ACM 105: Problem set 3

Due: April 25, 2008

1. Let X be a normed linear space and let $x_0 \neq 0$ be an element of X. Prove that there exist a bounded linear functional f on X such that

$$\|f\| = \frac{1}{\|x_0\|}, \quad f(x_0) = 1.$$

2. Let $M \neq \phi$ be a subset of a normed linear space X. The annihilator M^a of M is defined to be the set of all bounded linear functionals on X which are zero everywhere on M, that is

$$M^a := \{f \in X' \mid f(x) = 0, \quad \forall x \in M\}.$$

   (a) Show that M^a is a subspace of X' and that M^a is closed.

   (b) Let X and Y be normed linear spaces, and let $T : X \to Y$ be a bounded linear operator. Set $M = \overline{\mathcal{R}(T)}$. Show that $M^a = \mathcal{N}(T^\times)$.

3. Let X be a Banach spaces and Y be a normed liner space. Let $T_n : X \to Y$ be a sequence of bounded linear operators. Assume that for each $x \in X$ and each $f \in Y'$, there is a constant $c_{x,f} > 0$ (which depends on x and f) such that

$$|f(T_n x)| \leq c_{x,f}.$$

   Show that, there is a constant $c > 0$ (which does NOT depend on x and f) such that

$$\|T_n\| \leq c.$$

4. Let X and Y be Banach spaces and $T : X \to Y$ be an injective bounded linear operator. Show that $T^{-1} : \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.