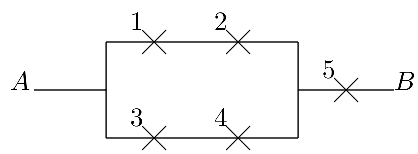
ACM116 - Fall 2007 - Problem Set #2 Solutions

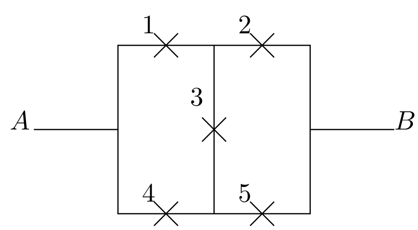
Distributed: 23 Oct 2007, Due: 30 Oct 2007

A note on grading: Your raw score out of 45 points will be multiplied by 2 to give a score out of 90 points.

1. Refer to the networks in figure 1. Switches are labelled with X's, where each numbered switch, i, has probability p_i of being closed (providing a connection). The states of the switches are independent of each other. For each network, what is its probability of successfully connecting points A and B? Network (a) (2 Points), network (b) (3 Points).



Network a)



Network b)

Figure 1: Networks for problem 1.

Solution:

(a) Let S_i be the event that switch i is closed. Let S_{12} be the event that the path through switches 1 and 2 is connected, and likewise define S_{34}. Then,

$$S_{12} = S_1 \cap S_2, \qquad \mathbb{P}(S_{12}) = p_1 p_2, \tag{1}$$

$$S_{34} = S_3 \cap S_4, \qquad \mathbb{P}(S_{34}) = p_3 p_4, \tag{2}$$

since the switches are independent. Let S_{1234} be the event that the path through switches 1 through 4 is connected. Consider the event that this path is open:

$$S_{1234}^c = S_{12}^c \cap S_{34}^c \tag{3}$$

$$\mathbb{P}(S_{1234}) = 1 - (1 - \mathbb{P}(S_{12}))(1 - \mathbb{P}(S_{34})) \tag{4}$$

$$= p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4 \tag{5}$$

Finally, the event S_{AB} that A connects to B is $S_{AB} = S_{1234} \cap S_5$, so by independence

$$\mathbb{P}(S_{AB}) = p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4). \tag{6}$$

(b) Condition on S_3:

$$\mathbb{P}(S_{AB}) = \mathbb{P}(S_{AB}|S_3)\mathbb{P}(S_3) + \mathbb{P}(S_{AB}|S_3^c)\mathbb{P}(S_3^c). \tag{7}$$

When S_3 is open, the network resembles the path through switches 1 through 4 in part (a), so

$$\mathbb{P}(S_{AB}|S_3^c) = p_1p_2 + p_4p_5 - p_1p_2p_4p_5. \tag{8}$$

With switch 3 closed, let S_{14} be the event that A connects to switch 3, and let S_{25} be the event that switch 3 connects to B. Then

$$S_{14} = (S_1^c \cap S_4^c)^c, \qquad \mathbb{P}(S_{14}) = p_1 + p_4 - p_1p_4, \tag{9}$$

$$S_{25} = (S_2^c \cap S_5^c)^c, \qquad \mathbb{P}(S_{25}) = p_2 + p_5 - p_2p_5, \tag{10}$$

$$\{S_{AB}|S_3\} = S_{14} \cap S_{25}, \qquad \mathbb{P}(S_{AB}|S_3) = (p_1 + p_4 - p_1p_4)(p_2 + p_5 - p_2p_5). \tag{11}$$

Combining these results,

$$\mathbb{P}(S_{AB}) = (1 - p_3)(p_1p_2 + p_4p_5 - p_1p_2p_4p_5) + p_3(p_1 + p_4 - p_1p_4)(p_2 + p_5 - p_2p_5). \tag{12}$$

□

2. (a) (3 Points) If N is a random variable taking non-negative integer values, show that

$$\mathbb{E}(N) = \sum_{n=0}^{\infty} \mathbb{P}(N > n). \tag{13}$$

(b) (4 Points) An urn contains b blue balls and r red balls. Balls are removed at random until the first blue ball is drawn. Show that the expected number drawn is $(b + r + 1)/(b + 1)$.

(c) (3 Points) We return all the balls to the urn. Balls are now removed from the urn at random until all the balls in the urn are the same colour. Find the expected number of balls in the urn.

