semilogy(dataplot(:,2),dataplot(:,3));
xlabel('delta'); ylabel('log(error)'); title('Relative residual error as a function
figure(2); clf; semilogy(dataplot(:,2),dataplot(:,4));
xlabel('delta'); ylabel('log(cond)'); title('Condition number as a function of delt
figure(3); clf; plot(dataplot(:,2),dataplot(:,5));
xlabel('delta'); ylabel('nb iterations'); title('Number of iterations required to i
```

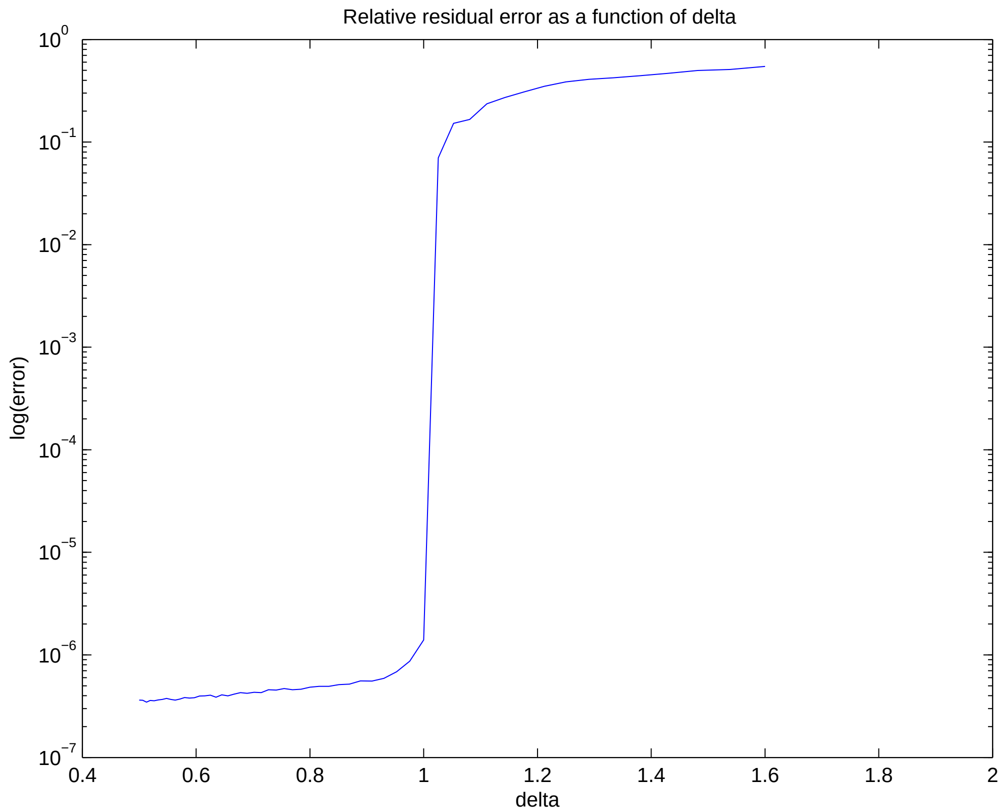

Figure 1: Relative accuracy of the interpolation as a function of δ (residual error of the least-squares). Here $T = 1$. Notice the log scale.

