O. Bruno **ACM 101a Final Exam** December 4, 2006

1. Due on **Friday December 8th at noon**.
2. The honor code is in effect.
3. There is no time limit for this work—work at your own pace.
4. The only materials that may be used are the Ablowitz and Fokas and the Carrier, Krook and Pearson texts, course notes of each student's handwriting, problem sets and graded quizes.
5. Use of computers, web searches and calculators is not permitted.
6. Discussion of the exam problems is not permitted until noon on 12/8.
7. The exam consists of five questions.
8. Use standard blue books. Please, write clearly your name on the front of the book.
9. Return the completed exam to Sheila Shull (217 Firestone) by noon on Friday December 8.

1. (20 Points.) Evaluate

$$\int_0^\infty \frac{dx}{1+x^{300}}.$$

2. (30 Points.)

(a) (10 Points.) Expand $1/\sin(z)$ in powers of z for $0 < |z| < \pi$ and also for $\pi < |z| < 2\pi$. Compute a few terms in the expansion explicitly, and give a recursive relation for the general coefficient.

(b) (20 Points.) In what annular regions centered on $-i$ could expansions in powers of $z+i$ be obtained for each of the following functions: $(z^2-1)^{1/2}$, $(z^2+1)^{1/2}$, $\log\left((z+1)/(z-1)\right)$, $[z(z-i)(z^2-1)]^{1/2}$? Obtain an expansion for the last of these that is valid for all sufficiently large values of z.

3. (20 Points.) Find a pair of independent solutions of the equation

$$(2z+z^2)w'' + 2(1+z)w' - \nu(\nu+1)w = 0$$

around $z = 0$.

4. (20 Points.) Let $I(x) = \int_x^\infty e^{-t^4} dt$.

(a) (5 Points.) Show that $I(0) = \Gamma(5/4)$. Obtain an asymptotic expansion for $I(x)$ around $x = 0$.

(b) (15 Points.) Use integration by parts to find an asymptotic expansion for $I(x)$ valid as $x \to +\infty$.

5. (10 Points.) Show that

$$\int_0^\infty \log\left(\frac{u}{1-e^{-u}}\right)\frac{e^{-ku}}{u}du \sim \frac{1}{2k} \quad \text{as} \quad k \to +\infty.$$