Solution:

(a) Let $p_n = \mathbb{P}(N = n)$. Work directly from the definition of $\mathbb{E}(\cdot)$:

$$\begin{aligned}
\mathbb{E}(N) &= \sum_{n=0}^{\infty} n\mathbb{P}(N = n) &(14)\\
&= p_1 + (p_2 + p_2) + (p_3 + p_3 + p_3) + \ldots &(15)\\
&= (p_1 + p_2 + p_3 + \ldots) + (p_2 + p_3 + p_4 + \ldots) + \ldots &(16)\\
&= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} p_k &(17)\\
&= \sum_{n=0}^{\infty} \mathbb{P}(N > n). &(18)
\end{aligned}$$

Note: we can only exchange the order of summation if we know that $\mathbb{E}(|N|) < \infty$. Otherwise, this result does not hold.

(b) Let N be the number of balls drawn from the urn. We can directly compute $\mathbb{P}(N > n)$, since we know that the game continues as long as no blue ball has been drawn so far:

$$\mathbb{P}(N > 1) = \frac{r}{r+b} \tag{19}$$

$$\mathbb{P}(N > 2) = \frac{r}{r+b} \times \frac{r-1}{r+b-1} \tag{20}$$

$$\vdots \tag{21}$$

$$\mathbb{P}(N > n) = \frac{r}{r+b} \times \ldots \times \frac{r-(n-1)}{r+b-(n-1)} \tag{22}$$

$$\vdots \tag{23}$$

$$\mathbb{P}(N > r+1) = 0. \tag{24}$$

Now we apply the formula from part (a), noting that $\mathbb{P}(N > 0) = 1$:

$$\mathbb{E}(N) = 1 + \frac{r}{r+b} + \left(\frac{r}{r+b} + \frac{r-1}{r+b-1}\right) + \ldots + \left(\frac{r}{r+b} \times \ldots \times \frac{1}{b+1}\right) \tag{25}$$

$$= 1 + \frac{r}{r+b}\left(1 + \frac{r-1}{r+b-1}\left(1 + \frac{r-2}{r+b-2}\left(\cdots\right)\right)\right). \tag{26}$$

To compute this sum, denote $S_k = \frac{k}{k+b}(1 + S_{k-1})$, and $S_1 = \frac{1}{1+b}$. Computing a few of the S_k reveals by induction that $S_k = \frac{k}{b+1}$ for all $1 \leq k \leq r$, so

$$\mathbb{E}(N) = 1 + S_r \tag{27}$$

$$= 1 + \frac{r}{b+1} \tag{28}$$

$$= \frac{r+b+1}{b+1}. \tag{29}$$

(c) The game of removing red balls from the urn until a blue ball is removed is the same as the game of removing balls from the urn until all balls in the urn are the same colour, in the following sense. Consider each event of the probability space to be a list of blue and red balls, where balls are removed from the urn until the urn is empty. A list that begins with all red balls (only red balls are removed) occurs with equal probability as a list that ends with all red balls (only red balls remain in the urn). Let R be the event that only red balls remain in the urn, and let N be the number of balls in the urn. By the symmetry of game, we can re-use the result from part (b) to say that

$$\mathbb{E}(N, R) = \frac{r}{b+1}. \tag{30}$$

We have subtracted 1 from the result in part (b) to reflect the fact that we are not counting the one blue ball that stops the game. We can compute the total expectation by conditioning on the colour of the remaining balls, using the fact that the event that there are only blue balls remaining is disjoint from the event that there are only red balls remaining:

$$\mathbb{E}(N) = \mathbb{E}(N|R)\mathbb{P}(R) + \mathbb{E}(N|B)\mathbb{P}(B) \tag{31}$$

$$= \mathbb{E}(N, R) + \mathbb{E}(N, B) \tag{32}$$

$$= \frac{r}{b+1} + \frac{b}{r+1}. \tag{33}$$

□

3. A drunkard is wandering back and forth on a road. At each step, he moves two feet east with probability 1/2, or one foot west with probability 1/2. There is a pub at the west end of the road, and if the drunkard gets to the pub, he stops. Let a_k denote the probability of the drunkard getting to the pub when he is k feet away from the pub. Let $k > 0$ represent locations east of the pub.

(a) (4 Points) Show $a_k = \frac{1}{2}a_{k+2} + \frac{1}{2}a_{k-1}$.

