ACM 106a, Problem Set 3: Solutions

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Theory

1. (a) Between two knots x_i, x_{i+1}, the energy functional is given by the single term

$$E[y] = \int_{x_i}^{x_{i+1}} \left[\left(y''(x)\right)^2 + \lambda_i^2 \left(y'(x)\right)^2 \right] dx.$$

Now take a variation δy of y; to stay in the space of interpolating functions, we must have $\delta y(x_i) = \delta y(x_{i+1}) = 0$ at the knots. The variation of the energy functional is then

$$\delta E[y] = \int_{x_i}^{x_{i+1}} \left[2y''(x)\delta y''(x) + 2\lambda_i^2 y'(x) \delta y'(x) \right] dx.$$

Integrating by parts, the boundary terms vanish since δy is zero at the knots, and so

$$\delta E[y] = \int_{x_i}^{x_{i+1}} 2\left[y^{(4)}(x) - \lambda_i^2 y''(x) \right] \delta y(x)\, dx.$$

The energy functional is minimized when $\delta E = 0$ for an arbitrary variation δy, and so the spline y satisfies the ordinary differential equation

$$y^{(4)}(x) = \lambda_i^2 y''(x).$$

It can be verified directly that the answer given in the book solves this equation.

(To actually obtain the solution form ourselves, we can notice that substituting $u = y''$ satisfies the ODE $u'' = \lambda_i^2 u$, which has solutions of the form $u(x) = a_i \cosh(\lambda_i x) + b_i \sinh(\lambda_i x)$. Integrating twice to recover y, we conclude that y contains the cosh and sinh terms as well, along with a linear term and a constant term, which agrees with the form given in the book.)

(b) As $\lambda_i \to 0$, these become cubic splines. This can be observed by noticing that the energy functional $E[y]$ approaches that for cubic splines, containing only the $\left(y''(x)\right)^2$ term. Equivalently, the ODE satisfied by the exponential splines becomes $y^{(4)}(x) = 0$, which says that the splines are cubic.

(c) Using the identities $\cosh x = \cos ix$ and $\sinh x = -i \sin ix$, we can conclude that taking $\lambda_i = i$ (or any pure imaginary number) gives us splines that are expressible in terms of ordinary, real trigonometric functions.

2. (a) Looking at the Chebyshev polynomials, we can see that $T_j(x)$ has the leading term $2^{j-1}x^j$ for $j \geq 1$. However, the recursive definition of $p_j(x)$ requires the leading term to be x^j, so we scale the Chebyshev polynomials by a constant to get

$$p_j(x) = \frac{1}{2^{j-1}} T_j(x) \text{ for } j \geq 1.$$

Note that this is *not* true for the special case $j = 0$, where we have $p_0(x) = T_0(x)$.
We can now divide the Chebyshev recursion $T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$ by 2^j on both sides to get

$$\begin{aligned} p_{j+1}(x) &= x\frac{1}{2^{j-1}} T_j(x) - \frac{1}{4}\frac{1}{2^{j-2}} T_{j-1}(x) \\ &= xp_j(x) - \frac{1}{4} p_{j-1}(x) \text{ for } j \geq 2, \end{aligned}$$

and with the special $j = 1$ case $p_2(x) = xp_1(x) - \frac{1}{2}p_0(x)$.
Therefore, the tridiagonal matrix has $\delta_j = 0$ for $j \geq 1$, $\gamma_j = \pm\frac{1}{2}$ for $j \geq 2$, and $\gamma_2 = \pm\frac{1}{\sqrt{2}}$:

$$\begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} & & & \\ \pm\frac{1}{\sqrt{2}} & 0 & \pm\frac{1}{2} & & \\ & \pm\frac{1}{2} & 0 & \ddots & \\ & & \ddots & \ddots & \pm\frac{1}{2} \\ & & & \pm\frac{1}{2} & 0 \end{pmatrix}.$$

