Anomalous diffusion and Homogenization on an infinite number of scales

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— To my parents Rezvan and Elaheh, my brother Ramine and to those who have been deprived from the right to study —

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ABSTRACT

It is now well known that natural Brownian Motions on various disordered or complex structures are anomalously slow, and that convection in a turbulent flow can create anomalously fast diffusion. In this work we try to understand the basic mechanisms of anomalous diffusion using and developing the tools of homogenisation. These mechanisms of the slow diffusion for instance are well understood for very regular strictly self-similar fractals. The archetypical specific example of a deep problem being the one solved by Barlow and Bass on the Sierpinski Carpet (which is infinitely ramified, a codeword for hard to understand rigorously). It appears that the main feature is the existence of an infinite number of scales of obstacle (with proper size) for the diffusion. We can show that one can implement the common idea that this last feature (infinitely many scales) is the key for the possibility of anomalous diffusion, fast and slow, in a general context, using the tools of homogenisation.

Résumé

Il est maintenant bien connu que des mouvements Browniens naturels sur diverses structures complexes et désordonnées sont anormalement lentes, et qu'une convection dans un écoulement turbulent peut créer une diffusion anormalement rapide. Dans ce travail, nous essayons de comprendre les mécanismes fondamentaux des diffusions anormales en utilisant et développant les outils de l'homogénéisation. Ces mécanismes, pour les diffusions lentes par exemple, sont bien comprises pour des Fractals très régulier et self-similaires. L'exemple spécifique archétype d'un profond problème étant celui résolu par Barlow et Bass sur le Tapis de Sierpinski (qui est infiniment ramifié, un nom de code signifiant difficile à comprendre rigoureusement). Il apparaît que la caractéristique essentielle est l'existence d'un nombre infini d'échelles d'obstacles (avec des tailles convenables) pour la diffusion. On peut montrer que l'on peut implémenter l'idée commune que cette dernière caractéristique (un nombre infini d'échelles) est la clé pour la possibilité d'une diffusion anormale, rapide et lente, dans un contexte général, en utilisant les outils de l'homogénéisation.

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0. INTRODUCTION

One might think that homogenization has as much to do with rigorous proofs of anomalous diffusion as with the price of cheese. Indeed to expect that a sequence of central limit theorems would cause the anomaly of a diffusion might not be a natural idea and in fact this work did not started with the purpose of using homogenization theory to prove the anomaly of a diffusion evolving in an infinite number of scales.

0.1 Origin of this work

In a series of six articles M.T. Barlow and R. Bass [BB89], [BB90a], [BB90b], [BB92], [BB93a], [BB97] have constructed a reflecting Brownian motion on the Sierpinski carpet and extended their work to the Sierpinski sponge; this work was a breakthrough since it was the first rigorous and complete analysis of Brownian motion constructed on an infinitely ramified fractal (codeword for hard to analyze). One of the interesting behavior of the Brownian motion on the Sierpinski sponge was its sub-diffusive behavior.
This work started on the basis of the following idea: the Brownian motion on the Sierpinski carpet is sub-diffusive, not because of the particular geometry of that object, but because it encounters obstacles at every scales (each scale manifesting its influence at a specific moment).
Thus the scientific objective was to analyze and understand the behavior of a diffusion evolving in a medium characterized by an infinite number of scales that are not self symmetric and have no symmetries. Of course, the initial quest was to find such a model (or a medium) presenting no symmetries and no self similarity and characterized by an infinite number of scales of reflecting obstacles; construct on it a Brownian motion, and prove its sub-diffusive behavior, this would lead to the proof of the following universal result "if a diffusion encounters obstacles at every scales then it becomes sub-diffusive".

Now a close look at the problem showed that only homogenization theory could handle obstacles without symmetries and it was a well known fact that homogenization on a periodic potential drift has the property to decrease the diffusivity. This was an interesting path to explore and reflecting obstacles were replaced by a smooth drift that allowed to use Ito calculus without wondering about the irregularities of the boundaries of some hard obstacles (it was also a way to avoid some pathologies appearing with hard obstacles without symmetries). Then the next natural idea was what happens with a potential drift characterized by a two periods of fluctuations? a short analysis showed that the diffusivity of the Brownian motion would decrease the product of the effective diffusivities associated to each period if the ratios between them were big enough. This was quite interesting because it meant that if homogenization on a periodic potential U generates a Brownian process with effective diffusivity aI_d ($a < 1$), homogenization ever n scales of U (with a large ratio separating them) would give a Brownian process with effective diffusivity $a^n I_d$, then what would happen with an infinite number of scales?
A short heuristic analysis pointed out that it would generate the anomaly of the diffusion, the key was the geometric decrease of the effective diffusivities with the number of scales and the fact that with an infinite number of scales homogenization is never finite. Moreover for a given time t one could easily separate the medium into effective scales (smaller ones) and drift scales (larger ones) and

the number of effective scales were growing with the time leading to the sub-diffusivity of the process. The model to explore were found, nevertheless a closer look showed that to obtain interesting results one had to improve some tools used in homogenization theory.
Homogenization theory gives only an asymptotic image in the sense that it gives an exact result only when the ratio between scales grows towards infinity. Here there were an infinite number of scales and the ratios between them were bounded. Thus it was necessary to find a way to do computations over an infinite number of scales with bounded ratios and to obtain sharp estimates of the rate of convergence towards the asymptotic process in homogenization theory.
An other difficulty appeared, and this one was inherent to the generality of the model, without a priori knowledge on the shapes of the fluctuations some intermediate scales can not be considered as drift scales or effective scales, one had to control their influence. This control is in a sense equivalent to compare the Green functions associated to two different elliptic operators (non isotropic and non homogeneous in the space) and it goes beyond conventional knowledge on the subject that concerns mainly comparison with the Laplace operator.

0.2 Realizations

Sub-diffusivity is proven in all dimensions $d \geq 1$ starting from the invariant measure with full generality and starting from any points for $d = 1$. The proof of the sub-diffusivity starting from any points for $d \geq 2$ depends on a stability condition of the multi-scale process.
To obtain those results, one is lead to develop the tools of homogenization theory. Indeed in reiterated homogenization, or DEM theories, in order to obtain estimates on the effective diffusivity associated to the multi-scale medium one has to assume that the ratios between those scales goes to infinity with their numbers. Here those estimates will be obtained, with an arbitrarily large number of scales and with bounded ratios between them.
Moreover, one has to obtain sharp estimates on the behavior of the heat kernel associated to a periodic operator. This question is directly liked to Davies conjecture (see section 5.3) on the rate of convergence of diffusion in a periodic medium towards its limit process. An answer will be given to this conjecture by showing that the homogenized behavior of the heat kernel $p(t,x,y)$ associated to a elliptic periodic operator starts for $t >> |x-y|$.

To replace the universal sentence "a diffusion becomes anomalous because it encounters obstacles at every scales" by "a diffusion becomes anomalous because homogenization operates on an infinite number of scales" is not a tautology. Indeed the word obstacles suggests a symmetric diffusion (with an associated Dirichlet form) whose generator is characterized by Neumann conditions on the boundaries of some reflecting obstacles whereas homogenization can operate on a very large variety of generators which can be non symmetric.
Moreover it was a well known fact that homogenization on a divergence-free drift has the property to enhance the diffusion, thus the natural idea that follows the study of a Brownian motion evolving in a potential drift characterized by an infinite numbers of fluctuations is to study the Brownian motion evolving in a divergence-free drift characterized by an infinite numbers of fluctuations and it is natural to expect that a super-diffusive behavior will arise. The precise super-diffusive behavior at this stage is rigorously proven in the shear flow model (with the techniques developed with potential diffusions).

0.3 Perspectives

The proof of the sub-diffusive behavior in the medium characterized by an infinite number of scales of potential drifts and starting from any point in dimension one is based on a new analytical inequality: let Ω be a smooth open bounded subset of $\mathbb{R}^1$ and $A \in C^\infty(\bar{\Omega})$, $A > 0$ on $\bar{\Omega}$ then

Theorem 0.3.1. *(d=1) For all ψ and ϕ sub harmonic with respect to the operator $-\nabla(A\nabla)$ and null on the boundary of Ω one has*

$$\int_\Omega |\nabla\psi A\nabla\phi| dx \leq 3 \int_\Omega \nabla\psi A\nabla\phi dx \tag{0.1}$$

We believe that this theorem might also be also true in higher dimensions and it appears as an interesting path to explore. Indeed its proof would give the anomaly of a diffusion starting from any point, moreover it has also strong consequences on the Green functions associated to elliptic operators and an interesting signification in terms of electrostatic theory (see the chapter 13).

An other path that we would like to explore is the extension of those results to the case where at each scale, the medium is not periodic but ergodic.

Next the study of super-diffusivity in the divergence-free case with full generality in dimension $d \geq 2$ appears as an interesting work to undertake.

0.4 History

Actually the more this work was progressing the more it was becoming clear that the idea to associate homogenization (or renormalization) on large number of scales with the anomaly of a physical system had already been applied on an heuristic point of view to several physical models.
May be one of the oldest one is Differential Effective Medium theories which was first proposed by Bruggeman to calculate the conductivity of a two-component composite structure formed by successive substitutions ([Bru35] and [AIP77]) and generalized by Norris ([Nor85]) to materials with more than two phases. For instance this theory has been applied to compute the anomalous electrical and acoustic properties of fluid-saturated sedimentary rocks [SSC81]. More recently this problem has been analyzed from a rigorous point of view by Avellaneda [Ave87] and Kozlov [Koz95]; by Allaire, Briane [AB96] and Jikov, Kozlov [JK99].
The heuristic application of this idea to prove the anomalous behavior of a diffusion seems to have been done only for the super-diffusive case that is to say for a diffusion evolving among a large number of divergence-free drifts. May be this is explained by the strong motivation to explore convective transports in turbulent flows which are known to be characterized by a large number of scales of eddies. The first observation was empirical: in 1926 when Richardson ([Ric26]) analyzed available experimental data on diffusion in air. Those data varied about 12 orders of magnitude. On that basis, Richardson phenomenologically conjectured that the diffusion coefficient D_λ in turbulent air depend on the scale length λ of the measurement. The Richardson law,

$$D_\lambda \propto \lambda^{\frac{4}{3}} \tag{0.2}$$

was related to Kolmogorov-Obukhov turbulence spectrum, $v \propto \lambda^{\frac{1}{3}}$, by Batchelor [Bat52]. The super-diffusive law of the root-mean-square relative displacement $\lambda(t)$ of advected particles

$$\lambda(t) \propto (D_{\lambda(t)} t)^{\frac{1}{2}} \propto t^{\frac{3}{2}} \tag{0.3}$$

was derived by Obukhov [Obu41] from a dimensional analysis similar to the one that led Kolmogorov [Kol41b] to the $\lambda^{\frac{1}{3}}$ velocity spectrum.
More recently physicists and mathematicians have started to investigate on the super-diffusive phenomenon (from both heuristic and rigorous point of view) by using the tools of homogenization or renormalization (the first cousin of multi-scale homogenization): M. Avellaneda and A. Majda [AM90]; J. Glimm and Al. [FGLP90], [FGL+91], [GLPP92], J. Glimm and Q. Zhang [GZ92], Q. Zhang [Zha92], M.B. Isichenko and J. Kalda [IK91], A. Fannjiang and G.C. Papanicolaou [FP94],[FP96]; M. Avellaneda [Ave96]; A. Fannjiang [Fan99]; Rabi Bhattacharya [Bha99] (see

also [BDG99] by Bhattacharya - Denker and Goswami).
Certainly this panorama is not complete, it reflects only the limited knowledge of the author who apologize if someone feels left over.

0.5 Map of the work

This work is divided into four parts,

- The first one gives our models and the main results. The chapter 1 presents the sub-diffusive model and the chapter 2 the super-diffusive one. For the clarity of the presentation, all the results are not present; they are not given with full generality and without any comments (this will be done in the part three). It is also important to note that the sub-diffusive model has been more deeply analyzed that the super-diffusive one. The complete investigation of the super-diffusive one is postponed to a sequel work.

- The second one is bibliographical. The chapter 3 presents the state of the art in the study of normal diffusions, it is also an introduction to the techniques of comparison between operators (the reference operator being Laplace operator or a Gaussian diffusion), this point is important because to be able prove something for a diffusion evolving in a medium characterized by an infinite number of scales, one must be able to compare elliptic operators (and the reference point is no more the Laplace operator, but actually those territories seems virgin for exploration). The chapter 4 presents a landscape on anomalous diffusion and focus mainly on diffusions in fractals. A short presentation of super-diffusion in Turbulence is given. The chapter 5 is an introduction to the tools of homogenization and multi-scale homogenization (both rigorous and heuristic such as DEM theories).
This second part is certainly not a complete survey giving all the contributions in those fields; it has been conceived only to help the reader non familiar with them to enter quickly into the subject.

- The third one has been conceived to explain our models, give the though process (the strategy) an insight on the proofs and the significance of the results. It also contains some results that will not be given in the first part because the clarity of the presentation has been privileged. The chapter 6 present the sub-diffusive model and the chapter 7 the super-diffusive one. It is advised to read this part before reading the proofs.

- The fourth one contains the proofs and the new tools. The chapter 8 gives the proofs of sub-diffusivity in dimension one. The chapter 9 presents the proofs of the control of the effective diffusivities associated to a multi-scale medium with an arbitrary large number of scales (with bounded ratios). The chapter 10 presents the proofs of the sub-diffusive behavior in all dimensions. The chapter 11 gives the proofs of super-diffusivity in the shear-flow model. The chapter 12 gives the proofs of a new exponential martingale inequality, and concerning Davies conjecture on the behavior of the heat kernel $p(t, x, y)$ in a periodic medium, shows that its homogenized behavior starts for $t >> |x - y|$. The chapter 13 presents a new analytical inequality and shows how it is linked with the deformation of elliptic operators and the notion of localization of energy in electrostatics. In the appendix, one will find the chapter B (theorems on elliptic operators in divergence form with discontinuous coefficients, controls of the infinite norms and gradients of solutions associated to elliptic operators) which presents the analytical tools used in this work and in the chapter C one will find the probabilistic tools (level three large deviations, thermodynamic formalism, deformation of harmonic functions).