(b) (3 Points) Show that $a_k = q^k$ for all $k \geq 0$ with $q = \frac{1}{2}(\sqrt{5}-1)$.

(c) (3 Points) Use these results to compute the probability that the number of heads ever exceeds twice the number of tails when a fair coin is tossed over and over.

Solution:

(a) Let R be the event that the drunkard returns to the pub, and let K_i be his location at step i. Suppose the drunkard is at step n. Condition on the direction of step $n+1$:

$$\begin{aligned} & \mathbb{P}(R|K_i = k) && (34) \\ & = \mathbb{P}(R|K_i = k, K_{i+1} = k-1)\mathbb{P}(K_{i+1} = k-1) && (35) \\ & \quad + \mathbb{P}(R|K_i = k, K_{i+1} = k+2)\mathbb{P}(K_{i+1} = k+2) && (36) \\ & = \frac{1}{2}\mathbb{P}(R|K_{i+1} = k-1) + \frac{1}{2}\mathbb{P}(R|K_{i+1} = k+2). && (37) \end{aligned}$$

Removal of the dependence on K_i follows because future steps taken by the drunkard are independent of previous steps. Identifying $a_k = \mathbb{P}(R|K = k)$, irrespective of which step he is at, gives the result.

(b) Apply the anzatz $a_k = q^k$ to the difference equation of part (a) and solve for q:

$$q^k = \frac{1}{2}q^{k-1} + \frac{1}{2}q^{k+2} \quad (38)$$

$$q^3 - 2q + 1 = 0. \quad (39)$$

The roots of this equation are $q = -(\sqrt{5}+1)/2, (\sqrt{5}-1)/2, 1$. We reject the first and last roots because they lie outside the range $0 < q < 1$.

$q \neq 1$ for a subtle reason. Given the asymmetry in the drunkard's walk, the expectation of the displacement of the drunkard with each step is positive 1/2. This fact, combined with the law of large numbers, can be used to show that the drunkard's distance from the pub increases with probability 1 after a large number of steps. We haven't studied the law of large numbers much yet in the course, so don't worry if you didn't get this part.

(c) Let $N = 2\#T - \#H$ where $\#T$ is the number of tails thrown and $\#H$ is the number of heads. We are looking for the probability that $N < 0$ ever occurs. This is equivalent to considering the drunkard's walk, where a tail represents two steps east and a head represents one step west. Let R be the event that N returns to zero if the first flip is a tail, and condition on the first flip:

$$\begin{aligned} \mathbb{P}(N<0) &= \mathbb{P}(N<0|H)\mathbb{P}(H) + \mathbb{P}(N<0|T)\mathbb{P}(T) && (40) \\ &= \frac{1}{2} + \frac{1}{2}\mathbb{P}(N<0)\mathbb{P}(R) && (41) \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2 \mathbb{P}(N<0). && (42) \end{aligned}$$

Solving gives $\mathbb{P}(N<0) = 2/(1+\sqrt{5})$.

Another way to see this is to let $N_k = 2\#T - \#H + 1$ at the kth toss, and compute $\mathbb{P}(N_{k+1} = 0|N_k = 1)$. By inspection, this is $q = \frac{\sqrt{5}-1}{2} = \frac{2}{1+\sqrt{5}}$.

□

4. **Monty Hall revisited.** Recall the Monty Hall problem: you are a contestant at a game show. The game show host shows you three doors, and tells you behind one of the doors is a prize (a new car, say), and behind the other two doors are gag prizes (donkeys, for example). You select one of the

doors, and the host opens one of the other doors behind which he knows there is a gag prize. You are given the option of changing your choice to the other closed door, and your goal is to select a strategy that gives you the best odds of winning the prize.

Consider now some variations to this game:

(a) (2 Points) Suppose you learn beforehand that there is a 0.2 probability of the prize being behind door 1, a 0.3 probability of its being behind door 2, and a 0.5 probability of its being behind door 3. Your strategy is to choose door 1 in the first round of the game, then to switch doors after the host opens a gag prize door. What is your new probability of winning the prize?

(b) (2 Points) Suppose now there are 5 doors, and the host promises to open two of the gag prize doors after you have chosen a door. Calculate the probability of winning by switching doors.

