ACM 101a November 14, 2007

# Problem Set VI

1.- The form of the differential equation

$$y'' + \left(1 + \frac{1}{x^2}\right) y = 0$$

suggest that one could set $y \sim e^{\pm ix}$ for large real values of $x$. Assume an asymptotic expansion of the form

$$y \sim e^{\pm ix} \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\right)$$

and determine the coefficients by substitution into the differential equation.

2.- Use Laplace's method to obtain an asymtpotic expansion valid for $x \to \infty$ of the complementary error function

$$\mathit{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-2tx} e^{-t^2}\, dt,$$

and compare your result with that obtained in class by means of integration by parts. (Note the factor $2/\sqrt{\pi}$ used here which is not included in the convention used in class.) Compare the asymtpotic results with the exact values $\mathit{erfc}(2) = 0.004677735...$ and $\mathit{erfc}(4) = 0.00000\ 00154\ 173....$ Is the asymptotic expansion convergent? [Hints: Note that the function $t$ in the exponent of the second integral above has its maximum at an endpoint of the integration interval. Expand $\exp(-t^2)$ in a power series around the origin.]

3.- Find the leading behavior of

$$\int_0^{2\pi} (1 + t^2) e^{x \cos(t)} dt$$

as $x \to \infty$. Note that two maxima contribute to this leading behavior.

4.- Consider the integral

$$f(x) = \int_a^b e^{xh(t)} q(t)\, dt$$

where the function $h$ is real-valued and where $h$ and $q$ are assumed to be differentiable as many times as needed in the analysis. Assume, further, that $h(t)$ attains an absolute maximum at the interior point $t_0$ for which we have $h''(t_0) < 0$. Show that the first term in the asymptotic expansion of $f$ as $x \to \infty$ is

$$q(t_0) e^{xh(t_0)} \left[\frac{-2\pi}{xh''(t_0)}\right]^{1/2}$$

and write out the next few terms.

Due November 21 at noon.