0.6 Applications

The first application of this work is clearly the problem of diffusion and transport in disordered media which is of an enormous physical interest by its diversity and the number of its applications. Indeed, equations of the following type (in R^2 or R^3):

$$\frac{\partial u(x,t)}{\partial t} = \kappa \Delta u + \xi(t,x)u + s(t,x) + \vec{h}(t,x).\nabla u$$

may be considered as universal (the term $\kappa\Delta u$ is a model of diffusive transport, $\xi(t,x)u$ is an interaction with the media, $s(t,x)$ is a source term and $\vec{h}(t,x).\nabla u$ represents a transport by convection). The examples are numerous and important [Bal92] :

- The propagation of heat in a solid or a fluid of non uniform temperature.
- The diffusion of neutrons in a nuclear reactor.
- The migration of impurities in a heated up solid (doping).
- The electrical current as a transfer of charged particles under the action of a spatial variation of the chemical potential (ions in a electrolyte, electrons in a metal, electrons of conduction and holes in a semi-conductor)
- Fluid mechanics with the Navier Stokes equations
- Magneto-hydrodynamics which concerns a large range of physical objects from liquid metals to cosmic plasma, such as the evolution of a magnetic field in a conducting media submitted to a random motion (this is also that kind of equation which governs the evolution of the temperature field in the coupled system: atmosphere + ocean)
- Burger's equation which is fundamental in hydrodynamics and in astrophysics since it describes self-gravitating matter where the attraction between the liquid particles replace the repulsion.
- The heat equation in a turbulent media (turbulent diffusion)

More precisely the techniques developed in this work can be applied and adapted to answer to the following question what happens when the medium is characterized by a large number of scales? For instance a physicist can see the sub-diffusive model, as a system whose potential energy landscape is characterized by a large (infinite) number of scales of potential pit (overlapping with each other) evolving with the thermal noise and under the propensity to minimize the energy (see figure 0.1).
The supper diffusive model can be seen as a model of turbulent flow.

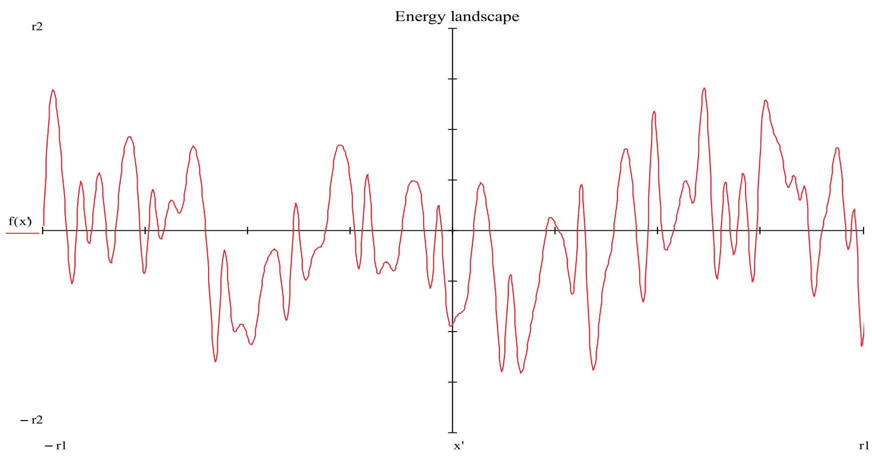

Fig. 0.1: *Multi-scale energy landscape*

Part I

OUR MODELS AND MAIN RESULTS

1. SUB-DIFFUSIVE MODEL

In this chapter the sub-diffusive model will be introduced and some of the main results will be given without any comments. For the clarity of the presentation all the results on the sub-diffusive model are not present here and they are not given in their full general form; this will be done in the chapter 6 (which also contains the meaning of those results).

1.1 Infinitely homogenized potential diffusion

1.1.1 The Model

The purpose of this model is to study in $\mathbb{R}^d$ the behavior of solutions the stochastic differential equation:

$$\begin{cases} dy_t = d\omega_t - \nabla V(y_t)dt \\ y_0 = 0 \end{cases} \tag{1.1}$$

(where ω is a standard Brownian motion in $\mathbb{R}^d$) when $V \in C^\infty(\mathbb{R}^d)$ is not bounded and has fluctuations over an infinite number of scales which are periodic but non symmetric and non self-similar. More precisely

$$V = \sum_{n=0}^{\infty} U_n(\frac{x}{R_n}) \tag{1.2}$$

Each potential U_n is smooth and periodic of period T_1^d (is in $\mathcal{C}^\infty(T_1^d)$, T_1^d is the torus of dimension d and side 1) and reflects the particular shape of the fluctuations (of the heat capacity) of U at the scale R_n.

Each integer $R_n \in \mathbb{N}^*$ reflects the length of the scale n and grows with n. More precisely

$$R_n = \prod_{k=0}^{n} r_k \tag{1.3}$$

Where r_n are integers element of $\mathbb{N}^*$ for $n \geq 1$. It is assumed that the small scale has length $r_0 = 1$. Moreover it is assumed that all the gradient of the potentials U_n are uniformly bounded. Thus there exist $K_0, K_1 > 0$ such that ($\text{Osc}(U)$ stands for $\sup U - \inf U$)

$$\sup_{n\in\mathbb{N}} \|\operatorname{Osc}(U_n)\|_\infty \leq K_0, \quad \sup_{n\in\mathbb{N}} \|\nabla U_n\|_\infty \leq K_1 \tag{1.4}$$

It is also assumed that there exist $\rho_{\min} \in \mathbb{R}_+^*$ and $\rho_{\max} \geq \rho_{\min}$, $\rho_{\min} \in \mathbb{R}_+^* \cup \{\infty\}$ such that

$$\forall n \geq 1 \quad 2 \leq \rho_{\min} \leq r_n \leq \rho_{\max} \leq \infty \tag{1.5}$$

Each potential U_n is chosen so that

$$\forall n \in \mathbb{N}, \quad U_n(0) = 0 \tag{1.6}$$

Thus V is a well defined Lipschitz potential such that

$$\|\nabla V\|_\infty \leq \sum_{n=0}^{\infty} \frac{K_1}{\rho_{\min}^n} \leq \frac{K_1 \rho_{\min}}{\rho_{\min} - 1} \tag{1.7}$$

$$|V(x)| \leq \frac{K_1 \rho_{\min}}{\rho_{\min} - 1} |x| \tag{1.8}$$

Thus it is well known that the solution of the stochastic differential equation 1.1 exists; is unique up to sets of measure 0 with respect to the Wiener measure and is a strong Markov continuous Feller process.

Effective diffusivities To each potential U_n is associated an invariant measure m_{U_n} by the equation 5.1 and an effective diffusivity $D(U_n)$ by the variational formulation 5.21. It is assumed that there exist $0 < \lambda_{\min} \leq \lambda_{\max} < 1$ such that for all $n \in \mathbb{N}$

$$0 < \lambda_{\min} \leq D(U_n) \leq \lambda_{\max} < 1 \tag{1.9}$$

Note that the effective diffusivity is a scale-invariant matrix that is to say an homogenization on the periodic potential $U(.)$ produce the same effective diffusivity as $U(\frac{.}{R_n})$ (this is an easy exercise).

Aggregation of scales In the sequel V_k^m will designate an aggregation of the scales $k, k+1, \ldots, m$ that is to say

$$V_k^m(x) = \sum_{n=k}^{m} U_n(\frac{x}{R_n}) \tag{1.10}$$

where $k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$
Note that since all the ratio r_n between scales are integers, the aggregation V_k^m is periodic (for $m < \infty$) of period $R_m \times T_1^d$. It is thus natural to relate this aggregation with an effective diffusivity written $D(V_k^m)$ associated to an homogenization on the medium V_k^m. It is important to note that since the effective diffusivity is scale invariant, all that is needed is to know that V_k^m is periodic and a rescaling of its period doesn't influence the effective diffusivity. That is to say

$$D(V_k^m(x)) = D(V_k^m(x.R_n)) \tag{1.11}$$

and since $V_k^m(x.R_n)$ is of period T_1^d the variational formulation 5.21 will naturally be used to define its effective diffusivity.

Infinitely Homogenized Potential Diffusion A solution of the stochastic differential equation 1.1 such that conditions 1.2, 1.5, 1.6, 1.7, 1.9 are satisfied will be called an Infinitely Homogenized Potential Diffusion with parameters: $\rho_{\min}$, $\rho_{\max}$, $\lambda_{\min}$, $\lambda_{\max}$, K_0, K_1 and written

$$IHPD(\rho_{\min}, \rho_{\max}, \lambda_{\min}, \lambda_{\max}, K_0, K_1) \tag{1.12}$$

The IHPD will be said self-similar if for all k, $U_k = U \in C^\infty(T_1^d)$ and $\rho_{\min} = \rho_{\max} = \rho = R$.

1.1.2 Some remarks

Note also that by the Voigt Reiss inequality and since homogenization on a periodic potential decrease the diffusivity, one has always $0 < D(U_n) \leq 1$. Note also that by the equivalent cell problem definition of the effective diffusivity, the condition $D(U_n) < 1$ is equivalent to the fact that $l.\nabla U_n$ is not the null function for all non null direction $l \in (\mathbb{R}^d)^*$. Thus if all the scales are associated to a finite number of pattern: $\forall n, U_n \in \{W_1, \ldots, W_d\}$ with non identically null gradient in all the directions, then the conditions 1.9 and 1.4 are trivially satisfied.

1.2 Anomalous behavior in dimension one

1.2.1 Exit times

1.2.1.i Self-similar case

The following corollary is the corollary 8.3.2 of chapter 8.

Corollary 1.2.1. *Let y_t be a self-similar infinitely homogenized potential diffusion in dimension one. Then*

$$\mathbb{E}_0[\tau(0,r)] = r^{2+\nu(r)} \tag{1.13}$$

with

$$\nu(r) = \frac{\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)}{\ln \rho} + \epsilon(r) \tag{1.14}$$

with $\epsilon(r) \to 0$ as $r \to \infty$.

Here $\mathcal{P}_\rho$ is the topological pressure associated to the shift s_ρ (see section C.1). The following theorem corresponds to the theorem 6.2.1.

Theorem 1.2.1. *For a self-similar IHPD in dimension one, if U is not a constant function, there exists a constant $\rho_0(K_1, D(U))$ such that for $\rho > \rho_0$,*

$$\mathbb{E}_0[\tau(0,r)] = r^{2+\nu+\epsilon(r)} \tag{1.15}$$

with $\nu > 0$ given by the topological pressure

$$\nu = \frac{\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)}{\ln \rho} \tag{1.16}$$

and $\epsilon(r) \to 0$ as $r \to \infty$. Moreover there are examples of U such that there exists ratios ρ_1, ρ_2 ($\rho_1 + 10 < \rho_2$) in the interval $(1, \rho_0]$ such that if $\rho = \rho_1$ or ρ_2 then $C_1 r^2 \leq \mathbb{E}[\tau(0,r)] \leq C_2 r^2$ and if $\rho \in (\rho_1, \rho_2) \cap \mathbb{N}$, $\mathbb{E}[\tau(0,r)]$ follows the anomalous behavior given in the equation 1.15 with $\nu > 0$ as above.

1.2.1.ii Non self-similar case with bounded ratios between the scales

The following theorem corresponds to the corollary 8.3.3 of chapter 8.

Theorem 1.2.2. *Let y_t be an infinitely homogenized potential diffusion such that, $\rho_{\min} > C_{3,K_0,K_1}$, $\rho_{\max} < \infty$ and $\lambda_{\max} < 1$. Then*

$$C_1 r^{2+\nu(r)} \leq \mathbb{E}_0[\tau(0,r)] \leq C_2 r^{2+\nu(r)} \tag{1.17}$$

where C_1, C_2 depends only on K_0, K_1 and $\rho_{\min}$ and

$$0 < -\frac{\ln \lambda_{\max}}{\ln \rho_{\max}} - \frac{C_{3,K_0,K_1}}{\rho_{\min} \ln \rho_{\max}} \leq \nu(r) \leq -\frac{\ln \lambda_{\min}}{\ln \rho_{\min}} + \frac{C_{3,K_0,K_1}}{\rho_{\min} \ln \rho_{\min}} \tag{1.18}$$

1.2.1.iii Non self-similar case with fast separation between the scales

The following theorem is the corollary 8.3.4 of chapter 8.3.4

Theorem 1.2.3. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$C_1 r^2 e^{g(r)} \leq \mathbb{E}_0[\tau(0,r)] \leq C_2 r^2 e^{g(r)} \tag{1.19}$$

where C_1, C_2 depends only on K_0, K_1, ρ, α and

$$g(r) = (\ln r)^{\frac{1}{\alpha}} \frac{\ln\left(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\right)}{(\ln \rho)^{\frac{1}{\alpha}}} \tag{1.20}$$

1.2.2 Mean squared displacement

1.2.2.i Bounded ratios between the scales

The following theorem corresponds to the theorem 8.5.3.

Theorem 1.2.4. *Assume $\lambda_{\max} < 1$, $\rho_{\min} > C_{1,K_1,K_0,\lambda_{\max}}$, $t > R_9$ and $\rho_{\max} < \infty$ then*

$$\mathbb{E}[y_t^2] = t^{1-\nu(t)} \tag{1.21}$$

$$\nu(t) \leq -\frac{\ln \lambda_{\min}}{2 \ln \rho_{\min}} + \frac{C_{2,K_1,K_0}}{(\ln \rho_{\min})^2} + \epsilon(t) \tag{1.22}$$

$$\nu(t) \geq -\frac{\ln \lambda_{\max}}{2 \ln \rho_{\max}} - \frac{C_{2,K_1,K_0}}{\ln \rho_{\min} \ln \rho_{\max}} - \epsilon(t) \tag{1.23}$$

where $\epsilon(t) \to 0$ as $t \to \infty$ and

$$-\frac{\ln \lambda_{\max}}{2 \ln \rho_{\max}} - \frac{C_{2,K_1,K_0}}{\ln \rho_{\min} \ln \rho_{\max}} > 0 \tag{1.24}$$

1.2.2.ii Fast separation between the scales

The following theorem corresponds to the theorem 8.5.4

Theorem 1.2.5. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$C_1 t e^{-g(t)} \leq \mathbb{E}_0[y_t^2] \leq C_2 t e^{-g(t)} \tag{1.25}$$

where C_1, C_2 depends only on K_0, K_1, ρ, α and

$$g(t) = (\ln t)^{\frac{1}{\alpha}} \frac{\ln\left(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\right)}{(2\ln \rho)^{\frac{1}{\alpha}}}(1+\epsilon(t)) \tag{1.26}$$

with $\epsilon(t) \to 0$ as $t \to \infty$.