(c) (3 Points) Suppose there are 4 doors. You choose a door, and the host opens another door, behind which is a gag prize. You are given an opportunity to switch doors. Regardless of whether or not you switch, the host opens yet another door (not your current choice) behind which is another gag prize. What is your best strategy for playing this version of the game?

(d) (3 Points) Suppose there are 2 doors, behind each of which is an amount of money. You know that the amount of money behind each door is in the range $[m, M]$, where $m, M > 0$. You choose a door and the host shows you how much money is behind the door. Now you secretly choose a random number between m and M. Only if your secret choice is greater than the amount shown behind the open door do you change doors. Show that this strategy selects the largest prize with probability greater than 1/2.

Solution:

(a) Let C be the event your first guess is correct. Let W be the event you win the prize, and condition on C:

$$\mathbb{P}(W) = \mathbb{P}(W|C)\mathbb{P}(C) + \mathbb{P}(W|C^c)\mathbb{P}(C^c) \tag{43}$$
$$= 0 \times 0.2 + 1 \times 0.8 = 0.8. \tag{44}$$

If you switch doors, you probability of winning is now 0.8.

(b) Again, let C be the event your first guess is correct, let W be the event you win the prize, and condition on C:

$$\mathbb{P}(W) = \mathbb{P}(W|C)\mathbb{P}(C) + \mathbb{P}(W|C^c)\mathbb{P}(C^c) \tag{45}$$
$$= 0 \times \frac{1}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}. \tag{46}$$

Your chance of winning is $\frac{2}{5}$.

(c) The game has three stages: after your initial pick, after the first door is opened, and after the second door is opened. Let C_1, C_2, C_3 be the events you are correct at these first, second and third stages, respectively. Let S_i be the event you switch after the host opens his ith door, so the four strategies are $\{S_1, S_2\}, \{S_1, S_2^c\}, \{S_1^c, S_2\}$, and $\{S_1^c, S_2^c\}$. Let us evaluate each strategy in turn. In all cases, W is the event you win, so $W = C_3$.

$\{S_1, S_2\}$ It is impossible to be wrong at the last two stages with this strategy, as is it impossible to be correct two stages in a row. Therefore, the only non-null events are $\{C_1, C_2^c, C_3\}, \{C_1^c, C_2, C_3^c\}$, and $\{C_1^c, C_2^c, C_3\}$. Condition on C_1:

$$\mathbb{P}(W) = \mathbb{P}(W|C_1)\mathbb{P}(C_1) + \mathbb{P}(W|C_1^c)\mathbb{P}(C_1^c) \tag{47}$$
$$= 1 \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{4} = \frac{5}{8}. \tag{48}$$

$\{S_1^c, S_2\}$ In this case, you guess once, let the host open two doors, and then switch. To make the calculation clear, condition on C_1:

$$\mathbb{P}(W) = \mathbb{P}(W|C_1)\mathbb{P}(C_1) + \mathbb{P}(W|C_1^c)\mathbb{P}(C_1^c) \tag{49}$$

$$= 0 \times \frac{1}{4} + 1 \times \frac{3}{4} = \frac{3}{4}. \tag{50}$$

$\{S_1, S_2^c\}$ Again, condition on C_1:

$$\mathbb{P}(W) = \mathbb{P}(W|C_1)\mathbb{P}(C_1) + \mathbb{P}(W|C_1^c)\mathbb{P}(C_1^c) \tag{51}$$

$$= 0 \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}. \tag{52}$$

$\{S_1^c, S_2^c\}$ Your first guess must be correct for you to win. Knowing about the original Monty Hall problem, this is probably not a good strategy, but for completeness,

$$\mathbb{P}(W) = \mathbb{P}(W|C_1)\mathbb{P}(C_1) + \mathbb{P}(W|C_1^c)\mathbb{P}(C_1^c) \tag{53}$$

$$= 1 \times \frac{1}{4} + 0 \times \frac{3}{4} = \frac{1}{4}. \tag{54}$$

Comparing the solutions, switching only once, after the host has opened two doors, is the best strategy, and your probability of winning with this strategy is 3/4.

