ACM 106C: Problem set 5

Due: May 15, 2008

1. Consider the following boundary value problem

$$-u_{xx} = 1, \quad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

   (a) Find the exact solution u to the above problem.

   (b) Find an approximate solution u_h to the above problem by the piecewise linear finite element method. You may download the sample code **fem.m** from the course web page. Use $M = 20$. Plot the exact and the approximate solution on the same graph.

2. Consider the following boundary value problem (D)

$$-u_{xx} + q(x)u = f, \quad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

Here $q(x) \geq q_0 > 0$ is a continuous function and q_0 is a positive constant. We define the following:

$$V = \Big\{ v \mid v \text{ is continuous on}[0,1], v' \text{ is piecewise continuous and bounded on } [0,1], \text{ and } v(0) = v(1) = 0 \Big\}$$

$$a(v,w) = \int_0^1 (v'(x)w'(x) + q(x)\, v(x)w(x))\, dx.$$

$$(v,w) = \int_0^1 v(x)w(x)\, dx.$$

$$F(v) = \frac{1}{2}a(v,v) - (f,v).$$

The minimization problem (M) is: find $u \in V$ such that

$$F(u) \leq F(v), \quad \forall\, v \in V.$$

The variational problem (V) is: find $u \in V$ such that

$$a(u,v) = (f,v), \quad \forall\, v \in V.$$

Show that (D), (M) and (V) are equivalent by proving the following.

   (a) If u is a solution of (D), then u is also a solution of (V).

   (b) Problems (M) and (V) have the same solution.

   (c) Problem (V) has a unique solution.

   (d) Assume that u is a solution of (V) and u is twice continuously differentiable. Show that u is also a solution of (D).