1.2.3 Heat kernel tail

1.2.3.i Bounded ratios between the scales

The following theorem corresponds to the theorem 8.6.2

Theorem 1.2.6. *Assume* $\rho_{\max} < \infty$, $\lambda_{\max} < 1$,
$\rho_{\min} > C_{16}$

$$\frac{h^2}{t} \geq C_{11}(\frac{t}{h})^{\frac{\ln \lambda_{\max}}{2 \ln \rho_{\max}} + \frac{C_{12}}{(\ln \rho_{\min})^2}} \tag{1.27}$$

and $(h > 0)$

$$\frac{t}{h} \geq C_{13} \tag{1.28}$$

then for $l \in \mathbb{S}^d$

$$\mathbb{P}[l.y_t \geq h] \leq C_{14} e^{-C_{15} \frac{h^2}{t} (\frac{t}{h})^{\nu}} \tag{1.29}$$

with

$$\nu = -\frac{\ln \lambda_{\max}}{\ln \rho_{\max}} - \frac{C_6}{\ln \rho_{\min} \ln \rho_{\max}} > 0 \tag{1.30}$$

Where C_{16}, C_{15} *depend on* $K_0, K_1, \rho_{\min}, \rho_{\max}, \lambda_{\max}$*;* C_{11} *depends on* K_0, K_1, $\rho_{\max}, \rho_{\min}$*;* C_{13} *on* K_0, K_1, R_2 *and* C_6, C_{12} *on* K_0, K_1

1.2.3.ii Fast separation between the scales

The following theorem corresponds to the theorem 8.6.3

Theorem 1.2.7. *Assume that for all* k, $U_k = U$ *(*U *non constant) and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ *then for*

$$C_1 < \frac{t}{h} < C_2 h \tag{1.31}$$

one has

$$\mathbb{P}[l.y_t \geq h] \leq C_3 e^{-C_4 \frac{h^2}{t} g(\frac{t}{h})} \tag{1.32}$$

with

$$g(x) = (\frac{1}{\lambda})^{(\frac{x}{\ln \rho})^{\frac{1}{\alpha}}(1+\epsilon(x))} \tag{1.33}$$

and $\epsilon(x) \to 0$ *as* $x \to \infty$
Where the constants C_1, C_2 *depend on* ρ, α, K_0, K_1 *and* C_4 *on* ρ, K_0, K_1, λ.

1.3 Anomalous behavior in all dimensions

1.3.1 Exit times starting from the invariant measure

1.3.1.i Bounded ratios between scales

For a smooth bounded open subset Ω of $\mathbb{R}^d$, let's write $\tau(\Omega)$ the exit time from Ω and m_U^Ω the following probability measure on Ω:

$$m_U^\Omega(dx) = \frac{e^{-2U(x)}\,dx}{\int_\Omega e^{-2U(x)}\,dx} \tag{1.34}$$

The following theorem corresponds to the theorem 10.1.1.

Theorem 1.3.1. *One has for $r > C_{16}$,*

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) = r^{2+\nu(r)} \tag{1.35}$$

with for $\rho_{\min} > C_{13}$

$$\nu(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}\big(1 + \frac{C_7}{\ln \rho_{\min}}\big) + \frac{1}{\ln r} C_6 \tag{1.36}$$

and

$$\nu(r) \geq \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}\big(1 - \frac{C_{12}}{\ln \rho_{\min}}\big) - \frac{1}{\ln r} C_{11} > C_{15} > 0 \tag{1.37}$$

Where the constants C_{11}, C_{12}, C_7, C_6 depends on d, K_0, K_1; C_{13} on d, K_0, K_1, $\lambda_{\max}$ and C_{15}, C_{16} on $d, K_0, K_1, \lambda_{\max}, \rho_{\max}$

1.3.1.ii Unbounded ratios between scales

the following theorem corresponds to the theorem 10.1.2.

Theorem 1.3.2. *Assume that $R_n = R_{n-1}[\frac{\rho^{n^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ and $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ then*

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) = \frac{r^2}{\lambda^{\beta(r)}} \tag{1.38}$$

with for $r > C_{16}(d, K_0, K_1)$

$$\beta(r) = \big(\frac{\ln r}{\ln \rho}\big)^{\frac{1}{\alpha}}(1 + \epsilon(r)) \tag{1.39}$$

with $\epsilon(r) \to 0$ as $r \to \infty$

1.4 Multi-scale homogenization with bounded ratios between scales

1.4.1 All dimensions

The following theorem corresponds to the theorem 9.2.2.

Theorem 1.4.1. *If $\rho_{min} \geq C_{1,d,K_0,K_1}$ then for all $n \geq 1$*

$$\lambda_{\max}(D(V_0^{n-1})) \leq (1 + \frac{C_{2,d,K_0,K_1}}{\rho_{\min}^{\frac{1}{2}}})^n \prod_{k=0}^{n-1} \lambda_{\max}(D(U_k)) \tag{1.40}$$

and

$$\lambda_{\min}\big(D(V^n)\big) \geq (1 + \frac{C_{2,d,K_0,K_1}}{\rho_{\min}^{\frac{1}{2}}})^{-n} \prod_{k=0}^{n-1} \lambda_{\min}\big(D(U_k)\big) \tag{1.41}$$

$$C_{1,d,K_0,K_1} = C_d e^{(6d+16)K_0}(1+K_1)^3 \tag{1.42}$$

and

$$C_{2,d,K_0,K_1} = C_d e^{(3d+8)K_0}(1+K_1)^{\frac{1}{2}} \tag{1.43}$$

This theorem is a corollary of more general results which allow to control the whole matrix $D(V_0^{n-1})$ (see propositions 9.3.5 and 9.4.1).

1.4.2 Self-similar case in dimension one

Assume that the IHPD is self-similar with ratio between scales $\rho \in \mathbb{N}/\{0,1\}$ and periodic potential $U \in C^\infty(T_1^1)$.
The following theorem is the theorem 8.2.1

Theorem 1.4.2.

$$\lim_{n\to\infty} -\frac{1}{n} \ln\big(D(V^{n-1})\big) = \mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) \tag{1.44}$$

1.4.3 Dimension two

the following theorem corresponds to the theorem 9.3.1.

Theorem 1.4.3. *For $d = 2$ one has*

$$\begin{aligned}\lambda_{\max}\big(D(U)\big)\lambda_{\min}\big(D(-U)\big) &= \lambda_{\min}\big(D(U)\big)\lambda_{\max}\big(D(-U)\big)\\ &= \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}\end{aligned} \tag{1.45}$$

from which one deduces that if $D(U) = D(-U)$ then

$$\lambda_{\max}\big(D(U)\big) = \lambda_{\min}\big(D(U)\big) = \frac{1}{\sqrt{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}} \tag{1.46}$$

Moreover

Theorem 1.4.4. *In the self-similar case, if $d = 2$ and for all n, $D(V_0^n) = D(-V_0^n)$ then*

$$\lim_{n\to\infty} -\frac{1}{n} \ln\big(\lambda(D(V_0^{n-1}))\big) = \frac{\mathcal{P}_R(2U) + \mathcal{P}_R(-2U)}{2} \tag{1.47}$$

where $\mathcal{P}_R$ is the topological pressure associated to the shift s_R.
As an example of medium satisfying the condition of the previous theorem one can give the following corollary

Corollary 1.4.1. *In the self-similar case, if $d = 2$ and for all n, $U_n(-x) = -U_n(x)$ then*

$$\lim_{n\to\infty} -\frac{1}{n} \ln\big(\lambda(D(V_0^{n-1}))\big) = \frac{\mathcal{P}_R(2U) + \mathcal{P}_R(-2U)}{2} \tag{1.48}$$

1.5 Davies conjecture, exponential martingales, homogenization and fast rate of convergence towards the limit process

1.5.1 A martingale inequality

Consider M_t a continuous square integrable $\mathcal{F}_t$ adapted martingale such that $M_0 = 0$ and for $\lambda, t > 0$, $\mathbb{E}[e^{\lambda M_t}] < \infty$.
Assume that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t_2 > t_1 \geq 0$ one has a.s.

$$\mathbb{E}[\int_{t_1}^{t_2} d < M, M >_s |\mathcal{F}_{t_1}] \leq \int_0^{t_2 - t_1} f(s) ds$$

With $f(s) = f_1$ for $s < t_0$ and $f(s) = f_2$ for $s \geq t_0$ with $t_0 > 0$ and $0 < f_2 < f_1$.

Theorem 1.5.1. *For the martingale given above one has*

1. *for all*

$$0 < |\lambda| < \frac{1}{\big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}} \tag{1.49}$$

one has

$$\mathbb{E}[\exp(\lambda M_t)] \leq e^{3(1-1/g(\lambda))} \exp(\frac{g(\lambda)}{2}\lambda^2 f_2 t) \tag{1.50}$$

with $g(\lambda) = \frac{1}{1-\lambda^2(f_1-f_2)t_0 e}$ *which verify* $1 \leq g \leq 2$

2. *for all*

$$0 < \nu < \frac{1}{2e(f_1 - f_2)t_0} \tag{1.51}$$

one has

$$\mathbb{E}[\exp(\nu < M, M >_t)] \leq \exp(\nu f_2 t) \frac{\exp\big(\nu t_0 (f_1 - f_2)\big)}{((f_1 - f_2)\nu t_0)^2} \tag{1.52}$$

1.5.2 Davies conjecture

As an example, the theorem 1.5.1 will be applied here to obtain estimates on heat kernel $p(t, x, y)$ associated to the following periodic operator showing that its homogenized behavior starts for $|x - y| << t$ which gives an answer to Davies conjecture concerning the rate of convergence towards the limit process in a multidimensional periodic medium.

$$dy_t = d\omega_t - \nabla U(y_t) dt \tag{1.53}$$

where $U \in C^1(T_1^d)$ $(U(0) = 0)$.
It will be important to remember that this is only an example selected for the clarity of the presentation and in the chapter 6 it will be shown that one can consider a wide range of operators as soon as a cell problem is well defined and Aronson estimates available.

The following corollary corresponds to the corollary 12.1.2.

Corollary 1.5.1. *Consider $p(t,x,y)$ the transition density probabilities of the diffusion 1.53 with respect to the measure*
$m_U(dx) = e^{-2U(x)}dx/(\int_{T_1^d} e^{-2U(x)}dx)$*. then for*

$$20k_1|x-y| < t, \qquad k_2 < \frac{|x-y|}{\sqrt{t}}, \qquad |x-y| > 4C_\chi \tag{1.54}$$

one has

$$p(t,x,y) \leq \frac{E_1}{t^{\frac{d}{2}}} \exp\big(-(1-E)\frac{|y-x-2C_\chi|^2}{2D(e_{y-x})t}\big) \tag{1.55}$$

where k_1, k_2, C_χ, E_1 are constants depending only on d and $\mathrm{Osc}(U)$. Moreover

$$E = 8(\frac{k_1|x-y|}{t})^2 + 2\frac{\sqrt{t}}{|x-y|} \leq \frac{1}{10} \tag{1.56}$$

Actually, it will be shown in the chapter 6 that one needs only $U \in L^\infty(T_1^d)$ for the above corollary and the below theorem.

The following theorem corresponds to the theorem 12.2.1.

Theorem 1.5.2. *For $l \in \mathbb{S}^d$, $\lambda > C_6(d, \mathrm{Osc}(U))$ and*

$$C_7(d, \mathrm{Osc}(U))\lambda < t \tag{1.57}$$

one has

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{4\sqrt{2\pi}} \int_X^\infty e^{-z^2/2} dz \tag{1.58}$$

with

$$X = \frac{\lambda}{\sqrt{{}^t l D(U) l t}}(1+E) \tag{1.59}$$

and

$$E = \frac{C_8(d, \mathrm{Osc}(U))}{\lambda} + C_5(d, \mathrm{Osc}(U))\sqrt{\frac{\lambda}{t}} \leq \frac{1}{10} \tag{1.60}$$

1.6 Anomalous behavior in all dimensions starting from any point

1.6.1 Stability condition

Let $U, P \in C^\infty(\overline{B(z,r)})$. Write $\mathbb{E}^U$, $\mathbb{E}^{U+P}$ the expectations associated to the diffusions generated by $L_U = \frac{1}{2}\Delta - \nabla U \nabla$ and L_{U+P} and $\tau(B(z,r))$ the exit time from the d dimensional ball $B(z,r)$ An IHPD is said to satisfy the stability condition 1.6.1 if and only if ($\mathrm{Osc}_{B(z,r)}(U)$ stands for $\sup_{B(z,r)} U - \inf_{B(z,r)} U$):

Condition 1.6.1. *There exists $\mu > 0$ such that for all $n \in N$, all $z \in \mathbb{R}^d$, and all $r > 0$,*

$$E_z^V\big[\tau(B(z,r))\big] \leq \mu e^{\mu \,\mathrm{Osc}_{B(z,r)}(V_{n+1}^\infty)} \sup_{x \in B(z,r)} E_x^{V_0^n}\big[\tau(B(z,r))\big] \tag{1.61}$$

and

$$E_z^V\big[\tau(B(z,r))\big] \geq \frac{1}{\mu} e^{-\mu \,\mathrm{Osc}_{B(z,r)}(V_{n+1}^\infty)} \inf_{x \in B(z,\frac{r}{2})} E_x^{V_0^n}\big[\tau(B(z,r))\big] \tag{1.62}$$

1.6.2 Anomalous exit times starting from any point

1.6.2.i Bounded ratios

Then the following theorem corresponds to the theorem 10.2.1.

Theorem 1.6.1. *If the IHPD satisfy the stability condition 1.6.1 and* $\lambda_{\max} < 1$*, then for* $\rho_{\min} > C_{1,d,K_0,K_1,\lambda_{\max},\mu}$*,* $r > C_{2,d,K_0,K_1,\rho_{\max},\mu}$ *one has*

$$\begin{aligned} \mathbb{E}_x\big[\tau(B(x,r))\big] &\leq C_{32,d,K_0,K_1,\mu} r^{2+\sigma(r)(1+\gamma)} \\ &\geq C_{33,d,K_0,K_1,\mu} r^{2+\sigma(r)(1-\gamma)} \end{aligned} \tag{1.63}$$

$$\gamma = C_{2,d}\frac{K_0}{\ln \rho_{\min}} < 0.5 \tag{1.64}$$

$$\frac{\ln\frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}(1+\frac{C_{34,d,K_0,K_1,\mu}}{\ln \rho_{\min}})^{-1} \leq \sigma(r) \tag{1.65}$$

and

$$\sigma(r) \leq \frac{\ln\frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}(1+\frac{C_{35,d,K_0,K_1,\mu}}{\ln \rho_{\min}}) \tag{1.66}$$

see the theorem 10.2.1 for a more general form of this theorem and the meaning of $\sigma(r)$.