(d) Let W_1 be the event you choose the smaller amount on the first step, and let W_2 be the event you choose the larger. Let W be the event you win the larger amount in the end. Let X_1, X_2 be the values of the smaller and larger amounts, respectively, and let Y be the amount you choose, where Y has cumulative distribution $F_Y(y)$. Suppse you know X_1, X_2, but you don't know which doors they're behind. Condition on your initial choice:

$$\mathbb{P}(W|X_1, X_2) = \mathbb{P}(W|W_2)\mathbb{P}(W_2) + \mathbb{P}(W|W_1)\mathbb{P}(W_1). \tag{55}$$

Now, we have

$$\mathbb{P}(W|W_2) = \mathbb{P}(\text{don't switch}|X_2) = \mathbb{P}(Y < X_2|X_2) = F_Y(X_2) \tag{56}$$

$$\mathbb{P}(W|W_1) = \mathbb{P}(\text{switch}|X_1) = \mathbb{P}(Y > X_1|X_1) = 1 - F_Y(X_1) \tag{57}$$

Hence, we have that

$$\mathbb{P}(W|X_1, X_2) = \frac{1}{2}\left(1 + F_Y(X_2) - F_Y(X_1)\right). \tag{58}$$

Since $X_2 > X_1$, $0 \leq F_Y(X_2) - F_Y(X_1) \leq 1$, and $1/2 \leq \mathbb{P}(W) \leq 1$. Furthermore, $\mathbb{P}(W) = 1/2$ only if $F_Y(X_2) = F_Y(X_1)$, and this can only happen for $X_1 < X_2$ if $F_Y(y)$ is not strictly increasing. You get to choose the distribution for Y so you can always guarantee $\mathbb{P}(W|X_1, X_2) > 1/2$. Since strict inequality holds for any X_1, X_2, we know that $\mathbb{P}(W) = \mathbb{E}(\mathbb{P}(W|X_1, X_2)) > 1/2$.

□

5. At a party n people take off their hats (everybody arrives wearing a hat). The hats are then mixed up, and each person randomly selects a hat. We say that a match occurs if a person selects their own hat. What is the probability of

(a) (6 Points) no matches;

(b) (4 Points) exactly k matches?

(Hint: Let E be the event that no matches occur, and let $\alpha_n = \mathbb{P}(E)$, where n is the number of people. Condition on the event that the first person selects their own hat to compute a recursive relationship between α_n, α_{n-1}, and α_{n-2}.)

Solution:

(a) Let M_1 be the event the first person chooses their own hat. Condition on M_1:

$$\mathbb{P}(E) = \mathbb{P}(E|M_1)\mathbb{P}(M_1) + \mathbb{P}(E|M_1^c)\mathbb{P}(M_1^c). \tag{59}$$

$\mathbb{P}(E|M_1) = 0$, so we need only consider the second term. Now we form a recursive relationship for the α_i by considering $\mathbb{P}(E|M_1^c)$.

If the first person doesn't choose their own hat, they form a *cycle* with other people in the following sense. By choosing somebody else's hat, that other person will take yet another person's hat, and so on, until the last person in the cycle takes the first person's hat. For example, if person 1 takes person 3's hat, person 3 takes person 5's hat, and person 5 takes person 1's hat, people 1,3 and 5 form a cycle. In this example, the remaining people constitute a matching problem of size $n-3$ and from this observation we build a recursion.

Condition on the number C, the size of the cycle the first person belongs to when they don't choose their own hat.

$$\mathbb{P}(E) = \sum_{k=2}^{n} \mathbb{P}(E|C=k)\mathbb{P}(C=k). \tag{60}$$

From the above discussion, we have that $\mathbb{P}(E|C=k) = \alpha_{n-k}$. $\mathbb{P}(C=k)$ is the probability that the last person in the cycle chooses the first person's hat, and nobody else in the cycle chooses the first person's hat:

$$\begin{aligned} \mathbb{P}(C=k) &= \frac{n-1}{n} \times \frac{n-2}{n-1} \times \ldots \times \frac{n-k+1}{n-k+2} \times \frac{1}{n-k+1} && (61) \\ &= \frac{1}{n}. && (62) \end{aligned}$$

Hence,

$$\begin{aligned} \alpha_n &= \frac{1}{n}\sum_{k=2}^{n} \alpha_{n-k} && (63) \\ n\alpha_n &= \sum_{k=2}^{n} \alpha_{n-k}. && (64) \end{aligned}$$