(b) For $n = 3$, $p_3(x) = x^3 - \frac{3}{4}x$ has the roots $x_1 = 0, x_2 = \frac{\sqrt{3}}{2}, x_3 = -\frac{\sqrt{3}}{2}$. This leads to the system

$$\begin{aligned} w_1 + w_2 + w_3 &= \pi \\ \frac{\sqrt{3}}{2} w_2 - \frac{\sqrt{3}}{2} w_3 &= 0 \\ -\frac{1}{2} w_1 + \frac{1}{4} w_2 + \frac{1}{4} w_3 &= 0, \end{aligned}$$

which has the solution $w_1 = w_2 = w_3 = \frac{\pi}{3}$.

MATLAB

1. With $\gamma_i = i$, the exponential/trigonometric splines interpolate the circle perfectly (or at least up to machine precision). See the plots in Figure 1 and Figure 2.

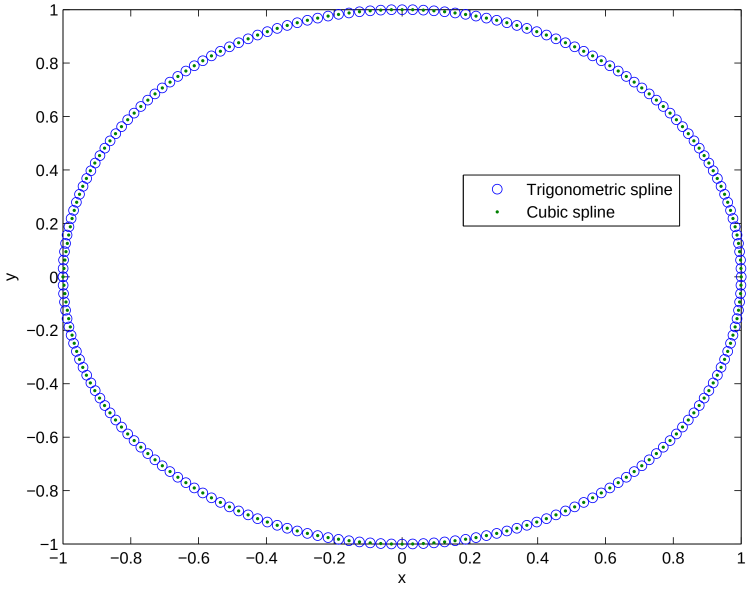

Figure 1: Interpolation plot for trigonometric and cubic splines.

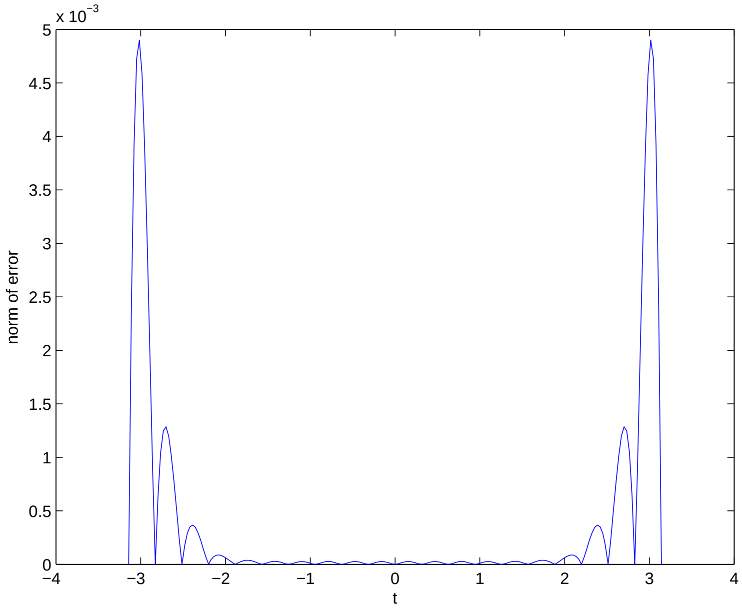

Figure 2: Error in the cubic spline interpolation.