1.6.2.ii Unbounded ratios

Theorem 1.6.2. *If the IHPD satisfies the stability condition 1.6.1,* $R_n = R_{n-1}[\frac{\rho^{n^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ *and* $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ *then*

$$\mathbb{E}_0\big[\tau(B(0,r)\big] = \frac{r^2}{\lambda^{\beta(r)}} \tag{1.67}$$

with for $r > C_{16}(d, K_0, K_1)$

$$\beta(r) = \big(\frac{\ln r}{\ln \rho}\big)^{\frac{1}{\alpha}}(1+\epsilon(r)) \tag{1.68}$$

with $\epsilon(r) \to 0$ *as* $r \to \infty$

1.6.3 Anomalous heat kernel tail starting from any point

1.6.3.i Bounded ratios

The following theorem corresponds to the corollary 10.3.2

Theorem 1.6.3. *If the IHPD satisfy the stability condition 1.6.1,* $\rho_{\max} < \infty$ *and* $\lambda_{\max} < 1$*. Then for* $\rho_{\min} > C(d, K_0, K_1)$ *and* $(r > 0)$

$$C_{40} r \leq t \leq C_{41} r^{2+\sigma(r)(1-3\gamma)}$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_7 \frac{r^2}{t}\big(\frac{t}{r}\big)^{\nu'}$$

with

$$0 < c < \frac{\ln\frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}(1-\frac{C_{50,d,K_0}}{\ln \rho_{\min}}) \leq \nu'(r) \leq \frac{\ln\frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}(1-\frac{C_{50,d,K_0}}{\ln \rho_{\min}}) \tag{1.69}$$

$C_{50,d,K_0} < 0.5 \ln \rho_{\min}$ *and the constants* C_{40}, C_{41}, C_{42} *depend on* $d, K_0, K_1, \rho_{\max}, \rho_{\min}$*. All the constants depending on* K_0 *also depend on* μ*.*

$\sigma(r)$ and γ are those given in the theorem 1.6.1

1.6.3.ii Unbounded ratios

Theorem 1.6.4. *If the IHPD satisfy the stability condition 1.6.1,* $R_n = R_{n-1}[\frac{\rho^{n^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ *and* $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ *then for*

$$C_{60} r \leq t \leq C_{61} r^2$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_{63} \frac{r^2}{t} g(\frac{t}{r})$$

with

$$g(x) = (\frac{1}{\lambda})^{(\frac{x}{\ln \rho})^{\frac{1}{\alpha}}(1+\epsilon(x))} \tag{1.70}$$

where $\epsilon(x) \to 0$ *as* $x \to \infty$ *and the constant* C_{60} *to* C_{63} *depends on* $\rho, \alpha, K_0, K_1, d$. *All the constants depending on* K_0 *also depend on* μ.

1.6.4 Conjecture

Strong stability conjecture II Under the notations of the condition 1.6.1.

Conjecture 1.6.1. *The exists* $C_d > 0$ *a constant depending only on the dimension such that for all* $U, P \in C^\infty(B(\bar{0}, 1))$

$$E_0^{U+P}\big[\tau(B(0,1))\big] \leq C_d e^{C_d \operatorname{Osc}(P)} \sup_{x \in B(0,1)} E_x^U\big[\tau(B(0,1))\big] \tag{1.71}$$

and

$$E_0^{U+P}\big[\tau(B(0,1))\big] \geq C_d e^{-C_d \operatorname{Osc}(P)} \inf_{x \in B(0,\frac{1}{2})} E_x^U\big[\tau(B(0,1))\big] \tag{1.72}$$

In fact with this conjecture 6.4.2 says that all the IHPD do satisfy the stability condition 1.6.1.

1.6.5 A new analytical inequality; deformation of elliptic operators

The following theorem corresponds to the corollary 13.5.1.

Theorem 1.6.5. *For* $\Omega \subset \mathbb{R}^1$ *an open subset of* $\mathbb{R}$ $(d = 1)$, *there exist a constant* $C_{d,\Omega}$ *depending only on the dimension of the space and the open set such that for* $\lambda \in C^\infty(\bar{\Omega})$ *such that* $\lambda > 0$ *on* $\bar{\Omega}$ *and* $\phi, \psi \in C^2(\bar{\Omega})$ *null on* $\partial\Omega$ *and both sub harmonic with respect to the operator* $-\nabla(\lambda\nabla)$, *one has*

$$\int_\Omega \lambda(x) |\nabla\phi(x).\nabla\psi(x)|\, dx \leq 3 \int_\Omega \lambda(x) \nabla\phi(x).\nabla\psi(x)\, dx \tag{1.73}$$

The following corollary corresponds to the corollary 13.5.4.

Corollary 1.6.1. *Let* Ω *be a smooth bounded open subset of* $\mathbb{R}^d$. *Assume that* ϕ, ψ *are both convex or both concave and null on* $\partial\Omega$, *then*

$$\int_\Omega |\nabla_x\phi(x).\nabla_x\psi(x)|\, dx \leq 3 \int_\Omega \nabla_x\phi(x).\nabla_x\psi(x)\, dx \tag{1.74}$$

Conjecture 1.6.2. *For* $\Omega \subset \mathbb{R}^d$*an open subset with smooth boundary, there exist a constant* $C_{d,\Omega}$ *depending only on the dimension of the space and the open set such that for* $\lambda \in C^\infty(\bar{\Omega})$ *such that* $\lambda > 0$ *on* $\bar{\Omega}$ *and* $\phi, \psi \in C^2(\bar{\Omega})$ *null on* $\partial\Omega$ *and both sub harmonic with respect to the operator* $-\nabla(\lambda\nabla)$, *one has*

$$\int_\Omega \lambda(x) |\nabla\phi(x).\nabla\psi(x)|\, dx \leq C_{d,\Omega} \int_\Omega \lambda(x) \nabla\phi(x).\nabla\psi(x)\, dx \tag{1.75}$$

This conjecture is true in dimension one with $C_{d,\Omega} = 3$ (this constant is an homotopy invariant, this is proven by the corollary 13.5.1). In dimension d it does imply the conjecture 1.6.1.

2. SUPER-DIFFUSIVE MODEL

In this chapter the super-diffusive model will be introduced. Nevertheless, it has not been analyzed as deeply as the sub-diffusive model; this investigation is postponed to a sequel work. The results given here are obtained for the shear flow model characterized by an infinite number of scales.
As for the sub-diffusive models, the results will be given without any comments, the insight on the results and the physical meaning of this model will be given in the chapter 7.

2.1 Infinitely homogenized eddy diffusion

2.1.1 The model

The purpose of this model is to analyze a convective transport in an incompressible fluid characterized by an infinite number of periodic but non-symmetric and non-self-similar scale of eddies; that is to say, in $\mathbb{R}^d$, the behavior of solutions the stochastic differential equation:

$$\begin{cases} dy_t = d\omega_t - \nabla\Gamma(y_t)dt \\ y_0 = 0 \end{cases} \tag{2.1}$$

(where ω is a standard Brownian motion in $\mathbb{R}^d$) where $\Gamma \in (C^\infty(\mathbb{R}^d))^{d(d-1)/2}$ is a skew-symmetric $d \times d$ matrix. The notation $\nabla\Gamma$ designate the (left) divergence of Γ which is an horizontal vector such that for all $i \in \{0, \dots, d\}$

$$(\nabla\Gamma)_i = \sum_{j=1}^{d} \partial_j \Gamma_{ji} \tag{2.2}$$

This stream matrix Γ can be un-bounded and has fluctuations over an infinite number of scales which are periodic but non symmetric and non self-similar.
More precisely

$$\Gamma = \sum_{n=0}^{\infty} \gamma_n \Gamma^n(\frac{x}{R_n}) \tag{2.3}$$

Each stream matrix Γ^n is a smooth, periodic (of period T_1^d) skew-symmetric $d \times d$ matrix (in $\mathcal{C}^\infty(T_1^d)^{d(d-1)/2}$) and reflects the particular shape of the eddies (of the incompressible flow) of Γ at the scale R_n.
Each strictly positive real number $\gamma_n \in \mathbb{R}_+^*$ is called the diffusivity power of the scale n.
As in the sub-diffusive model, each integer $R_n \in \mathbb{N}^*$ reflects the length of the scale n and grows with n. More precisely

$$R_n = \prod_{n=0}^{n} r_n \tag{2.4}$$

Where r_n are integers element of $\mathbb{N}^*$ for $n \geq 1$. It is assumed that the small scale has length $r_0 = 1$. Moreover it is assumed that all the gradient of the elements of the stream matrices Γ^n are uniformly bounded. Thus there exist $K_0, K_1 > 0$ such that for all $i, j \in \{1, \dots, d\}$

$$\sup_{n\in\mathbb{N}} \| \operatorname{Osc}(\Gamma_{ij}^n)\|_\infty \leq K_0, \quad \sup_{n\in\mathbb{N}} \|\nabla\Gamma_{ij}^n\|_\infty \leq K_1 \tag{2.5}$$

It is also assumed that there exist $\rho_{\min} \in \mathbb{R}_+^*$ and $\rho_{\max} \geq \rho_{\min}$, $\rho_{\min} \in \mathbb{R}_+^* \cup \{\infty\}$ such that

$$\forall n \geq 1 \quad 2 \leq \rho_{\min} \leq r_n \leq \rho_{\max} \leq \infty \tag{2.6}$$

Each potential stream matrix Γ^n is chosen so that

$$\forall n \in \mathbb{N}, \quad \Gamma^n(0) = 0 \tag{2.7}$$

Thus Γ is a well defined Lipschitz stream matrix such that

$$\|\nabla\Gamma\|_\infty \leq \sum_{n=0}^{\infty} d\frac{K_1}{\rho_{\min}^n} \leq d\frac{K_1\rho_{\min}}{\rho_{\min}-1} \tag{2.8}$$

$$|\Gamma(x)_{ij}| \leq \frac{K_1\rho_{\min}}{\rho_{\min}-1}|x| \tag{2.9}$$

Thus it is well known that the solution of the stochastic differential equation 2.1 exists; is unique up to sets of measure 0 with respect to the Wiener measure and is a strong Markov continuous Feller process.

Effective diffusivities Each stream matrix Γ^n is associated with an effective diffusivity $D(\Gamma^n)$ by the equation 5.30.
It is assumed that there exist $1 < \lambda_{\min} \leq \lambda_{\max} < \infty$ such that for all $n \in \mathbb{N}$

$$1 < \lambda_{\min} \leq D(\Gamma^n) \leq \lambda_{\max} < \infty \tag{2.10}$$

Note that the effective diffusivity is a scale-invariant matrix that is to say an homogenization on the periodic stream $\Gamma(.)$ produce the same effective diffusivity as $\Gamma(\frac{.}{R_n})$ (this is an easy exercise). This explains the name diffusivity power given to the numbers γ_n. Moreover γ_0 is be chosen equal to 1 and it is assumed that there exist $0 \leq \gamma_{\min} \leq \gamma_{\max} \leq \infty$ such that for all $n \in \mathbb{N}$

$$\gamma_{\min} \leq \frac{\gamma_{n+1}}{\gamma_n} \leq \gamma_{\max} \tag{2.11}$$

Aggregation of scales In the sequel Γ^{km} will designate an aggregation of the scales $k, k+1, \ldots, m$ that is to say

$$\Gamma^{km}(x) = \sum_{n=k}^{m} \gamma_n \Gamma(\frac{x}{R_n}) \tag{2.12}$$

where $k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$
Note that since all the ratio r_n between scales are integers, the aggregation Γ^{km} is periodic (for $m < \infty$) of period $R_m \times T_1^d$. It is thus natural to relate this aggregation with an effective diffusivity written $D(\Gamma^{km})$ associated to an homogenization on the medium Γ^{km}. It is important to note that since the effective diffusivity is scale invariant, all that is needed is to know that Γ^{km} is periodic and a rescaling of its period doesn't influence the effective diffusivity. That is to say

$$D(\Gamma^{km}(x)) = D(\Gamma^{km}(x.R_m)) \tag{2.13}$$

and since $\Gamma^{km}(x.R_m)$ is of period T_1^d the variational formulation 5.30 will naturally be used to define its effective diffusivity.

Infinitely Homogenized Eddy Diffusion A solution of the stochastic differential equation 2.1 such that conditions 2.3, 2.6, 2.7, 2.8, 2.10 and 2.11 are satisfied will be called an Infinitely Homogenized Eddy Diffusion with parameters: $\rho_{\min}$, $\rho_{\max}$, $\lambda_{\min}$, $\lambda_{\max}$, $\gamma_{\min}$, $\gamma_{\max}$, K_0, K_1 and written

$$IHED(\rho_{\min}, \rho_{\max}, \lambda_{\min}, \lambda_{\max}, \gamma_{\min}, \gamma_{\max}, K_0, K_1) \tag{2.14}$$

Some remarks Note also that since homogenization on a periodic potential increase the diffusivity, one has always $1 \leq D(\Gamma^n) \leq \infty$. Note also that by the equation 5.29, the condition $D(\Gamma^{0,n}) > 1$ is equivalent to the fact that $l.\nabla\Gamma^n$ is not the null function for all non null direction $l \in (\mathbb{R}^d)^*$. Thus if all the scales are associated to a finite number of pattern of eddies: $\forall n, \Gamma^n \in \{H_1, \dots, H_d\}$ with non identically null gradient in all the directions, then the conditions 2.10 and 2.5 are trivially satisfied.

2.2 The shear flow model

The shear flow model is a particular case of IHED in dimension two.

2.2.1 The model

Consider the solution in $\mathbb{R}^2$ of

$$\begin{cases} dy_t = d\omega_t - \nabla\Gamma(y_t)dt \\ y_0 = 0 \end{cases} \tag{2.15}$$

(where ω is a standard Brownian motion in $\mathbb{R}^2$) where $\Gamma \in (C^\infty(\mathbb{R}^2))$ is a skew-symmetric 2×2 matrix.
The purpose of this chapter is to prove the super-diffusive transport in the turbulent shear flow model. More precisely, in all this chapter, y_t will be an infinitely homogenized shear-flow diffusion, that it is to say, it has all the characteristics of an infinitely homogenized eddy diffusion except for its associated stream matrix Γ which has the following particular structure ($\Gamma \in (C^\infty(\mathbb{R}^2))$ is a skew-symmetric 2×2 matrix):

$$\Gamma(x_1, x_2) = \begin{pmatrix} 0 & h(x_1) \\ -h(x_1) & 0 \end{pmatrix} \tag{2.16}$$

This stream matrix Γ can be un-bounded and has fluctuations along the $(0, x_1)$-axis over an infinite number of scales which are periodic but non symmetric and non self-similar.
More precisely

$$h(x_1) = \sum_{n=0}^{\infty} \gamma_n h^n(\frac{x_1}{R_n}) \tag{2.17}$$

With for all n, $h_n \in C^\infty(T_1^1)$ and

$$h^n(0) = 0 \qquad \mathrm{Var}(h^n) = \int_0^1 (h(x) - \int_0^1 h(y)dy)^2 dx = 1 \tag{2.18}$$

(more generally for a continuous function j on $\mathbb{R}$ of period R, $\frac{1}{R}\int_0^R (j(x) - \int_0^R j(y)dy)^2 dx$ will be written $\mathrm{Var}(j)$) Each stream matrix

$$\Gamma^n(x_1, x_2) = \begin{pmatrix} 0 & h^n(x_1) \\ -h^n(x_1) & 0 \end{pmatrix} \tag{2.19}$$

is a smooth, periodic smooth skew-symmetric 2×2 matrix and reflects the particular shape of the shear-flow (of the incompressible flow) of Γ at the scale R_n.
Each strictly positive real number $\gamma_n \in \mathbb{R}_+^*$ is called the diffusivity power of the scale n. This name is explained by the fact that since $\mathrm{Var}(h^n) = 1$

$$D(\gamma_n\Gamma^n) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + 4\gamma_n^2 \end{pmatrix} \tag{2.20}$$

Moreover γ_0 is be chosen equal to 1 and it is assumed that there exist $0 \leq \gamma_{\min} \leq \gamma_{\max} \leq \infty$ such that for all $n \in \mathbb{N}$

$$\gamma_{\min} \leq \frac{\gamma_{n+1}}{\gamma_n} \leq \gamma_{\max} \tag{2.21}$$

As in the IHED it is assumed that there exist $K_0, K_1 > 0$ such that

$$\sup_{n\in\mathbb{N}} \| \operatorname{Osc}(h_n)\|_\infty \leq K_0, \quad \sup_{n\in\mathbb{N}} \|h'_n\|_\infty \leq K_1 \tag{2.22}$$

and each integer $R_n \in \mathbb{N}^*$ reflects the length of the scale n and grows with n. More precisely

$$R_n = \prod_{n=0}^{n} r_n \tag{2.23}$$

where r_n are integers element of $\mathbb{N}^*$ for $n \geq 1$ (it is assumed that the small scale has length $r_0 = 1$) and there exist $\rho_{\min} \in \mathbb{R}^*_+$ and $\rho_{\max} \geq \rho_{\min}$, $\rho_{\min} \in \mathbb{R}^*_+ \cup \{\infty\}$ such that

$$\forall n \geq 1 \quad 2 \leq \rho_{\min} \leq r_n \leq \rho_{\max} \leq \infty \tag{2.24}$$

Observe that Γ is a well defined Lipschitz stream matrix the solution of the stochastic differential equation 2.15 exists; is unique up to sets of measure 0 with respect to the Wiener measure and is a strong Markov continuous Feller process.