Apply the same argument to a group of size $n-1$ to get

$$(n-1)\alpha_{n-1} = \sum_{k=2}^{n-1} \alpha_{n-1-k}. \tag{65}$$

Subtract these two expressions to get the recursion:

$$\begin{aligned} n\alpha_n - (n-1)\alpha_{n-1} &= \alpha_{n-2} && (66) \\ \alpha_n - \alpha_{n-1} &= -\frac{1}{n}\left(\alpha_{n-1} - \alpha_{n-2}\right). && (67) \end{aligned}$$

Now we observe that $\alpha_1 = 0$ and $\alpha_2 = 1/2$. From this starting point, we have

$$\alpha_n - \alpha_{n-1} = \left(-\frac{1}{n}\right) \times \left(-\frac{1}{n-1}\right) \times \ldots \times \left(\frac{1}{2}\right) = \frac{(-1)^n}{n!}. \tag{68}$$

Since $\alpha_n = \sum_{k=2}^{n}(\alpha_i - \alpha_{i-1})$,

$$\mathbb{P}(E) = \alpha_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{69}$$

Note that as $n \to \infty$, $\mathbb{P}(E) \to e^{-1}$.

There is at least one other way to compute the recursive relationship (66). Let $M_i^{(j)}$ be the event person i chooses hat j, and begin by conditioning on the choice made by the first person:

$$\mathbb{P}(E) = \mathbb{P}(E|M_1^{(2)})\mathbb{P}(M_1^{(2)}) + \ldots + \mathbb{P}(E|M_1^{(n)})\mathbb{P}(M_1^{(n)}) \tag{70}$$

$$= \frac{n-1}{n}\mathbb{P}(E|M_1^{(2)}). \tag{71}$$

The second step follows since the situation is symmetric about the choice the first person makes for a hat, provided it is not their hat. Now condition on the choice made by the second person:

$$\mathbb{P}(E|M_1^{(2)}) = \mathbb{P}(E|M_1^{(2)}, M_2^{(1)})\mathbb{P}(M_2^{(1)}) + (n-2)\mathbb{P}(E|M_1^{(2)}, M_2^{(3)})\mathbb{P}(M_2^{(3)}) \tag{72}$$

$$= \frac{1}{n-1}\mathbb{P}(E|M_1^{(2)}, M_2^{(1)}) + \frac{n-2}{n-1}\mathbb{P}(E|M_1^{(2)}, M_2^{(3)}). \tag{73}$$

With the first term, people 1 and 2 form a cycle, and what remains is a matching problem of size α_{n-2}.

We claim that the second term is equal to α_{n-1}. The event $M_2^{(1)}, M_3^{(2)}$ happens with equal probability to the event $M_2^{(2)}, M_3^{(1)}$. This precludes the probability of no matches over all n hats (person 2 gets their own hat) but is equal to $\mathbb{P}(E_{n-1}|M_3^{(1)})$. By comparison with (71), this gives $\alpha_{n-1} = \frac{n-2}{n-1}\mathbb{P}(E_{n-1}|M_3^{(1)})$. Hence,

$$\alpha_n = \frac{1}{n}\alpha_{n-2} + \frac{n-1}{n}\alpha_{n-1}. \tag{74}$$

(b) Let S_k be the event that the first k people get their own hats, and let T_{n-k} be the event that there are no other matches

$$\mathbb{P}(S_k, T_{n-k}) = \mathbb{P}(T_{n-k}|S_k)\mathbb{P}(S_k). \tag{75}$$

The first event, $\{T_{n-k}|S_k\}$, is a matching problem of size $n-k$, and we know the probability of this event from part (a). The probability of event S_k can be computed by inspection:

$$\mathbb{P}(S_k, T_{n-k}) = \alpha_{n-k} \times \frac{1}{n} \times \frac{1}{n-1} \times \ldots \times \frac{1}{n-k+1} \tag{76}$$

$$= \frac{(n-k)!}{n!}\alpha_{n-k}. \tag{77}$$

We have selected a particular case where there are exactly k matches out of n people, so to compute the probability of $K = k$ matches, we permute over all ways k items can be selected from a set of n items:

$$\mathbb{P}(K = k) = \binom{n}{k}\frac{(n-k)!}{n!}\alpha_{n-k} = \frac{1}{k!}\alpha_{n-k} \tag{78}$$

$$= \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!}. \tag{79}$$

Note that for fixed k, as $n \to \infty$, $\mathbb{P}(K = k) \to \frac{1}{k!}e^{-1}$, and K approaches a Poisson variable with mean 1.

□