ACM 105: Homework 2
Selected Solutions

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1. Show that in a Banach space, an absolutely convergent series is convergent.

   SOLUTION:

   Let X be a Banach space, and $(x_n) \subseteq X$ an absolutely convergent series. This means, if we define the real numbers $t_n = \sum_{k=1}^{n} \|x_k\|$, then t_n converges to some real limit t.

   Define the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$. We wish to prove this is a Cauchy sequence, and hence, since a Banach space is complete, it converges to a limit $s \in X$. Fix any $\varepsilon > 0$. Since $t_n \to t$, we can find some N such that $p > N$ implies $|t - t_p| < \varepsilon$. Now, pick n and m greater than N, and wlog, assume $m > n$. Then

   $$\|s_m - s_n\| = \Big\| \sum_{k=n+1}^{m} x_k \Big\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=n+1}^{\infty} \|x_k\| = |t - t_n| < \varepsilon$$

   hence (s_n) is Cauchy, and hence convergent, so (x_n) is a convergent series. □

2. (a) Let B be the normed linear space of all bounded sequence of complex numbers with norm $\|x\| = \sup |x_j|$ where $x = (x_1, x_2, \ldots)$. Show that the operator $T : B \to B$ defined by $y = (\eta_1, \eta_2, \ldots) = Tx$, $\eta_j = x_j / j$, is linear and bounded.

   (b) Show that the range $\mathcal{R}(T)$ is not closed in B. (Hint: Consider the sequence $y_n = (1, 1/\sqrt{2}, \cdots, 1/\sqrt{n}, 0, 0, \cdots)$ in $\mathcal{R}(T)$.)

   SOLUTION to (b):

   Consider $y_n = (1, 1/\sqrt{2}, \ldots, 1/\sqrt{n}, 0, 0, \ldots)$. Then y_n is in $\mathcal{R}(T)$ because $Tx_n = y_n$ where $x_n = (1, \sqrt{2}, \ldots, \sqrt{n}, 0, 0, \ldots) \in B$. $x_n \to x$ where $x = (1, \sqrt{2}, \ldots, \sqrt{n}, \sqrt{n+1}, \ldots)$ and $x \notin B$. Since T is continuous, $Tx_n \to Tx = y \in \overline{\mathcal{R}(T)}$, where $y = (1, 1/\sqrt{2}, \ldots, 1/\sqrt{n}, 1/\sqrt{n+1}, \ldots)$. Since $x \notin B$, then $y \notin \mathcal{R}(T)$ and hence $\mathcal{R}(T) \neq \overline{\mathcal{R}(T)}$.

   Alternatively, use $y_n = (0, \log(2)/2, \ldots, \log(n)/n, 0, \ldots)$. Due to P. Mullen.

   Other options: use $y_n = ((1 + 1/n)^{-1}, (1 + 1/n)^{-2}, \ldots)$. Then $y_n \in \mathcal{R}(T)$ (you can check this) and $y_n \to y$ where $y = (1, 1, 1, \ldots)$ which is in B but not in $\mathcal{R}(T)$. In this case, if we define x_n s.t. $Tx_n = y_n$, then (x_n) doesn't converge at all (it's not Cauchy), so we have to be a bit more careful and use the fact that T^{-1} is injective (obvious from definition of T). If y were in $\mathcal{R}(T)$, then we can calculate $T^{-1}y$ explicitly and it's easy to see it is not in B. Due to A. Harvard.

   Along similar lines, we could let y_n be the sequence with n 1's and the rest zeroes, e.g. $y_3 = (1, 1, 1, 0, 0, \ldots)$. So $y_n \in \mathcal{R}(T)$ and $y_n \to y$ where $y = (1, 1, 1, \ldots)$ which is not in $\mathcal{R}(T)$. Again, $(x_n) = (T^{-1}y_n)$ is not convergent in B, so we calculate $T^{-1}y = (1, 2, \ldots, n, n+1, \ldots)$ explicitly and find it is not in B. □

3. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded. (Hint: Consider the bounded linear operator defined in Question 2, and find a sequence $(y_n) \in \mathcal{R}(T)$ such that $\|y_n\| = 1$ and $\|T^{-1}y_n\| \to \infty$.)