Aggregation of scales write for $x \in \mathbb{R}$, $k \leq p-1$

$$H^{k,p-1}(x) = \sum_{n=k}^{p-1} \gamma_n h_n(\frac{x}{R_n}) \tag{2.25}$$

with $H^{p-1} = H^{0,p-1}$ and

$$\kappa^{p,p} = \frac{1}{R_p}\int_0^{R_p} H^{p,p}(y)\,dy \quad \kappa^{p-1} = \frac{1}{R_{p-1}}\int_0^{R_{p-1}} H^{p-1}(y)\,dy \tag{2.26}$$

Infinitely Homogenized Shear Flow Diffusion A solution of the stochastic differential equation 2.15 such that conditions 2.17, 2.24, 2.18, 2.22 and 2.21 are satisfied will be called an Infinitely Homogenized Shear Flow Diffusion with parameters: $\rho_{\min}$, $\rho_{\max}$, $\gamma_{\min}$, $\gamma_{\max}$, K_0, K_1 and written

$$IHSFD(\rho_{\min}, \rho_{\max}, \gamma_{\min}, \gamma_{\max}, K_0, K_1) \tag{2.27}$$

2.2.2 The results

Those results are proven in the chapter 11.

2.2.2.i Multi-scale effective diffusivity

Consider a IHSFD, the following theorem corresponds to the theorem 11.2.1.

Theorem 2.2.1. *assume $\gamma_{min} > 1$ and*

$$\epsilon = \frac{2^{\frac{3}{2}}}{\rho_{\min}} \frac{K_1}{\gamma_{\min} - 1} < 1 \tag{2.28}$$

then for all $p \in \mathbb{N}$

$$D(\Gamma^{0,p}) = \begin{pmatrix} 1 & 0 \\ 0 & D(\Gamma^{0,p})_{22} \end{pmatrix} \tag{2.29}$$

with

$$1 + 4(1-\epsilon)\sum_{k=0}^{p} \gamma_k^2 \leq D(\Gamma^{0,p})_{22} \leq 1 + 4(1+\epsilon)\sum_{k=0}^{p} \gamma_k^2 \tag{2.30}$$

2.2.2.ii Mean squared displacement

Bounded ratios The following theorem corresponds to the theorem 11.3.1.

Theorem 2.2.2. *assume* $\gamma_{min} > 1$, $\gamma_{\max}, \rho_{\max} < \infty$, $\rho_{min} > \rho_0(\gamma_{min}, \gamma_{max}, K_0, K_1)$ *and* $t > t_0(\gamma_{min}, \gamma_{max}, R_1, K_0, K_1)$, *then*

$$\mathbb{E}_0[|y_t.e_2|^2] = t^{1+\nu(t)} \tag{2.31}$$

with

$$\nu(t) \leq \frac{\ln \gamma_{\max}}{\ln \rho_{min} + \ln \frac{\gamma_{\min}}{\gamma_{\max}}} + \frac{C_2}{\ln t} \tag{2.32}$$

$$\nu(t) \geq \frac{\ln \gamma_{\min}}{\ln \rho_{max} + \ln \frac{\gamma_{\max}}{\gamma_{\min}}} - \frac{C_1}{\ln t} \tag{2.33}$$

Where the constants C_1 *and* C_2 *depends on* $\rho_{\min}, \gamma_{\min}, \gamma_{\max}, \rho_{\max}, K_1, K_2$

Fast separating ratios The following theorem corresponds to the theorem 11.3.2.

Theorem 2.2.3. *assume* $\gamma_p = \gamma^p$ *and* $R_p = R_{p-1}[\frac{\rho^{p^\alpha}}{R_{p-1}}]$ *with* $\gamma, \rho > 1$ *and* $\alpha \geq 1$ *Then for* $t > t_0(\gamma_2, R_2, K_0, K_1)$

$$C_1 t \gamma^{\beta(t)} \leq \mathbb{E}_0[|y_t.e_2|^2] \leq C_2 t \gamma^{\beta(t)} \tag{2.34}$$

with

$$\beta(t) = 2(\frac{1}{2\ln \rho})^{\frac{1}{\alpha}} (\ln t)^{\frac{1}{\alpha}} \tag{2.35}$$

Where the constants C_1 *and* C_2 *depends on* $\rho, \gamma, \alpha, K_1, K_2$

Part II

STATE OF THE ART AND LANDSCAPE

3. NORMAL DIFFUSION

Diffusions are more general objects than the sum of independent random vectors. In smooth cases they can be seen as the solution of a Stochastic Differential Equation, and are analyzed mainly through Ito calculus; they are even more general objects than solutions of a stochastic differential equation. Indeed, the name of diffusion is for instance given to a Markov processes X_t such that for almost everywhere x and $\mathbb{P}_x$ almost surely the process has continuous paths.
Moreover,the family of symmetric diffusions on a locally compact separable metric space can be put into one to one correspondence with at least four different objects: the semigroup, the resolvent, the generator and the Dirichlet form. This correspondence allows to prove non trivial controls on the diffusion by choosing the right object, for instance the spectrum of the generator, the geometry of the semigroup or the resolvent or the local properties of the Dirichlet form.
It will be assumed in this chapter that the reader is familiar with these objects and the connections between them (which are summarized in the appendix A)

3.1 Criteria of normality

By normal diffusion it is meant that the diffusion, behaves like a Gaussian diffusion and there are several levels at which a diffusion can do so.

Criterion 3.1.1. The first one and more general one is when the mean square displacement grows linearly with time, in other words:
$\exists C_1, C_2 > 0$ such that for all $t > 0$ (or for t large enough) and all x

$$C_1 t \leq \mathbb{E}_x\left[|X_t - x|^2\right] \leq C_2 t \tag{3.1}$$

In most papers of applied sciences, this behavior is considered as sufficient to call a diffusion "normal" (this behavior often appears for $t > t_0$ large enough). The Fick Law reflects a sharper control:

$$\mathbb{E}_x\left[|X_t - x|^2\right] \sim Dt \tag{3.2}$$

where D is the diffusivity constant (in $d = 1$).

Criterion 3.1.2. The second one is less general and gives a control of the probabilities of "going far" by Gaussian bounds, that is to say:
there exists $C > 0$ such that for all z, t, x:

$$\frac{1}{C} e^{-C\frac{z^2}{t}} \leq \mathbb{P}_x\left[|X_t - x| \geq z\right] \leq C e^{-\frac{1}{C}\frac{z^2}{t}} \tag{3.3}$$

Criterion 3.1.3. The Third and most restricting one gives Gaussian bounds on the transition densities $p_t(x, y)$ of the diffusion, that is to say:
there exists $C > 0$ such that for all y, t, x:

$$\frac{1}{C t^{\frac{t}{2}}} e^{-C\frac{(x-y)^2}{t}} \leq p_t(x, y) \leq \frac{C}{t^{\frac{t}{2}}} e^{-\frac{1}{C}\frac{(x-y)^2}{t}} \tag{3.4}$$

Normality is defined with comparison to Gaussian Bounds and to the Fick law because those behaviors appear as soon as the diffusion can be characterized as the sum of independent, uniformly bounded increments, which is often encountered in nature. Indeed, for the Fick law this is a simple consequence of the absence of correlations between increments, and the Gaussian Bounds then appear thanks to the Central Limit theorem.

3.2 Central Limit Theorem Revisited

Central limit theorems are an active field of research in the theory of homogenization that will be discussed in a following chapter. Here some examples will be given to show that they can deal with highly non trivial and non regular random media. The key point in those proof is the (explicit or implicit) use of the Dirichlet form.
When the medium where the diffusion takes place is non periodic and ergodic, the usual trick used to obtain a CLT is to replace the diffusion y_t starting from 0 on a space ξ which is chosen in an ergodic way and at random in a space of configuration $\mathcal{C}$, by the process ξ_t which is the medium seen by the particle y_t as it moves, this process ξ_t starts from a measure μ on the configuration space which remains invariant under the dynamic of the process (that's why a CLT is expected). Then the CLT is proven for a functional X of the process ξ_t.

3.2.1 C. Kipnis-S.R.S. Varadhan's central limit theorem

One of the first major result in this field is the paper of C. Kipnis and S.R.S. Varadhan [KV86]; in which functional central limit theorems for additive functionals of stationary reversible Markov processes are proven and applied to the study of the asymptotic normality of tagged particles of simple exclusion processes. The key point in the proof is to assume the square integrability of the velocity function of the process, as well as a the condition of integrability of the velocity autocorrelation function (equivalent to a condition on the spectral measure of the velocity function.)
Indeed in this paper, the medium seen by a tagged particle at integer times n is a stationary, reversible, ergodic Markov chain ξ_n and ξ_0 is distributed according to the invariant measure μ. The mean velocity V is a function on the space of configurations $\mathcal{C}$ such that $\mathbb{E}_\mu[V(\xi_0)] = 0$, $D = \mathbb{E}_\mu\left[\left(V(\xi_0)\right)^2\right] < \infty$ and such that $\sum_{n=0}^{+\infty} E_\mu[V(\xi_0)V(\xi_n)]$ converges. Then it is proven, under these weak conditions, that $\frac{1}{n^{\frac{1}{2}}}\sum_{k=1}^{[nt]} V(\xi_k)$ converges weakly to a Brownian motion with effective diffusivity D.

3.2.2 A. De Masi, P. A. Ferrari, S. Goldstein and W.D. Wick's approach

A. De Masi, P.A. Ferrari, S. Goldstein and W.D. Wick, in [MFGW89], consider ξ_t (the configuration state seen by the tagged particle) a stationary reversible Markov process (in continuous or discrete time). The CLT is proven for functionals X_I on the path space of ξ, indexed by intervals I, which satisfy an additivity and an antisymmetry property.
Additivity says that $X_{I\cup J} = X_I + X_J$ when $I \cap J$ consists of a single point, and antisymmetry requires $X_I(\xi) = -X_I(R_c(\xi))$, where $R_c(\xi)$ is the reflected sample path with $R_c(\xi)_t = \xi_{2c-t}$ for c the midpoint of I.
Moreover, it is show that the condition on the square integrability of the velocity function of the process was unnecessary and the second condition of integrability of the velocity autocorrelation function is automatically satisfied by considering only antisymmetric functionals of the process. Indeed, in this paper explicit assumptions on mixing or decay of correlations are replaced by an assumption on the symmetry properties of the variables under time reversal. Then it is proven that an antisymmetric function $X_{[0,t]}$ of a time-symmetric, stationary, ergodic Markov process converge to a Brownian motion with effective diffusivity matrix D when appropriately rescaled, in other words $\epsilon X_{[0,\epsilon^{-2}t]}$ converges weakly to a Brownian motion as $\epsilon \downarrow 0$. In typical applications, $X_{[0,t]}$ is an increment of a component made in time t and is obviously antisymmetric.
A simple example of such X_I in the discrete time case when $I = [k,n]$ is $X_{[k,n]} = \sum_{i=k}^{n-1} f(\xi_i, \xi_{i+1})$ for a function f satisfying $f(u,v) = -f(v,u)$ (for instance $f(u,v) = v-u$ if ξ is real-valued, in which case X_I is the displacement over the time interval I)
A formula for the effective diffusivity D of the Brownian motion is given but it does not show that

the limit process is nonsingular (D has strictly positive eigenvalues). This is one of the difficulties in the applications however since no mixing conditions and only very few integrability conditions are imposed this makes it easy to apply the results to a wide class of random motions in random environments.
For instance

- A walker moving on the infinite cluster of the two-dimensional bond percolation model.
- A d-dimensional walker moving in a symmetric random environment
- A tagged particle in a d-dimensional symmetric lattice gas which allows interchanges.
- A tagged particle in a d-dimensional system of interacting Brownian particles.

For instance in [Tan93]; spherical obstacles are distributed on R^d, where $d \geq 2$, according to some Gibbs state with sufficiently small "activity". The CLT is proved for a random walk in R^d with reflection on these obstacles. A similar conclusion is reached for the motion of a tagged particle moving in a system of particles with hard-core interaction (note that the application is not direct and needs some work).
In [Tan94] these spherical obstacles are distributed on R^d at each point of a Poisson process. The CLT is obtained for a reflecting Brownian motion on the cluster of spheres containing the origin of the continuum percolation process under the measure obtained by conditioning on the event "the cluster containing the origin is unbounded". The hard work is to show that the diffusion coefficient of the limit process is strictly positive.

The first two main theorems of [MFGW89] will be given here in details.

3.2.2.i An invariance principle for reversible Markov processes

Let ξ_t be a (discrete or continuous time: $t \in \mathbb{R}$ or $t \in \mathbb{Z}$) Markov process on a measurable state space $\mathcal{C}$ (in applications the state ξ of the Markov process represents the environment seen from a "tagged" particle).
Let Ω be the space of trajectories of the process. Let $\mathcal{F}_I$ (index by intervals, $I \subset \mathbb{R}$ or $I \subset \mathbb{Z}$), be the σ-algebra generated by ξ_t, $t \in I$ with $\mathcal{F}_t = \mathcal{F}_{(-\infty,t]}$. Let μ be a measure on the configuration $\mathcal{C}$ and $\mathbb{P}_\mu$ the law of the process ξ on Ω with initial measure μ.
It is assumed that $\xi : [o, \infty) \to \mathcal{C},\ \xi = \xi(t, \omega)$ is jointly measurable.
It follows that the probability semigroup $P_t : L^p(\mu) \to L^p(\mu)$ given by

$$\mathbb{E}_\mu[f(\xi_t)|\mathcal{F}_0] = P_t f \quad a.s. \tag{3.5}$$

is strongly continuous (in t) in $L^p(\mu)$, for both $p = 1$ and $p = 2$ and thus can be associated a Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on $L^2(\mu)$.
Let θ_τ and R_τ denote the time-translation operator and the time-reflection operator (in τ) acting on that space and defined by:

$$(\theta_\tau \xi)(t) = \xi(t - \tau), \qquad (R_\tau \xi)(t) = \xi(2\tau - t) \tag{3.6}$$

It is assumed that ξ_t is reversible with respect to the invariant measure μ, that is to say that $\mathbb{P}_\mu$ is invariant under R_τ for all τ.
It is also assumed that the process is ergodic with respect to μ, that is to say that $\mathbb{P}_\mu$ is ergodic under the time-translation group.

