ACM 105: Midterm Exam
Solutions

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1. Let $X = C^\alpha[a,b]$, $0 < \alpha < 1$, be the space of all continuous functions defined on the closed interval $[a,b]$ with the following property: for each $x \in X$, there is a constant L_x such that

$$|x(t) - x(s)| \le L_x |t-s|^\alpha, \quad \forall\, t,s \in [a,b].$$

Note that X is a normed linear space with norm defined by

$$\|x\| = \max_{t\in[a,b]} |x(t)| + \max_{s,t\in[a,b]} \frac{|x(t)-x(s)|}{|t-s|^\alpha}.$$

Show that X is complete.

SOLUTION:

First, some notation and setup. Let's relabel the above norm to $\|\cdot\|_\alpha$ to avoid confusion with the standard norm on $C[a,b]$. Let the standard norm on $C[a,b]$ be $\|\cdot\|_\infty$, i.e. $\|f\|_\infty = \max_t |x(t)|$. Note that $X \subset C[a,b]$, although we must be careful due to the different norms. By definition, if $f \in X$, then $\|f\|_\alpha = \|f\|_\infty + \max \frac{|f(t)-f(s)|}{|t-s|^\alpha}$.

Now, suppose we have a Cauchy sequence $(x_n) \subset X$. Then (x_n) is also a Cauchy sequence in $C[a,b]$ with the $\|\cdot\|_\infty$ norm, since $\|x_n - x_m\|_\infty \le \|x_n - x_m\|_\alpha$. We know from class that $C[a,b]$ is complete, so there is some continuous function x such that $\lim_n \|x - x_n\|_\infty = 0$. It remains to show that $x \in X$ and that $\lim_n \|x - x_n\|_\alpha = 0$.

We first show $\|x - x_n\|_\alpha \to \infty$. Fix $\epsilon > 0$, and since (x_n) is Cauchy, we can find N such that

$$\|x_n - x_m\|_\infty + \max_{s,t} \frac{(x_m(t) - x_n(t)) - (x_m(s) - x_n(s))}{|t-s|^\alpha} < \epsilon$$

for all $m, n > N$. In particular, the expression holds for any fixed s and t. Since $x_n \to x$ pointwise, we have

$$\lim_{m\to\infty} \Big(\|x_m - x_n\|_\infty + \frac{(x_m(t) - x_n(t)) - (x_m(s) - x_n(s))}{|t-s|^\alpha} \Big) = \|x - x_n\|_\infty + \frac{(x(t) - x_n(t)) - (x(s) - x_n(s))}{|t-s|^\alpha} < \epsilon$$

and since this holds for any s and t, it also holds for the max. Hence $\|x - x_n\|_\alpha < \epsilon$ for all $n > N$.

We now claim $x \in X$. Because $\|x - x_n\|_\alpha \to \infty$, we can find some n such that $\|x - x_n\|_\infty < \epsilon$, hence, for any s and t,

$$\frac{|\,(x_n(t) - x(t)) - (x_n(s) - x(s))\,|}{|t-s|^\alpha} \le \|x_n - x\|_\infty + \max_{s,t} \frac{|\,(x_n(t) - x(t)) - (x_n(s) - x(s))\,|}{|t-s|^\alpha} < \epsilon$$

and hence

$$\begin{aligned} |x(t) - x(s)| &\le |\,(x_n(t) - x(t)) - (x_n(s) - x(s))\,| + |x_n(t) - x_n(s)| \\ &\le \epsilon |t-s|^\alpha + L_{x_n} |t-s|^\alpha \\ &= \tilde{L} |t-s|^\alpha \end{aligned}$$

Hence $x \in X$ since we had already proved that x is continuous. □

Note that the space $C^\alpha[a,b]$ is called a Hölder space, or a space of Hölder continuous functions. It is a generalization of a Lipschitz space.

2. Let X be a normed linear space and let $f \neq 0$ be a bounded linear functional on X. Set

$$d = \inf\{\, \|x\| \;:\; f(x) = 1\,\}.$$

Prove that $\|f\| = 1/d$.

