

ACM 105: Problem set 1

Due: April 11, 2008

1. Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$. Prove that
 - (a) The open ball $B(x_0; r)$ is open.
 - (b) The closed ball $\tilde{B}(x_0; r)$ is closed.
2. Let (X, d) be a metric space and let $G \subset X$. Prove that G is open if and only if G is a union of open balls.

3. Let X be the set of all bounded real-valued functions defined on a set S and let

$$d(f, g) = \sup_{t \in S} |f(t) - g(t)|.$$

- (a) Verify that (X, d) is a metric space.
 - (b) Prove that (X, d) is a complete metric space.
4. Let (X, d) and (Y, \tilde{d}) be two metric spaces and let $f : X \rightarrow Y$ be a function. Show that
 - (a) f is continuous if and only if $f^{-1}(F)$ is closed in X for any closed set F in Y .
 - (b) f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all subset $A \subset X$.

5. Let Y be the subset of all continuous functions on $[a, b]$ such that $f(a) = f(b)$ and let

$$d(f, g) = \max_{t \in [a, b]} |f(t) - g(t)|.$$

Show that (Y, d) is a complete metric space. (Hint: use a suitable theorem, direct proof is not required.)

6. Let (X, d) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \rightarrow x$ and $y_n \rightarrow y$ where $x, y \in X$. Show that $d(x_n, y_n) \rightarrow d(x, y)$.