ACM 105: Homework 3
Selected Solutions

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All theorems are from Kreyszig.

1. Let X be a normed linear space and let $x_0 \neq 0$ be an element of X. Prove that there exist a bounded linear functional f on X such that

$$\|f\| = \frac{1}{\|x_0\|}, \quad f(x_0) = 1.$$

SOLUTION:

The result follows immediately from Theorem 4.3-3 with x_0 replaced by $x_0/\|x_0\|$. The main idea is to use the Hahn Banach Theorem with the original subspace being $\text{span}(x_0)$. $\square$

2. Let $M \neq \phi$ be a subset of a normed linear space X. The annihilator M^a of M is defined to be the set of all bounded linear functionals on X which are zero everywhere on M, that is

$$M^a := \{f \in X' \mid f(x) = 0, \quad \forall x \in M\}.$$

(a) Show that M^a is a subspace of X' and that M^a is closed.

(b) Let X and Y be normed linear spaces, and let $T\colon X \to Y$ be a bounded linear operator. Set $M = \overline{\mathcal{R}(T)}$. Show that $M^a = \mathcal{N}(T^\times)$.

SOLUTION to (b):

First, we'll note that $M^a = \bar{M}^a$ (which means that, in the problem, we can shoose either $M = \mathcal{R}(T)$ or $M = \overline{\mathcal{R}(T)}$). By $\bar{M}^a$, we mean the annihilator of $\bar{M}$, not $\overline{M^a}$, although (via part (a)), they turn out to be equivalent.

Since $M \subset \bar{M}$, it follows that $\bar{M}^a \subset M^a$ (since the bigger M is, the smaller M^a is). Now suppose $f \in M$, so $f(x) = 0$ for all $x \in M$. For any $x \in \bar{M}$, take a sequence $(x_n) \subset M$ that converges to it (if x is not an accumulation point, then we can choose $x_n = x$ for all n). Since f is continuous (since bounded and linear), $f(x_n) \to f(x)$. But $f(x_n) = 0$, so $f(x) = 0$, hence f is zero on $\bar{M}$, and thus $f \in \bar{M}^a$.

Now, we show $M^a = \mathcal{N}(T^\times)$. Let 0_x denote the zero operator on X.

$$\begin{aligned} f \in M^a &\iff f(y) = 0 \quad \forall y \in T(X) \\ &\iff f(Tx) = 0 \quad \forall x \in X \\ &\iff T^\times f(x) = 0 \quad \forall x \in X \\ &\iff T^\times(f) = 0_x \\ &\iff f \in \mathcal{N}(T^\times) \end{aligned}$$

$\square$

3. Let X be a Banach spaces and Y be a normed liner space. Let $T_n\colon X \to Y$ be a sequence of bounded linear operators. Assume that for each $x \in X$ and each $f \in Y'$, there is a constant $c_{x,f} > 0$ (which depends on x and f) such that

$$|f(T_n x)| \leq c_{x,f}.$$

Show that, there is a constant $c > 0$ (which does NOT depend on x and f) such that

$$\|T_n\| \leq c.$$

SOLUTION:

The trick to this problem is applying the Uniform Boundedness Theorem twice. The first part of the proof is similar to the proof of part (c) in Lemma 4.8-3.

Fix x for now. For every n, define the functional $g_n\colon X' \to \mathbb{R}$ (that is, a functional on the space of functionals of X) by

$$g_n(f) = f(T_n x) \quad \forall f \in Y'$$

So, our assumption implies that

$$|g_n(f)| \le c_{x,f} \quad \forall n$$

Since Y' is a Banach Space (the dual space Y' of a normed space Y is always a Banach Space, even if Y is not a Banach Space), we can apply the Uniform Boundedness Theorem (and keeping x constant still). Thus we get

$$\|g_n\| \le c_x \quad \forall n$$

for some constant c_x.

By using the definition of a norm on a dual space, and using Theorem 4.3-4, we have

$$\|g_n\| = \sup_{f \in Y', f \neq 0} \frac{|g_n(f)|}{\|f\|} = \sup_{f \in Y', f \neq 0} \frac{|f(T_n x)|}{\|f\|} = \|T_n x\|$$

Hence, for every n, $\|T_n x\| \le c_x$.

Now, apply the Uniform Boundedness Theorem again (since X is Banach), and we get

$$\|T_n\| \le c \quad \forall n$$

The requirement that $c > 0$ is trivial, since if c is a bound, then so is $c + 1$ (think of the statement "$c > 0$" not as an inequality, but like a shorthand to say "$c \in \mathbb{R}$").

This proof used the terminology of Lemma 4.8-3, but we can also use the adjoint terminology. We note that $|f(T_n x)| = |(T_n{}^{\times} f)(x)| \le c_{x,f}$, then apply the Uniform Boundedness Theorem to get $\|(T_n{}^{\times} f)\| \le c_f$, and then apply once more to get $\|T_n{}^{\times}\| \le c_f$ and use the fact that $\|T^{\times}\| = \|T\|$. □

4. Let X and Y be Banach spaces and $T\colon X \to Y$ be an injective bounded linear operator. Show that $T^{-1}\colon \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.

SOLUTION:

Note that T^{-1} exists since T is injective, and that it is linear (Theorem 2.6-10).

First, suppose $R(T)$ is closed. Since Y is complete, this means $R(T)$ is complete, and hence a Banach Space (since the range is always a linear subspace). So, by the Open Mapping Theorem, $T\colon X \to R(T)$ is an open mapping, which means T^{-1} is continuous. Since T^{-1} is also linear, it follows that T^{-1} is bounded (Theorem 2.7-9).

Now suppose T^{-1} is bounded, hence continuous. Let $(y_n) \subset R(T)$ be a sequence converging to y. We wish to show $y \in R(T)$ as well. Since T^{-1} is continuous, $(T^{-1} y_n) \subset X$ is also convergent, with some limit x. Because T is continuous, $T(x) = \lim_n T(x_n) = \lim_n y_n = y$ and hence $y \in R(T)$ which means $R(T)$ is closed. □