SOLUTION:

Let $S = \{x \in X \;:\; f(x) = 1\}$. Then $d = \inf_{x \in S} \|x\|$. Note that $0 \notin S$ since f is linear. Thus

$$\|f\| = \sup_{x \neq 0} \frac{f(x)}{\|x\|} \geq \sup_{x \in S} \frac{f(x)}{\|x\|} = \sup_{x \in S} \frac{1}{\|x\|} = \frac{1}{d}.$$

To prove the reverse inequality, we let $\epsilon > 0$. Then there is $x \in X$ with $x \neq 0$ such that

$$\frac{|f(x)|}{\|x\|} > \|f\| - \epsilon.$$

Assume $f(x) \neq 0$, otherwise the proof is trivial. Write $y = f(x)^{-1}x$. Then $f(y) = 1$ and

$$\frac{1}{\|y\|} > \|f\| - \epsilon.$$

Thus,

$$\frac{1}{d} = \sup_{y \in S} \frac{1}{\|y\|} > \|f\| - \epsilon.$$

This shows that $1/d \geq \|f\|$. ☐

3. Let X be a normed linear space over a field K and let $\{x_1, x_2, \cdots, x_n\} \subset X$ be a linearly independent subset of X. Prove that there exist bounded linear functionals $f_1, f_2, \cdots, f_n$ on X such that

$$f_i(x_j) = 1, \quad i = j; \quad f_i(x_j) = 0, \quad i \neq j.$$

(Hint: if $\{x_1, x_2, \cdots, x_n\}$ is linearly independent, then there is a constant $C > 0$ such that

$$|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| \leq C\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|$$

for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$.)

SOLUTION:

We let $M = \operatorname{span}\{x_1, x_2, \cdots, x_n\}$. For any $x \in M$, we have $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. For each i, we define

$$f_i(x) = \alpha_i, \quad x \in M.$$

Then f_i is a linear functional on M. Moreover, it is bounded since

$$|f_i(x)| = |\alpha_i| \leq C\|x\|$$

by the given inequality. Also, we have

$$f_i(x_i) = 1; \quad f_i(x_j) = 0, \quad j \neq i.$$

Finally, by the Hahn-Banach theorem, we can extend f_i to X. This finishes the proof. ☐

A proof of the hint can be found in Lemma 2.4-1 in Kreyszig.

4. Let X be a normed linear space and let $\{x_n\}$ be a sequence in X. Suppose that x_n converges weakly to $x \in X$. Show that
$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

SOLUTION:

If $x = 0$, then $\|x\| = 0 \leq \liminf \|x_n\|$. If $x \neq 0$, then a corollary of the Hahn-Banach Theorem proves that $\exists f \in X'$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$. Since $x_n \rightharpoonup x$,

$$\|x\| = f(x) = \lim_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f(x_n)$$

since $\lim a_n = \limsup a_n$ if $\{a_n\}$ is convergent. Since f is linear and bounded and $\|f\| = 1$, we have $\|f(x_n)\| \leq \|f\| \cdot \|x\| = \|x\|$, and hence

$$\|x\| = \liminf_{n\to\infty} f(x_n) \leq \liminf_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} \|x_n\|.$$

□

Due to P. Mullen. Why do we use $\liminf \|x_n\|$ and not just $\lim \|x_n\|$? We know $\liminf \|x_n\|$ exists because $\|x_n\| > 0$ (and $\limsup \|x_n\|$ also exists because $\{\|x_n\|\}$ is bounded - see Lemma 4.8-3 in Kreyszig). However, $\lim \|x_n\|$ need not exist, so we cannot talk about it. Also, note that both
$$\|x\| \geq \limsup_{n\to\infty} \|x_n\|.$$
and (if it exists)
$$\|x\| = \lim_{n\to\infty} \|x_n\|$$
are *not* true in general.

EXAMPLES of weak convergence:

Weak convergence makes more sense if you can find an example of a sequence that converges weakly but not strongly. Unfortunately, the most accessible examples involve L_p spaces, which have not yet been defined in class.

Consider $u_n(x) = \sin(nx)$ on $I = [0, 2\pi]$. Then $\{u_n\} \subset L_2(I)$, where L_2 can be roughly thought of as the set of functions f such that

$$\int_I |f(x)|^2 dx < \infty$$

It turns out that the dual space of L_2 is itself (L_2 is a Hilbert space; L_p, for $p \neq 2$, is not), in the sense that all bounded linear functionals g take the form

$$g(f) = \int_I f(x) \cdot g(x) dx$$

In this sense, $u_n \rightharpoonup u$ where u is the constant zero function, and hence $\|u\| = 0$, where the norm in $L_2(I)$ is
$$\|f\|^2 = \int_I |f(x)|^2 dx$$
However, $\forall n$, $\|u_n\| = \sqrt{\pi}$ and hence

$$0 = \|u\| < \liminf_{n\to\infty} \|u_n\| = \sqrt{\pi}.$$