Now let $X_I \in \mathbb{R}^d$ be a family (indexed by intervals $I = [a, b] \in \mathbb{R}$ or $\mathbb{Z}$) of functionals of the Markov process ξ_t, thus the X_I are assumed to be $\mathcal{F}_I$-measurable random variables.
Moreover it assumed that

- $X_I \in L^1(\mathbb{P}_\mu)$ for each bounded interval I.
- $X_{[a,b]} \circ \theta_\tau = X_{[a+\tau,b+\tau]} \quad a.s.$
- $X_I + X_{I'} = X_{I \cup I'}$ a.s. when intervals intersect in exactly one point.
- In the continuous time case it is assumed that $t \to X_t = X_{[0,t]}$ is an element of $D([0,\infty);\mathbb{R}^d)$
- The family X_I is assumed to be antisymmetric, that is to say $X_I \circ R_c = -X_I$ a.s. if c is the midpoint of I.

The CLT will be proven for this family.

Increment form CLT Let $X(= X_{[0,1]})$ be an $\mathcal{F}_{[0,1]}$-measurable, square-integrable, antisymmetric random variable. Define

$$X_n = X \circ \theta_{n-1}, \quad n = 1, 2, \dots \tag{3.7}$$

$$X_t^\epsilon = \epsilon \sum_{n=1}^{[\epsilon^{-2}t]} X_n \tag{3.8}$$

Then as $\epsilon \to 0$, X^ϵ converges weakly in μ-measure to a Brownian motion ω_D starting from 0 and with effective diffusivity D: that is to say for all bounded continuous function F on $\mathbb{D}([0,\infty),\mathbb{R}^d)$ (the space of "cad lag" functions) equipped with the Skorohod topology as $\epsilon \to 0$.

$$\mathbb{E}_\mu[F(X^\epsilon)|\xi_0 = \xi] \to \mathbb{E}(F(\omega_D)) \quad \text{in } \mu \text{ probability} \tag{3.9}$$

Furthermore

$$\lim_{\epsilon \to 0} \mathbb{E}_\mu[(X_t^\epsilon)^2] = Dt \tag{3.10}$$

D is given by

$$D = \mathbb{E}_\mu[X^2] - 2\big(\varphi, (1 - P_1)^{-1}\varphi\big) \tag{3.11}$$

where $(.,.)$ is the scalar product on the space $L^2(\Omega,\mu)$ and

$$\varphi = \mathbb{E}_\mu[X|\mathcal{F}_0] \tag{3.12}$$

Both terms in 3.11 are finite; the second term is the dual Dirichlet form associated with the self-adjoint, nonnegative operator $1 - P_1$ and may be expressed in terms of a power series.

Integral form CLT Let $X_\delta = X_{[0,\delta]}$ have values in $\mathbb{R}^d$. Assume that the mean forward velocity φ exists; that is,

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}_\mu[X_\delta|\mathcal{F}_0] = \varphi(\xi_0) \tag{3.13}$$

exists as a strong L^1 limit ($\varphi \in L^1(\mu)$ and has values in $\mathbb{R}^d$). In addition, assume that the martingale

$$M_t = X_t - \int_0^t \varphi(\xi_\tau) d\tau \tag{3.14}$$

is square integrable. Then $\varphi_i \in H^{-1}$, where H^{-1} is the dual space of $\mathcal{D}[\mathcal{E}]$ (the domain of Dirichlet form associated to the Markov process ξ) in $L^2(\mu)$; and the following hold:

1. Let the matrix D be given by $(1 \leq i, j \leq d)$

$$D_{ij} = C_{ij} - 2\int_0^\infty (\varphi_i, P_t\varphi_j)dt \tag{3.15}$$

where C is the symmetric matrix determined by: for all $l \in R^d$

$${}^t l.C.l = \mathbb{E}_\mu[(l.M_1)^2] \tag{3.16}$$

Let $X_t^\epsilon = \epsilon X_{\epsilon^{-2}t}$, and let ω_D be a Brownian motion with effective diffusivity D starting from 0. Then as $\epsilon \to 0$ the finite-dimensional distribution of X^ϵ converges in μ- measure towards those of ω_D. Furthermore

$$\lim_{\epsilon \to 0} \mathbb{E}_\mu[(l.X_t^\epsilon)^2] = ({}^t lDl)t \tag{3.17}$$

2. If in addition, $\varphi \in L^2(\mu)$ or, more generally, for some $T > 0$

$$\sup_{0 \leq t \leq T} |X_t| \in L^2(\mathbb{P}_\mu) \tag{3.18}$$

then as $\epsilon \to 0$, X^ϵ converges weakly in μ-measure to the Brownian motion ω_D.

3.2.3 Osada - Saitoh's result

In [KV86] and [MFGW89] the Markov processes describing the media are symmetric and mean forward velocities are functions. H. Osada and T. Saitoh, in [OS95], show that these conditions can be weakened:

- the symmetry condition is relaxed to "near-symmetry" as embodied in the sector condition: Call $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ the non-symmetric Dirichlet form (see [MR92]) associated to the ergodic Markov process ξ with stationary distribution μ, then it is assumed that there exists a constant $K \geq 1$ such that for all $u, v \in \mathcal{D}[\mathcal{E}]$

$$|\mathcal{E}(u,v)| \leq K\mathcal{E}(u,u)^{\frac{1}{2}}\mathcal{E}(v,v)^{\frac{1}{2}} \tag{3.19}$$

- The additive functional $\{X_t,\ t \geq 0\}$ of the Markov process is assumed to satisfy $\mathbb{E}_\mu[|X_t|^2] < \infty$ for all t and the following rather weak mean forward velocity condition:
For all $p > 0$, μ-a.e. in the initial configuration ξ the function

$$\chi_p(\xi) = p^2\mathbb{E}_\xi\Big[\int_0^\infty e^{-pt}|X_t|dt\Big] < \infty \tag{3.20}$$

is well defined and an element of $L^2(\mu)$.
And the linear functional $\mathcal{D}[\mathcal{E}] \ni f \mapsto \int f\chi_p\, d\mu$ converges weakly as $p \to \infty$ to an element φ of the dual space $\mathcal{D}[\mathcal{E}]'$

Under these assumptions, $\epsilon X_{\frac{t}{\epsilon^2}}$ converges in finite dimensional distributions in μ-measure to the distribution of a d-dimensional continuous martingale Z such that

$$< Z_i, Z_j >_t = D_{ij}t \quad Z_0 = 0 \tag{3.21}$$

where D is a constant matrix.
This result is applied to study homogenization of reflecting diffusion and tagged particles of infinitely many particle systems with hard core interaction, where the additive functionals contain local time type drifts and mean forward velocities are not functions. It is interesting to notice that in these applications, the condition on the mean velocity and the strict positivity of the matrix D are ensured by the strict positivity of an isoperimetric constant associated to the medium (direct geometrical properties of the medium can be used to control the diffusion).

3.3 Global Gaussian Bounds for diffusions

Central Limit Theorems are not the unique reason explaining why Normal Diffusion should appear. As it shall be shown with several examples linked to second order elliptic operators, the diffusion can be constrained to have a Gaussian behavior by analytical inequalities describing the geometrical properties of its generator. A simple example will be given below: the Brownian motion in a bounded potential drift.
In fact for a medium where a CLT takes place, usually, those analytical inequalities give Gaussian bounds for the behavior diffusion when the time t is small and the CLT gives the precise asymptotic behavior of the Diffusion for large time by characterizing the convergence towards the limit process. However if the medium is such that no CLT takes place, those Gaussian bounds remain valid and allow to control the diffusion.

3.3.1 A simple example: Aronson's estimates for elliptic operators in potential form

Let $\{P_t : t > 0\}$ be the semigroup associated to the second order partial differential operator:

$$L_U = \frac{1}{2}\Delta - \nabla U.\nabla = \frac{1}{2}e^{2U}\nabla(e^{-2U}\nabla) \tag{3.22}$$

With $U \in C^\infty(R^d)$ bounded, and with derivates bounded at all orders.
Under this assumption $\{P_t : t > 0\}$ is the unique Feller continuous Markov semigroup on $C_b(\mathbb{R}^d)$ with the property that

$$[P_t\phi](x) - \phi(x) = \int_0^t [P_s L\phi](x)ds, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{3.23}$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$. Moreover, under these assumptions, there is a function $p \in \cup_{n=1}^\infty C_b^\infty\big([\frac{1}{n}, n] \times \mathbb{R}^d \times \mathbb{R}^d\big)$ with values in $(0,\infty)$ such that

$$[P_t\phi](x) = \int_{\mathbb{R}^d} \phi(y)p(t,x,y)dm_U(y) \tag{3.24}$$

with

$$dm_U(y) = e^{-2U(y)}dy \tag{3.25}$$

p_t is the transition probability density with respect to the measure m_U which is the invariant measure associated to the strong symmetric Markov process having for generator L and solution of the Stochastic Differential Equation :

$$\begin{cases} dy_t = d\omega_t - \nabla U(y_t)dt \\ y_0 = x \end{cases} \tag{3.26}$$

where ω is a standard Brownian motion in $\mathbb{R}^d$
Now it will be shown that the transition probability densities p_t are constrained to have Gaussian Bounds. The key point leading to those bounds is the fact that U is bounded.

$$p_t(x,y)e^{-2U(y)} \leq Ce^{(4+d)\,\mathrm{Osc}(U)}\frac{1}{t^{\frac{d}{2}}}\exp\big(-\frac{|x-y|^2}{4t}\big) \tag{3.27}$$

and

$$p_t(x,y)e^{-2U(y)} \geq e^{-Z}\frac{1}{t^{\frac{d}{2}}}\exp\big(-Z\frac{|x-y|^2}{t}\big) \tag{3.28}$$

with

$$Z = Ce^{10(4+d)\,\mathrm{Osc}(U)} \tag{3.29}$$

Initially those bounds called "Aronson estimates" [Aro67] have been proved for operators in the divergence form

$$L = \nabla.(a\nabla) \tag{3.30}$$

where $a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable, symmetric matrix valued function which satisfies the ellipticity condition

$$\lambda I_d \leq a(.) \leq \frac{1}{\lambda} I_d \tag{3.31}$$

for some $\lambda \in (0, 1]$. It constitutes a beautiful summary of the results contained in the sequence of articles starting with those of E. De Giorgi [Gio57] and J. Nash [Nas58] and culminating in the article by J. Moser. [Mos64].
Since then, several methods are available for the proof of those bounds. The basic ideas of the adaptation of the method of E.B. Fabes and D. W. Stroock [FS86] to potential form operators $L = \frac{1}{2}\Delta - \nabla U.\nabla$ and the method of Davies [Dav87], [Dav89] (which can also be adapted) will be given below. The adaptation of the method of Fabes-Stroock can be found in [Sei98] (there are several mistakes in the preprint version cited here but they can easily be corrected)

3.3.1.i The upper bound

The main point is to consider the semigroup $\{P^\psi_t,\ t > 0\}$ defined by

$$P^\psi_t \phi(x) = e^\psi [P_t(e^{-\psi})] \tag{3.32}$$

Where ψ is a function to be optimized (this trick is due to Davies [Dav87]). This semigroup has a kernel

$$p^\psi(t, x, y) = e^{\psi(x)} p(t, x, y) e^{-\psi(y)} \tag{3.33}$$

If one expect p to have Gaussian bounds then it is natural to expect that (when $\psi = l.x$ has a linear behavior)

$$p^\psi(t, x, y) \leq \frac{C}{t^{\frac{d}{2}}} e^{tN(\psi)} \tag{3.34}$$

for some number $N(\psi)$ and some C independent of ψ. It follows immediately that

$$p(t, x, y) \leq \frac{C}{t^{\frac{d}{2}}} e^{\psi(y)-\psi(x)+tN(\psi)} \tag{3.35}$$

And an optimization on ψ gives the upper bound (with $\psi = l.x$ for some $l \in R^d$ the optimization is made on l). The hard point is to obtain the bound 3.34 and this is where the method of Faber-Stroock differs with the one of Davies.

The Nash Inequality method The method of E.B. Fabes and D.W. Stroock is based on the following Nash inequality: there is a constant $C_1 \in (0, \infty)$ such that for all $\phi \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$

$$\|\phi\|^{2+\frac{4}{d}}_{L^2(dx)} \leq C_1 \|\nabla\phi\|^2_{L^2(dx)} \|\phi\|^{\frac{4}{d}}_{L^1(dx)} \tag{3.36}$$

This inequality is used to give an upper bound for the L^{2q} norm of $P^\psi_t \phi$ for $q \in [1, \infty)$ and $\phi \in C^\infty_0(\mathbb{R}^d)^+$. And for $\psi = l.x$, this norm is big when "the process has good chances of going far" thus it is natural to find it in the strategy giving the upper bound for $p(t, x, y)$.