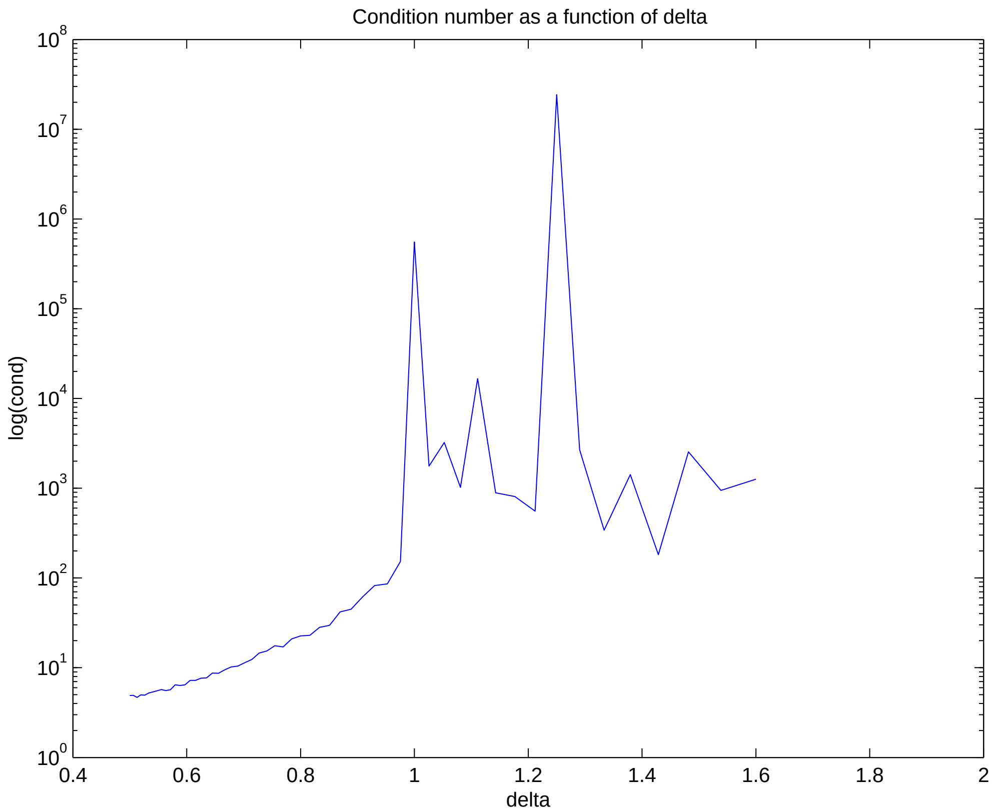

Figure 2: Condition number of $A^T A$ as a function of δ when $\delta \leq T$, and condition number of AA^T when $\delta > T$. Here $T = 1$. Notice the log scale.

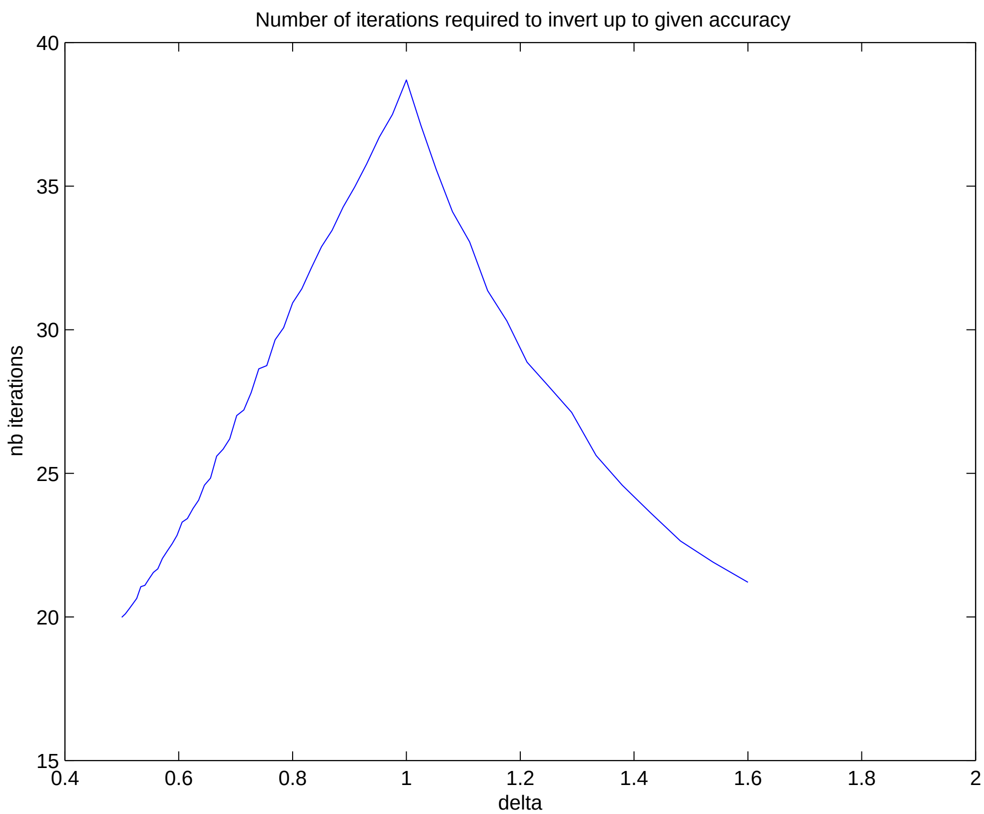

Figure 3: Number of CG iterations required to solve the least-square problem (up to tolerance accuracy 1e-12), as a function of δ. Again, the choice of least-squares strategy depends on the value of δ.