ACM 105: Problem set 2

Due: April 18, 2008

1. Show that in a Banach space, an absolutely convergent series is convergent.

2. (a) Let B be the normed linear space of all bounded sequence of complex numbers with norm $\|x\| = \sup |x_j|$ where $x = (x_1, x_2, \cdots)$. Show that the operator $T : B \to B$ defined by $y = (\eta_1, \eta_2, \cdots) = Tx$, $\eta_j = x_j/j$, is linear and bounded.

   (b) Show that the range $\mathcal{R}(T)$ is not closed in B. (Hint: Consider the sequence $y_n = (1, 1/\sqrt{2}, \cdots, 1/\sqrt{n}, 0, 0, \cdots)$ in $\mathcal{R}(T)$.)

3. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded. (Hint: Consider the bounded linear operator defined in Question 2, and find a sequence $(y_n) \in \mathcal{R}(T)$ such that $\|y_n\| = 1$ and $\|T^{-1} y_n\| \to \infty$.)

4. Find the norm of the linear functional f defined on $C[-1, 1]$ by

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt.$$

5. If Y is a subspace of a linear space X over K and f is a linear functional on X such that $f(Y) \neq K$, show that $f(y) = 0$ for all $y \in Y$.

6. Consider the normed linear space $C[0, 1]$ with norm defined by

$$\|x\| = \int_{0}^{1} |x(t)|dt, \quad x \in C[0, 1].$$

   Let f be a linear functional defined by $f(x) = x(1/2)$. Show that f is not bounded.