# ACM 105: Homework 1
# Selected Solutions

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Comments: as for any math class, clear and concise proofs are preferred (as is legible handwriting). Using LaTeX is great, but by no means necessary. There is no need to restate the problem in your solutions, unless you prefer to. Below are solutions to some of the problems; in general, I write up the solutions to the more difficult problems. If you would like to make sure that I write up the solution to a particular problem, please email me and I will be glad to do it.

1. Let $(X, d)$ be a metric space. Let $x_0 \in X$ and $r > 0$. Prove that

   (a) The open ball $B(x_0; r)$ is open.

   SOLUTION:
   Let $x \in B(x_0; r)$, so $d(x_0, x) < r$. Define $r' = r - d(x_0, x)$. Consider $B(x; r')$. If $y \in B(x; r')$, then by the triangle inequality

   $$d(x_0, y) \leq d(x_0, x) + d(x, y) < d(x_0, x) + r' = r$$

   and hence $y \in B(x_0; r)$, so $B(x; r') \subseteq B(x_0; r)$. Since this holds for any $x \in B(x_0; r)$, it follows that $B(x_0; r)$ is open. $\square$

   (b) The closed ball $\tilde{B}(x_0; r)$ is closed.

   SKETCH OF SOLUTION:
   Similar to part (a), but apply to the complement of the ball. Re-arrange the "plain" triangle inequality $d(a, c) \leq d(a, b) + d(b, c)$ into this form:

   $$d(a, b) \geq d(a, c) - d(b, c)$$

4. Let $(X, d)$ and $(Y, \tilde{d})$ be two metric spaces and let $f : X \to Y$ be a function. Show that

   (a) $f$ is continuous if and only if $f^{-1}(F)$ is closed in $X$ for any closed set $F$ in Y.

   SKETCH OF SOLUTION:
   Recall that $f$ is continuous iff $f^{-1}(U)$ is open (in $X$) for all open sets $U$ in $Y$. Show that $f^{-1}(F^c) = (f^{-1}(F))^c$.

   (b) $f$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subseteq $A \subseteq X$.

   SOLUTION:
   Let $f$ be continuous, and let $y \in f(\overline{A})$. Choose some $x \in \overline{A}$ such that $f(x) = y$ (note: $x$ may not be unique). Since $x \in \overline{A}$, there exists a sequence $(x_n) \subseteq A$ such that $x_n \to x$ (why? if $x \in A$, then choose $x_n = x$; if $x \notin A$, then $x$ is an accumulation point, and for every ball $B_n \equiv B(x; \frac{1}{n})$, choose $x_n \in B_n \bigcap A \neq \emptyset$). $f(x_n) \subseteq f(A)$ and $f$ is continuous, so $f(x_n) \to f(x)$, hence $f(x)$ is an accumulation point of $f(A)$ and thus $y = f(x) \in \overline{f(A)}$. So $f(\overline{A}) \subseteq \overline{f(A)}$.

   You can also prove this direction using part (a). Again, assume $f$ is continuous. Since $\overline{f(A)}$ is closed, then $f^{-1}(\overline{f(A)})$ is closed, as we proved in part (a). Note that

   $$A \subseteq f^{-1}(\overline{f(A)})$$

   so

   $$\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$$

   where the last equality follows since the set is closed. Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

Now for the second part of the proof. Assume $f(\overline{A}) \subseteq \overline{f(A)}$ for all sets $A \subseteq X$. Let $F$ be a closed set and consider $A = f^{-1}(F)$. So

$$f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(F))} \subseteq \overline{F} = F$$

$f(\overline{A}) \subseteq F$ implies $\overline{A} \subseteq f^{-1}(F) = A$, hence $A$ is closed (note: $F = f(f^{-1}(F))$ is not always true; for example, $F$ may not be wholly contained in the range of $f$). Thus the inverse image of a closed set is closed, and by part (a), $f$ is continuous. □

5. Let $Y$ be the subseteq of all continuous functions on $[a, b]$ such that $f(a) = f(b)$ and let

$$d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|.$$

Show that $(Y, d)$ is a complete metric space. (Hint: use a suitable theorem, direct proof is not required.)

SOLUTION:
We need only show that $Y$ is closed, since, as we saw in class, the space of continuous functions on $[a, b]$ with the above metric is complete. To show $Y$ is closed, it suffices to prove that $\overline{Y} \subseteq Y$. Let $f \in \overline{Y}$, so for any $\varepsilon > 0$, there is some $y \in Y$ such that $d(f, y) < \varepsilon$. Thus $|f(a) - y(a)| < \varepsilon$ and $|f(b) - y(b)| < \varepsilon$. Now, using the triangle inequality, we get

$$\begin{aligned} |f(a) - f(b)| &= |f(a) - y(a) + y(a) - f(b)| \\ &= |f(a) - y(a) + y(b) - f(b)| \\ &\leq |f(a) - y(a)| + |y(b) - f(b)| \\ &< 2\varepsilon \end{aligned}$$

and since $\varepsilon$ was arbitrary, we conclude $|f(a) - f(b)| \leq 0$, hence $f(a) = f(b)$ and thus $f \in Y$, so $\overline{Y} \subseteq Y$. □

6. Let $(X, d)$ be a metric space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that $x_n \to x$ and $y_n \to y$ where $x, y \in X$. Show that $d(x_n, y_n) \to d(x, y)$.

SOLUTION:
Given some $\varepsilon > 0$, we can find an integer $N$ such that for all $n > N$, $d(x, x_n) < \varepsilon/2$ and $d(y, y_n) < \varepsilon/2$. By applying the triangle inequality twice, we get

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, y) + d(y, y_n) \\ &\leq d(x_n, x) + d(x, y) + d(y, y_n) \\ &< d(x, y) + \varepsilon \end{aligned}$$

and a similar calculation shows $d(x, y) < d(x_n, y_n) + \varepsilon$ as well, hence $|d(x, y) - d(x_n, y_n)| < \varepsilon$ for any $\varepsilon > 0$, so

$$\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$$

□