   SOLUTION:

   Note that the inverse *can* be bounded, e.g. let T be the identity operator, so $T^{-1} = T$.

   To find an example, use the setup from problem 2. Let y_n be the sequence with n 1's and the rest zeroes, e.g. $y_3 = (1, 1, 1, 0, 0, \ldots)$. So $\|y_n\| = 1$ and $y_n \in \mathcal{R}(T)$, but $\|T^{-1}y_n\| = n$ so T^{-1} is unbounded. □

4. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt.$$

SOLUTION:

First, we'll bound the norm of f. Recall $\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|}$. Note that for all x,

$$\begin{aligned} |f(x)| &\leq \left| \int_{-1}^{0} x(t)dt \right| + \left| \int_{0}^{1} x(t)dt \right| \\ &\leq \int_{-1}^{0} |x(t)|dt + \int_{0}^{1} |x(t)|dt \\ &\leq \|x\| + \|x\| \leq 2\|x\| \end{aligned}$$

hence $\|f\| \leq 2$.

Is this bound sharp, i.e. does $\|f\| = 2$? Yes. Clearly, the function x that takes the value $+1$ on $[-1,0]$ and -1 on $[0,1]$ ($x = -\mathrm{sgn}(t)$) realizes this bound (that is, for this function x, $f(x) = 2\|x\|$), but this function is not continuous. Instead, we can produce a sequence of functions (x_n) such that $\lim_{n\to\infty} |f(x_n)|/\|x_n\| = 2$, and x_n is continuous.

For example, take $x_n = -\tan^{-1}(nt)/(\pi/2)$, or

$$x_n(t) = \begin{cases} +1 & t \leq -1/n \\ -nt & -1/n < t < 1/n \\ -1 & t \geq 1/n \end{cases}$$

5. If Y is a subspace of a linear space X over K and f is a linear functional on X such that $f(Y) \neq K$, show that $f(y) = 0$ for all $y \in Y$.

SOLUTION:

$f(Y) \neq K$ means $\exists\, k \in K$ s.t. $\nexists\, y \in Y$ with $f(y) = k$. Suppose $\exists\, x \in Y$ with $f(x) = \alpha \neq 0$. Then $y = kx/\alpha$ is in Y (α^{-1} exists because K is a field, and $y \in Y$ because Y is a subspace) and $f(y) = k$, a contradiction. □

6. Consider the normed linear space $C[0,1]$ with norm defined by

$$\|x\| = \int_{0}^{1} |x(t)|dt, \quad x \in C[0,1].$$

Let f be a linear functional defined by $f(x) = x(1/2)$. Show that f is not bounded.

SOLUTION:

To show f is not bounded, we can find a sequence (x_n) in $C[0,1]$ such that $\|x_n\| = 1$ (or is bounded by some constant) but $f(x_n)$ grows without bound.

For example, let x_n be a triangle-shaped function centered at $x = 1/2$ with height $2n$ and width $1/n$. The integral of x_n is just the area of a triangle, which is $1/2 \cdot \text{base} \cdot \text{height} = (1/2)(1/n)(2n) = 1$, so $\|x_n\| = 1$ but $f(x) = 2n$.

There are many other possible examples. If we shift everything by 1/2 to the left (so that $f(x) = x(0)$), then we could take (x_n) to be any sequence of continuous nascent delta functions. For example, $x_n(t) = \alpha_n e^{-(nt)^2}$ or $x_n(t) = \beta_n e^{-|nt|}$, where $\alpha_n = \frac{n}{\sqrt{\pi}}$ and $\beta_n = \frac{n}{2}$ are normalization constants that ensure $\|x_n\|$ is bounded.