The Log-Sobolev Inequality The method of Davies is based on the following Log-Sobolev inequality:
For all $\epsilon > 0$, for all $\phi \in H^1 \cap L^1 \cap L^\infty$

$$\int_{\mathbb{R}^d} \phi^2 \ln \phi m(dx) \leq \epsilon \int_{\mathbb{R}^d} |\nabla\phi|^2 m(dx) + M(\epsilon)\|\phi\|^2_{L^2(m(dx))} + \|\phi\|^2_{L^2(m(dx))} \ln \|\phi\|_{L^2(m(dx))} \tag{3.37}$$

Where the measure $m(dx)$ is the Lebesgue measure for divergence form operators and the invariant measure $m_U(dx)$ for potential form operators.
The Log-Sobolev inequality for m_U can be deduced by perturbing the Log-Sobolev inequality for m and noticing that this inequality compares the entropy of a squared function (for which a variational formula is available and can easily be perturbed) and its Dirichlet form.
This inequality is also a consequence of the ultracontractivity property of P_t that is to say

$$\|P_t\|_{\infty,2} \leq e^{M(t)} \tag{3.38}$$

for all $t > 0$ where $M(t)$ is a monotonically decreasing function of t. The beauty of the proof of Davies is to notice that the inequality 3.37 can be used and perturbed to obtain the L^p Log-Sobolev inequality for the operator associated to P^ψ.

$$\int_{\mathbb{R}^d} \phi^p \ln \phi m(dx) \leq \epsilon(p) \int_{\mathbb{R}^d} \big((-e^\psi L e^{-\psi})\phi\big)\phi^{p-1} m(dx) + \Gamma(p)\|\phi\|^p_{L^p(m(dx))} + \|\phi\|^p_{L^p(m(dx))} \ln \|\phi\|_{L^p(m(dx))} \tag{3.39}$$

for all $2 < p < \infty$. And the equation 3.39 gives an upper bound on the $\|.\|_{\infty,2}$ norm of P_t^ψ.

$$\|P_t^\psi\|_{\infty,2} \leq e^M \tag{3.40}$$

with

$$t = \int_2^\infty p^{-1}\epsilon(p)dp, \qquad M = \int_2^\infty p^{-1}\Gamma(p)dp \tag{3.41}$$

The strategy of Davies is interesting because it is quite robust, allows to obtain sharp estimates and leaves some flexibility in choice of $\epsilon(p)$ and $\Gamma(p)$.
Actually the equivalence between logarithmic Sobolev inequalities and hypercontractivity is due to L. Gross [Gro75]. Hypercontractivity is a smoothing property introduced in quantum field theory that roughly describes that P_t maps L^2 in to L^4 for some $t > 0$. It actually gives rise to bounds on the operator norm $\|P_t\|_{p,q}$ of P_t from L^q into L^p, $1 \leq q \leq p \leq \infty$. The equivalence between logarithmic Sobolev inequality and hypercontractivity is an important issue when studying rates of convergence to the equilibrium are characterizing the spectral gap of the generator (see [Gro93]). In fact, Log-Sobolev inequalities have become a major tool in analysis (this is particularly true in infinite dimension) and a good survey [ABC+99] is available on the subject.

3.3.1.ii The lower bound

To obtain the lower bound, it is sufficient to prove that there exists $c_1, c_2 > 0$ such that $|x-y| \leq c_1 t^{\frac{1}{2}}$ implies:

$$p(t,x,y) \geq c_2 t^{-\frac{d}{2}} \tag{3.42}$$

Then the chain rule says that

$$p(t,x,y) \geq \int_{B_1} \cdots \int_{B_n} p(\frac{t}{n+1}, x, z_1) \cdots p(\frac{t}{n+1}, z_n, y) m(dz_1) \cdots m(dz_n) \tag{3.43}$$

where $B_1 \dots B_n$ are balls joining x to y and n an integer chosen in an optimal way to give the lower Gaussian bound 3.28.
The hard and quite technical point is the proof of the inequality 3.42 and in the strategy of Fabes-Stroock (and its adaptation to potential form operator in [Sei98]) is based on the following Poincaré inequality: for all $\phi \in C_b^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} e^{-\pi|y|^2}\Big(\phi(y) - \int_{\mathbb{R}^d} e^{-\pi|z|^2}\phi(z)dz\Big)^2 dy \le 2\pi \int_{\mathbb{R}^d} e^{-\pi|y|^2}|\nabla\phi(y)|^2 dy \tag{3.44}$$

There are other ways to prove the Gaussian bounds for diffusions associated to second other elliptic operators.

3.3.2 Resolvent method

P. Auscher, in [Aus96], gives an other proof of Aronson's upper Gaussian bound on the heat kernel for parabolic equations with time-independent real measurable coefficients. This approach also gives Gaussian bounds in the case of a complex perturbation of real coefficients and in the case of uniformly continuous and complex valued coefficients.
The idea is to proceed from elliptic regularity theory to parabolic theory. The three steps of the proofs are:

1. An improvement of regularity by iteration of the resolvent, the idea is to proceed from elliptic regularity theory to parabolic theory.
2. The use of Davies' method to obtain decay for the kernels by exponential perturbation
3. A contour integral from functional calculus and a rescaling argument

In [AMT98] P. Auscher, A. McIntosh and P. Tchamitchian consider the heat kernels of second order elliptic operators in divergence form with complex bounded measurable coefficients on $\mathbb{R}^d$, difficulties which arise in the complex situation include the failure of the heat semi-group to be contractive on L^p spaces, the absence of symmetry or self-adjointness of the matrix a in 3.31. They obtain Gaussian bounds (on the modulus of the heat kernel) without further assumption if $d \le 2$. It is interesting to notice that if $d \ge 5$, there exists ([AMT98]) operators with complex measurable bounded coefficients whose heat kernels do not satisfy Gaussian bounds. However when the principal part (the matrix a in 3.31) has Hölder continuous coefficients when $d \ge 3$ Gaussian bounds are proven.

3.3.3 Parabolic Harnack Inequality

An other way to prove Gaussian bounds on the diffusion associated to an uniformly elliptic operator is through the Parabolic Harnack Inequality.
J. Moser, in [Mos64], proved a Harnack inequality for parabolic equations associated with second order uniformly elliptic divergence form operators in Euclidean space in [Mos71] he simplifies his proof. His approach has been used in many other situations because it rests only on two functional inequalities (Sobolev and Poincaré inequalities).
Let Ω be an open domain in the Euclidean space $\mathbb{R}^d$ and set $H = T \times \Omega$ where T is an open interval on the real line. For $(t, x) \in H$ Moser considers weak positive solutions of the parabolic equation

$$\frac{\partial \phi}{\partial t} - \nabla\big(a(t,x)\nabla\phi\big) = 0 \tag{3.45}$$

where a is a bounded measurable symmetric real matrix satisfying 3.31. Choose a compact and connected subdomain K of Ω. Assume that T is the interval (T_1, T_2). Choose t_1, t_2, t_3, t_4 such that

$$T_1 < t_1 < t_2 < t_3 < t_4 \le T_2 \tag{3.46}$$

Then Harnack Parabolic Inequality says that, there exists a constant $c > 1$ depending only on H, K and t_1, t_2, t_3, t_4 such that

$$\sup_{(t_1,t_2)\times K} \phi \leq c^{\frac{1}{\lambda}} \inf_{(t_3,t_4)\times K} \phi \tag{3.47}$$

where the symbols sup and inf stand for the essential supremum and the essential infimum.
The Parabolic Harnack Inequality allows to prove Gaussian bounds in the heat kernel, in fact it was the historical proof used by Aronson [Aro67]. This inequality is intuitively natural in the sense that it says that if at time t the supremum value of the heat in a compact set is above v then if one waits until time the time $t + \delta t$ the heat would have diffused and lower bound the value of the heat everywhere in all the compact set by $c_2 v$. The Gaussian bounds are hidden in the law of power between the time $\delta t = R^2$ one have to wait and the size of the compact set (R is its radius when it is a closed ball) if one wants the ratio c_2 to be independent of the size R (R^2 is for a Brownian motion the expectation of the time needed to exit a ball of radius R).
It is interesting to notice that reciprocally Gaussian bounds on the heat kernel are sufficient to prove the Parabolic Harnack Inequality [FS86]. Since the work of Moser an impressive number of articles on the Parabolic Harnack Inequality have appeared and it would be foolish to list them all here (see [SC95] for a good survey on the subject). P. Li and S. T. Yau, in [LY86], proved a sharp Parabolic Harnack Inequality the Laplacian on complete Riemannian manifold whose Ricci curvature is bounded below. Their proof are primarily based on the Bochner Weitzenbock formula

$$-\Delta|\nabla\phi|^2 + 2g(\nabla\phi, \nabla\Delta\phi) = 2\big(|\operatorname{Hess}\phi|^2 + \operatorname{Ric}(\nabla\phi,\nabla\phi)\big) \tag{3.48}$$

which partly explains the role played by the Ricci curvature.
In [Nag92] their method has been adapted to potential form operators $\frac{1}{2}\Delta - \nabla U.\nabla$.
Actually the parabolic Harnack principle for divergence form second order operators is characterized by two simple geometric properties:

1. The Poincaré inequality
2. The doubling property

3.4 Connections between geometric properties of the generator and the control on the diffusion

As it is starting to become clear, there exists deep and fruitful connections between the geometry of the operator associated to the diffusion and controls on that diffusion (see [Led00], [SC95], [Bak94], [Var91], [Dav93] for good surveys on the subject).
Consider the Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ associated to a strongly continuous Markov semigroup P_t and assume that this semigroup possesses a nice kernel $p(t,x,y)$ (see [CKS87] and [Var91] for instance for the properties given here). Then the following uniform estimate on decay of the heat kernel

$$p(t,x,y) \leq \frac{C}{t^{\frac{d}{2}}} \tag{3.49}$$

is equivalent to the Nash inequality

$$\|\phi\|_2^{2+\frac{4}{d}} \leq C\mathcal{E}(\phi,\phi)\|\phi\|_1^{\frac{4}{d}} \tag{3.50}$$

moreover for $d > 2$ a basic theorem of N. Varopoulos ([Var91]) shows that the Sobolev inequality

$$\|\phi\|^2_{2\frac{d}{d-2}} \leq C_2\mathcal{E}(\phi,\phi) + C_3\|\phi\|_2^2 \tag{3.51}$$

is equivalent to the decay 3.49 of the heat kernel for all $t > 0$ if $C_3 = 0$ and $0 < t < t_0$ elsewhere.
It is interesting to notice that Sobolev inequalities such as 3.51 are actually parts of more general families of inequalities considered by E. Gagliardo and J. Nirenberg in the late fifties (see [Led00] for this part):

$$\|\phi\|_r \leq \left(A\|\phi\|_2^2 + B\mathcal{E}(\phi,\phi)\right)^{\frac{\theta}{2}} \|\phi\|_s^{1-\theta} \tag{3.52}$$

where $0 < r, s \leq \infty$, $\theta \in (0,1]$. These families describe in an unified way several inequalities that appeared in the literature (Nash inequality, logarithmic Sobolev inequality, ...). Various cases have to be distinguished according to the value of the parameter $p \neq 0$ defined by

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{s} \tag{3.53}$$

which should be considered as a dimensional parameter (according to examples of $\mathbb{R}^d$ for which $\frac{1}{p} = \frac{1}{2} - \frac{1}{d}$). The choice of $r = 2, s = 1, \theta = \frac{d}{d+2}$ yields the Nash inequality

$$\|\phi\|_2^{2+\frac{4}{d}} \leq \left(A\|\phi\|_2^2 + B\mathcal{E}(\phi,\phi)\right) \|\phi\|_1^{\frac{4}{d}} \tag{3.54}$$

used by J. Nash [Nas58] to prove the Hölder regularity of solutions of divergence form uniformly elliptic equations.
In the subsequent work on the subject [Mos64], J. Moser considers $r = 2 + \frac{4}{d}$, $s = 2$ and $\theta = \frac{d}{d+2}$.
It is interesting that for $r = 2$ and $\theta \to 0$, 3.52 yields the logarithmic Sobolev inequality

$$\int (\phi)^2 \ln \phi^2 m(dx) \leq \frac{d}{2} \ln(A\|\phi\|_2^2 + B\mathcal{E}(\phi,\phi)) \tag{3.55}$$

for all ϕ with $\|\phi\|_2^2 = 1$ which is also implied by the Nash inequality 3.54.

3.5 The Dirichlet form revisited

Set for instance $\mathcal{E}, \mathcal{D}[\mathcal{E}]$ a symmetric strongly local Dirichlet form on a real Hilbert space $L^2(X,m)$ (X is a locally compact separable Hausdorff space and m is a Radon measure with support X)(see [Stu96] for this paragraph, see also [JKM+98], [Stu95]). That is to say (strongly local) $\mathcal{E}(u,v) = 0$ whenever $u \in \mathcal{D}[\mathcal{E}]$ is constant on a neighborhood of the support of $v \in \mathcal{D}[\mathcal{E}]$. Any such form can be written

$$\mathcal{E}(u,v) = \int_X d\Gamma(u,v) \tag{3.56}$$

where Γ is a positive semidefinite, symmetric bilinear form on $\mathcal{D}[\mathcal{E}]$ with values in the signed Radon measure on X (the so-called energy measure). It can be defined by the formulae

$$\int_X \phi d\Gamma(u,u) = \mathcal{E}(u, \phi u) - \frac{1}{2}\mathcal{E}(u^2, \phi) \tag{3.57}$$

for every $u \in \mathcal{D}[\mathcal{E}(u,u)] \cap L^\infty(X,m)$ and every $\phi \in \mathcal{D}[\mathcal{E}] \cap C_0(X)$ For instance for the operator $L = \frac{1}{2}\Delta - \nabla U \nabla$ this energy measure is

$$d\Gamma(u,v) = \nabla u(x).\nabla v(x)\, e^{-2U(x)} dx \tag{3.58}$$

Define $\mathcal{D}_{loc}[\mathcal{E}] = \{u \in L^2_{loc}(X,m)\,;\, \Gamma(u,u) \text{ is a Radon measure}\}$
The energy measure Γ defines in an intrinsic way a pseudo metric ρ on X by

$$\rho(x,y) = \sup\{u(x) - u(y) \,:\, u \in \mathcal{D}_{loc}[\mathcal{E}] \cap C(X),\ \Gamma(u,u) \leq m \text{ on } X\} \tag{3.59}$$

The condition $\Gamma(u,u) \leq m$ in 3.59 means that the energy measure $\Gamma(u,u)$ is absolutely continuous with respect to the reference measure m with Radon-Nikodym derivate $\frac{d}{dm}\Gamma(u,u) \leq 1$.
Assume also that $\mathcal{E}$ is strongly regular that is to say: $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is regular and ρ is a metric on X whose topology coincides with the original one.

3.5.1 Local Properties

Fix arbitrary open subset $Z \subset X$. In the sequel it will be shown that local properties (on Z) of the heat kernel (which is defined on the whole space X) associated to the Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ are closely connected to local properties (on Z) of the Dirichlet.

Property 3.5.1. *Z is said to verify the completeness property if for all balls $B_{2r} \subset Z$ the closed balls $\overline{B}_r(x)$ are complete (or, equivalently compact here because of the strong regularity)*

Property 3.5.2. *m is said to verify the doubling property if there exists a constant $d = d(\mathcal{D}[\mathcal{E}])$ such that for all balls $B_{2r}(x) \subset Z$*

$$m(B_{2r}(x)) \leq 2^d m(B_r(x)) \tag{3.60}$$

The number d (when it is optimized) plays the role of the dimension of the space Z and it can be a fractional number.

Property 3.5.3. *$\mathcal{E}$ is said to verify the (weak) Poincaré inequality if there exists a constant $C_P = C_P(Z)$ such that for all balls $B_{2r}(x) \subset Z$*

$$\int_{B_r(x)} |u - u_{x,r}|^2 dm \leq C_P . r^2 \int_{B_{2r}(x)} d\Gamma(u,u) \tag{3.61}$$

for all $u \in \mathcal{D}[\mathcal{E}]$ where $u_{x,r} = \frac{1}{m(B_r(x))} \int_{B_r(x)} u dm$.

