Problem Set 1

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1 "Bread & Butter": Distribution functions.

Express respectively the distribution of

1. $X^+ = \max\{0, X\}$
2. $X^- = -\min\{0, X\}$
3. $|X| = X^+ + X^-$
4. -X

in terms of the distribution function F of the random variable X.

2 "Appetizers": Dies

On a modern die, the face value 6 is opposite to the face value 1, the face value 5 to the face value 2, and the face value 4 to the face value 3. Now three fair modern dies are rolled one by one.

1. Describe the probability space $(\Omega, \mathcal{U}, \mathbb{P})$ in this problem.
2. The totals X is a random variable. Describe the σ-algebra $\mathcal{U}(X)$.
3. Compute $\mathbb{P}(X = 9)$ and $\mathbb{P}(X = 12)$.
4. Are two probabilities equal? Can you get the conclusion without computing the probabilities?
5. Old Etruscan die show 1 and 2, 3 and 4, 5 and 6 on opposite sides. If we use three fair Old Etruscan dies instead of morden ones, $\mathbb{P}(X = 9) = \mathbb{P}(X = 12)$? Why?

3 "Soups": Continuity of Probability Measure

1. Let $(\Omega, \mathcal{U}, \mathbb{P})$ be a probability space. Probability measure $\mathcal{P}$ is continuous in the sense described as follows.
   (a) $\{A_i\}_{i=1}^{+\infty}$, a sequence of measurable set, i.e. $A_i \in \mathcal{U}$, is decreasing in the sense that $A_1 \supset A_2 \supset A_3 \supset A_4 \supset \cdots$.
   Prove $\lim_{i\to+\infty} \mathbb{P}(A_i) = \mathbb{P}(A)$ where $A = \bigcap_{i=1}^{+\infty} A_i$
   (b) $\{A_i\}_{i=1}^{+\infty}$, a sequence of measurable set, is increasing in the sense that $A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots$.
   Prove $\lim_{i\to+\infty} \mathbb{P}(A_i) = \mathbb{P}(A)$ where $A = \bigcup_{i=1}^{+\infty} A_i$
2. A fair coin is tossed inifinite times.
   (a) Define in details the probability space $(\Omega, \mathcal{U}, \mathbb{P})$. (If you can not define the probability measure $\mathbb{P}$ rigorously, it is ok, but at least think and try.)
   (b) By applying the continuity of probability measure, show that, with probability one, a head turns up sooner or later.
   (c) Show similiarly that any given finite sequence of heads and tails occurs eventually with probability one

4 "Entrees": Blood testing

Suppose that a large number, n, of blood samples are to be screened for a relatively rare diseas. Each sample tests positive with probability p independent of each other. If each sample is assayed individually, n tests will be required. On the other hand, it is possible, assuming that the disease is rare, that some savings can be achieved through some pooling procedure. The purpose of this exercise is to examine some common pooling procedures.

Consider the following scheme for grouping testing. The original lot of samples is divided into two groups and each of the subgroups is tested as a whole. If either subgroup tests positive, it is divided in two and the procedure is repeated. If any of the groups thus obtained tests positive, test every member of that group. (We will assume that the test method is sensitive enough; A group tests positive if and only if at least one person is positive in that group)

1. Find the expected number of tests performed
2. Compare it to the number of tests performed with no groupings. For which value of p is this grouping testing scheme inferior to testing every individuals?
3. Consider now the following scheme. The n samples are first grouped into m gourps of k samples each, or $n = mk$. Each group is then tested; If a group tests positive, each individual in the group is tested. Find the expected number of tests performed.

5 "Desserts": Seating Problem

The n passengers for a Bell-Air flight in an airplane with n seats have been told their seat numbers. They get on the plane one by one. However, the first person sits in the wrong seat. Subsequent passengers sit in their assigned seats whenever they find them available, or otherwise in a randomly chosen empty seat. What is the probability that the last passenger finds his assigned seat free?

(Hint: Write F for the event that the last passenger sits on his assigned sit and K for the seat the first passagener takes. Let $\alpha_k = \mathbb{P}(F \,|K = k\,), k = 2 \ldots n$. Start with finding a recursive formula of α_k)