Problem Set 2 Solutions

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1. (a) Since BTFS is an implicit method, for each time step we solve a linear system $v^{n+1}A = v^n$.

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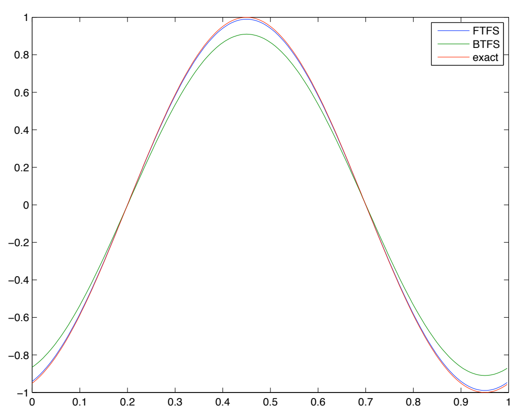
function [x, v] = btfs(a, lambda, M, u0, N)

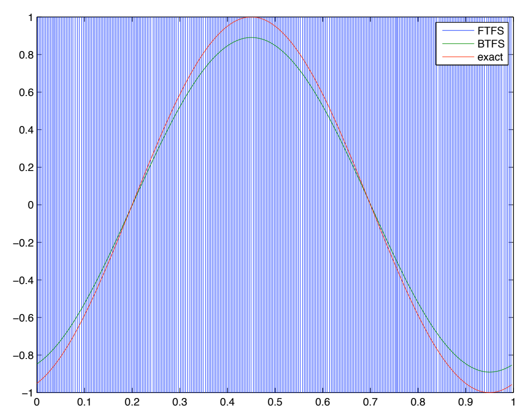
x = linspace(0,1,M+1);
x = x(1:M);

%set up iteration matrix
A = (1-a*lambda)*speye(M) + a*lambda*circshift(speye(M),1);

v = u0(x);
for n=1:N
    v = v/A;
end
```



(b) $\lambda = 0.8$



(c) $\lambda = 1.2$

Figure 1: Comparison of FTFS, BTFS, and exact solutions.

(d) For $\lambda = 0.8$, FTFS gives a closer approximation to the exact solution, with less numerical dissipation than BTFS. However, FTFS becomes unstable for $\lambda = 1.2$, while BTFS remains stable.

2. Given a consistent finite difference scheme

$$v_m^{n+1} = \alpha v_m^n + \beta v_{m-1}^n + \gamma_{m-2}^n$$

for the one way wave equation $u_t + au_x = 0$, it follows that

$$0 \le a\lambda \le 2$$

is a necessary condition for stability.

Proof. Consider the approximate solution $v_0^n = v(1,0)$. This approximation depends only on the initial data $v_0^0, v_{-1}^0, \ldots, v_{-2n}^0$. Furthermore, the exact solution is $u(1,0) = u(0,-a)$, so in order for the approximation to converge, this point must be contained in the interval

$$-2/\lambda = x_{-2n} \le -a \le x_0 = 0.$$

Therefore, $0 \le a\lambda \le 2$ is a necessary condition for convergence, and since convergence and stability are equivalent for consistent methods, the condition is also necessary for stability. □

3. The initial value problem

$$\begin{aligned} u_t + au_x + bu = 0, &\qquad -\infty < x < \infty, t > 0, \\ u(0,x) = u_0(x), &\qquad -\infty < x < \infty \end{aligned}$$

is well-posed.

Proof. Take the Fourier transform with respect to the spatial variable x, turning $u(t,x)$ into $\hat{u}(t,\omega)$. Furthermore, $\widehat{u_t} = \hat{u}_t$ and $\widehat{u_x} = i\omega\hat{u}$, so the transformed function $\hat{u}$ satisfies

$$\begin{aligned} \hat{u}_t + (b + ia\omega)\hat{u} = 0, \\ \hat{u}(0,\omega) = \hat{u}_0(\omega). \end{aligned}$$

This is now just an ODE, which has the unique solution

$$\hat{u}(t,\omega) = \hat{u}_0(\omega)e^{-(b+ia\omega)t},$$

and taking the inverse Fourier transform, we get the solution to the original problem

$$u(t,x) = u_0(x - at)e^{-bt}.$$

Hence, the problem is well-posed. (Note: instead of explicitly computing the inverse Fourier transform, you also could have used Parseval's identity to bound the L^2-norm of the solution.) □