ACM 105: Final Exam

Due: June 5, 2008 (Thur), 5pm

1. Suppose that $|E|_e < +\infty$. Prove that E is measurable if and only if for any $\epsilon > 0$, we have $E = (S \cup N_1) - N_2$, where S is a finite union of non-overlapping intervals, $|N_1|_e < \epsilon$ and $|N_2|_e < \epsilon$.

2. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E. Assume that $f_k \to f$ a.e. in E and there is $g \in L(E)$ such that $|f_k| \leq g$. Show that, for any $\epsilon > 0$, there is a measurable set $F \subset E$ such that $|E - F| < \epsilon$ and $\{f_k\}$ converges to f uniformly on F.

3. Find the following limit

$$\lim_{k\to\infty} \int_0^1 \frac{1 + kx^2}{(1 + x^2)^k}\, dx.$$

   Justify your answer.

4. (a) Let $0 < p < q < r \leq \infty$. Show that $L^p \cap L^r \subset L^q$ and

$$\|f\|_q \leq \|f\|_p^{\lambda}\, \|f\|_r^{1-\lambda},$$

   where $\lambda \in (0, 1)$ is defined by

$$\lambda = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}.$$

   (b) Let $f \in L^p \cap L^\infty$ for some finite p. Show that $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

5. (a) Let $f \in L(E)$ where E is measurable. Let $\epsilon > 0$. Prove that there exists a simple function $\phi = \sum_{i=1}^N a_i \chi_{E_i}$ such that each E_i is bounded and

$$\int_E |f - \phi| < \epsilon.$$

   (b) Prove the Riemann-Lebesgue lemma: for any $f \in L(\mathbb{R})$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}} f(x) \sin(2k\pi x)\, dx = 0.$$