

# ACM 105: Midterm Exam

Due: May 7, 2008; 5pm

1. Let  $X = C^\alpha[a, b]$ ,  $0 < \alpha < 1$ , be the space of all continuous functions defined on the closed interval  $[a, b]$  with the following property: for each  $x \in X$ , there is a constant  $L_x$  such that

$$|x(t) - x(s)| \leq L_x |t - s|^\alpha, \quad \forall t, s \in [a, b].$$

Note that  $X$  is a normed linear space with norm defined by

$$\|x\| = \max_{t \in [a, b]} |x(t)| + \max_{s, t \in [a, b]} \frac{|x(t) - x(s)|}{|t - s|^\alpha}.$$

Show that  $X$  is complete.

2. Let  $X$  be a normed linear space and let  $f \neq 0$  be a bounded linear functional on  $X$ . Set

$$d = \inf\{\|x\| : f(x) = 1\}.$$

Prove that  $\|f\| = 1/d$ .

3. Let  $X$  be a normed linear space over a field  $K$  and let  $\{x_1, x_2, \dots, x_n\} \subset X$  be a linearly independent subset of  $X$ . Prove that there exist bounded linear functionals  $f_1, f_2, \dots, f_n$  on  $X$  such that

$$f_i(x_j) = 1, \quad i = j; \quad f_i(x_j) = 0, \quad i \neq j.$$

(Hint: if  $\{x_1, x_2, \dots, x_n\}$  is linearly independent, then there is a constant  $C > 0$  such that

$$|\alpha_1| + |\alpha_2| + \dots + |\alpha_n| \leq C \|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\|$$

for all  $\alpha_1, \alpha_2, \dots, \alpha_n \in K$ .)

4. Let  $X$  be a normed linear space and let  $\{x_n\}$  be a sequence in  $X$ . Suppose that  $x_n$  converges weakly to  $x \in X$ . Show that

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$