Write L the negative self-adjoint operator associated to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$

Property 3.5.4. *The operator $L - \frac{\partial}{\partial t}$ is said to verify the Parabolic Harnack inequality if there exists a constant $C_H = C_H(Z)$ such that for all balls $B_{2r} \subset Z$ and all $t \in \mathbb{R}$*

$$\sup_{(s,y)\in Q^-} u(s,y) \leq C_H \inf_{(s,y)\in Q^+} u(s,y) \tag{3.62}$$

whenever u is a nonnegative local (weak) solution of the parabolic equation $(L - \frac{\partial}{\partial t})u = 0$ on $Q = (t - 4r^2, t) \times B_{2r}(x)$. Here $Q^- = (t - 3r^2, t - 2r^2) \times B_r(x)$ and $Q^+ = (t - r^2, t) \times B_r(x)$

(to be precise one should replace the extremum by the essential extremum however the Harnack Parabolic inequality imply the Hölder continuity of the local solutions in the equivalence sense).
Consider now $\{(\mathcal{E}_t, \mathcal{D}[\mathcal{E}]\}_{t\in\mathbb{R}}$ a family of regular, strongly local and symmetric Dirichlet form (with same domain as $\mathcal{E}$) uniformly parabolic with respect to in the initial Dirichlet form $\mathcal{E}$ in the following sense: there exists a constant κ such that for all $u \in \mathcal{D}[\mathcal{E}]$ and all $t \in \mathbb{R}$

$$\frac{1}{\kappa}\mathcal{E}(u,u) \leq \mathcal{E}_t(u,u) \leq \kappa\mathcal{E}(u,u) \tag{3.63}$$

The negative semidefinite selfadjoint operator on $L^2(X,m)$ associated with the Dirichlet form $\mathcal{E}_t$ is denoted by L_t and their family defines a parabolic operator $L_t - \frac{\partial}{\partial t}$.

Property 3.5.5. *The parabolic operator $L_t - \frac{\partial}{\partial t}$ (defined above) is said to verify the Parabolic Harnack inequality if for all $\kappa \geq 1$ and all $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ with $0 < \alpha < \beta < \gamma < \delta$ and $0 < \epsilon < 2s$ a constant $C_H^* = C_H^*(Z)$ such that for all balls $B_{2r} \subset Z$ and all $t \in \mathbb{R}$*

$$\sup_{(s,y)\in Q^-} u(s,y) \leq C_H \inf_{(s,y)\in Q^+} u(s,y) \tag{3.64}$$

whenever u is a nonnegative local (weak) solution of the parabolic equation $(L_t - \frac{\partial}{\partial t})u = 0$ on $Q = (t - \delta r^2, t) \times B_{2r}(x)$. Here $Q^- = (t - \gamma r^2, t - \beta r^2) \times B_{\epsilon r}(x)$ and $Q^+ = (t - \alpha r^2, t) \times B_{\epsilon r}(x)$.

3.5.2 Connections between these properties

In [Stu96] it is shown that

Theorem 3.5.1. *Under the completeness assumption 3.5.1 the following are equivalent:*

1. *The doubling property 3.5.2 and the Poincaré inequality 3.5.3 hold true on Z.*
2. *The parabolic Harnack inequality 3.5.4 for the (time-independent) operator $L - \frac{\partial}{\partial t}$ on $\mathbb{R} \times Z$ holds true.*
3. *The parabolic Harnack inequality 3.5.5 holds true for the all (time-dependent) operator $L_t - \frac{\partial}{\partial t}$ on $\mathbb{R} \times Z$ which satisfy 3.63.*

The parabolic Harnack constant C_H in 2 can be chosen as $C_H(d, C_P)$, i.e., only to depend on the doubling constant d and the Poincaré constant C_P. The constant C_H^* in addition depends on the parabolicity constant and the parameters $\alpha, \beta, \gamma, \delta, \epsilon$. In the converse direction, both constants d and C_P in 1 can be chosen to depend only on the parabolic Harnack constant C_H for $L - \frac{\partial}{\partial t}$ (e.g. $d = 4^{\frac{\ln C_H}{\ln 2}}$ and $C_P = C_H^2.2^d$)

It was a general knowledge quite a long time that it suffices to have a doubling property, a Sobolev inequality and a Weighted Poincaré inequality in order to prove parabolic Harnack inequality (for sub elliptic operators on $\mathbb{R}^d$ or for Laplace-Beltrami operators on Riemannian manifolds) using the method of Moser. Only recently, independently Grigor'yan [Gri92] and Saloff-Coste [SC92] could prove that (at least in Riemannian geometry), a doubling property and a Poincaré inequality already imply a Sobolev inequality. Indeed here the following theorem is true ([Stu96])

Theorem 3.5.2. *Assume that the completeness 3.5.1, the doubling 3.5.2 properties and the Poincaré inequality hold true and put $d^* = \max(d, 3)$. Then there exists a constant $C_S = C_S(Z)$ such that for all balls $B_{2r}(x) \subset Z$*

$$\int_{B_r(x)} |u|^{\frac{2d^*}{d^*-1}} dm \leq C_S \frac{r^2}{m(B_r(x))^{\frac{2}{d^*}}} \int_{B_r(x)} \big(d\Gamma(u,u) + r^{-2}u^2 dm\big) \tag{3.65}$$

for all $u \in \mathcal{D}[\mathcal{E}] \cap C_0(B_r(x))$. The constant C_S can be chosen to be $f(d).C_P$ where $f(d)$ depends only on the doubling constant d.

Moreover under the completeness assumption 3.5.1 if doubling property 3.5.2 and the Poincaré inequality 3.5.3 hold true on Z using the parabolic Harnack inequality 3.5.5 one easily derives pointwise estimates on $\mathbb{R} \times Z$ for the fundamental solution $p(t, y, s, x)$ of the parabolic operator $L_t - \frac{\partial}{\partial t}$ on $\mathbb{R} \times X$ satisfying 3.63. [Stu96] For every $\epsilon > 0$ there exists a constant C depending only on $d = d(Z)$, $C_P = C_P(Z)$ and $\epsilon > 0$ such that the following estimate holds true for all points (t_1, y_1) and $(t_2, y_2) \in \mathbb{R} \times Y$ with $t_1 < t_2$.

$$p(t_2, y_2, t_1, y_1) \leq C.m^{-1}\big(B_{\sqrt{t_s}}(y_1)\big) \exp\Big(- \frac{\rho^2(y_1, y_2)}{(4+\epsilon)\kappa(t_2 - t_1)}\Big) \tag{3.66}$$

and

$$\begin{aligned} p(t_2, y_2, t_1, y_1) \geq & \frac{1}{C}.m^{-1}\big(B_{\sqrt{t_i}}(y_1)\big) \\ & . \exp\Big(- C\kappa \frac{\rho^2(y_1, y_2)}{t_2 - t_1}\Big) \exp\big(- \frac{C\kappa}{R_i^2}(t_2 - t_1)\big) \end{aligned} \tag{3.67}$$

Here $t_s = \inf\{t_2 - t_1, R_s^2\}$ and $t_i = \inf\{t_2 - t_1, R_i^2\}$ with $R_s = \inf\{\rho(y_1, X \setminus Y), \rho(y_2, X \setminus Y)\}$ (being $+\infty$ if $X = Y$) and $R_i = \inf_{0 \leq s \leq 1} \rho(\gamma(s), X \setminus Y)$ (γ is the geodesic of length $\rho(y_1, y_2)$ joining y_1 to y_2).

If $Y = X$ and $L_t \equiv L$ then $p(t_2, y_2, t_1, y_1) = p(t_2 - t_1, y_2, y_1)$ where $p(t, y, x)$ is the heat kernel associated to L and

$$p(t,x,y) \leq C.m^{-1}\big(B_{\sqrt{t}}(x)\big) \exp\Big(- \frac{\rho^2(x,y)}{(4+\epsilon)t}\Big) \tag{3.68}$$

and

$$p(t,x,y) \geq \frac{1}{C}.m^{-1}\big(B_{\sqrt{t}}(x)\big). \exp\Big(- C\frac{\rho^2(x,y)}{t}\Big) \tag{3.69}$$

3.5.3 Important remarks

- Fabes and Stroock, in [FS86], have given a strategy (modeled on an argument given by Krylov) to deduce parabolic Harnack inequality from Gaussian bounds. But here [Stu96] it is shown that the parabolic Harnack inequality is equivalent to the doubling condition and the Poincaré inequality . This means that for the range of diffusions for which the strategy of Fabes-Stroock can be applied, Gaussian bounds on the diffusion with respect to the measure ρ are equivalent to the doubling condition for the measure m and the Poincaré inequality for the Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$.

- The distance ρ 3.59 issued from the energy measure appears as the intrinsic object describing the Gaussian behavior. However if the diffusion takes place on $\mathbb{R}^d$. All that is needed on ρ is that its topology coincides with the Euclidean one but this does not necessarily mean that the two distances are equivalent. This would mean that although the diffusion may have a Gaussian behavior with respect to the distance ρ; its behavior with respect to the Euclidean distance can be not Gaussian (anomalous). In other words if one has doubling condition and Poincaré inequality and the relation between ρ and the Euclidean distance is not linear then the diffusion reflects an anomalous behavior with respect to the Euclidean distance.

- The initial assumptions on m (positive Radon measure) and the Dirichlet form (symmetric, strongly local and regular) are quite weak in the sense that they allow to consider a wide range of diffusions.

- The parabolic Harnack inequality is a local property obtained from local estimates (Z is an open subset of X) with global consequences (bounds on heat kernel). And when the strategy of Fabes-Stroock holds, the global consequences give back the local properties.

Now,consider for instance the diffusion 3.26 associated with the operator $L = \frac{1}{2}\Delta - \nabla U \nabla$ on $L^2(\mathbb{R}^d, e^{-2U}dx)$ with $U \in \mathcal{C}^\infty(\mathbb{R}^d)$ (not necessarily bounded). Then it is easy to see that ρ is the Euclidean distance

$$\rho(x,y) = 2|x-y| \tag{3.70}$$

This means, that if $e^{-2U(x)}dx$ satisfies the doubling condition and $\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla f(x) \nabla g(x) e^{-2U(x)}$ satisfies the Poincaré inequality then the diffusion shows Gaussian behavior.
Conversely on every open subset Z of $\mathbb{R}^d$, U is bounded and it is easy to see that the strategy of Fabes-Stroock can be applied to obtain a parabolic Harnack inequality.
Thus the Gaussian behavior of the diffusion 3.26 is equivalent to the doubling property of the measure $m_U(dx) = e^{-2U(x)}dx$ with the same doubling constant $\hat{d}$ everywhere and the Poincaré inequality (with the same constant C_P everywhere.)
Actually in that situation the behavior of the heat kernel $p(t,x,x)$ on its diagonal is governed by the behavior of $\frac{1}{\sqrt{m_U(B_t(x))}}$ which is in $\frac{1}{t^{\frac{d}{2}}}$ for t small since U is smooth and is controlled by the volume growth associated to m_U for t large.

3.6 Control of the Green function

There exists an other way to characterize the "normality" of a diffusion (remember "normality" is defined with respect to the Brownian motion). This is done through the expectation $E_x[\tau(x,r)]$ of the time $\tau(x,r)$ put by the diffusion starting from x to exit a ball $B_r(x)$ of center x and radius r. For a Brownian motion this expectation grows like r^2 with the radius of the ball, thus it is natural to set a criterion of normality the following property:

Criterion 3.6.1. for all x there exists C_1 and C_2 such that for all $r > 0$ (or $r > r_0 > 0$ or $0 < r < r_1$ depending on the range of an observation)

$$C_1 r^2 \leq \mathbb{E}_x[\tau(x,r)] \leq C_2 r^2 \tag{3.71}$$

At the first sight this control might seem weaker than a control on transition probability densities, however it will be shown that it has strong consequences on those densities (if it holds for a wide range of radius r and the beginning of the sentence is inverted: "there exists C_1 and C_2 such that for all x ... ").
In the sequel it will be shown through a simple example that the exit times can be obtained from a control on the Green functions associated to the given diffusion killed when exiting $B_r(x)$. In this process, normality reflects an uniform equivalence in the comparison of the given operator L with the Laplace operator Δ in an equilibrium sense.

3.6.1 A simple example

Consider now the usual simple example of the diffusion y_t 3.26 associated to potential form operator $L_U = \frac{1}{2}\Delta - \nabla U.\nabla$ ($U \in \mathcal{C}^\infty(\mathbb{R}^d)$).
For simplicity of the notations the exit times will be characterized for balls centered on the origin O. Call $G_r(x,y)$ the Green function associated to the divergent form operator $-\nabla\big(e^{-2U}\nabla\big)$ on the open ball $B_r(O)$ with Dirichlet conditions on $\partial B_r(O)$.

$$-\nabla\big(e^{-2U}\nabla G_r(x,y)\big) = \delta(x-y) \tag{3.72}$$

Then it is easy to see that the expectation of the exit time $\tau(O,r)$ is given by the solution ψ of the Poisson equation

$$L\psi = -1 \tag{3.73}$$

with Dirichlet conditions on $\partial B_r(O)$. Thus

$$\mathbb{E}_x[\tau(0,r)] = 2\int_{B_r(0)} G_r(x,y) e^{-2U(y)}\,dy \tag{3.74}$$

This means that a control on the Green functions gives a control on the exit times.
Write $H_r(x,y)$ the Green function associated to the Laplace operator on $B_r(0)$.

G. Stampacchia, in [Sta66] (see also [Sta65], the proof is beautiful), shows that there exists a constant $C_d > 0$ depending only on the dimension $d \geq 3$ such that (for $x \neq y$).

$$\frac{1}{C_d}\exp\big(-C_d e^{C_d \operatorname{Osc}_r(U)}\big) \leq \frac{G_r(x,y)}{H_r(x,y)} \leq C_d \exp\big(C_d e^{C_d \operatorname{Osc}_r(U)}\big) \tag{3.75}$$

where $\operatorname{Osc}_r(U) = \sup_{B_r(O)} U - \inf_{B_r(O)} U$.
Thus

$$\frac{1}{C_d}\exp\big(-C_d e^{C_d \operatorname{Osc}_r(U)}\big) r^2 \leq \mathbb{E}_O[\tau(O,r)] \leq C_d \exp\big(C_d e^{C_d \operatorname{Osc}_r(U)}\big) r^2 \tag{3.76}$$

And if U is bounded then exit times exhibit necessarily a normal behavior.
Actually there is mainly two ways to obtain such a control on the Green functions. The first one is through the parabolic Harnack inequality, however the proof is unsatisfactory in the sense that dynamical properties of the operator L are used to deduce an equilibrium property. The second one is through the Harnack inequality (see [Sta65], [Anc97], [GW82], [Pin89]).

The Harnack inequality reflects an equilibrium property of harmonic functions: Given a second order differential operator A, it is said that A satisfies the Harnack inequality in the domain $\Omega \subset \mathbb{R}^d$, if for each compact $K \subset \Omega$, there exists a constant $C = C(K, \Omega)$ such that any positive A-harmonic function u in Ω satisfies

$$\sup_K u \leq C \inf_K U \tag{3.77}$$

Note that the parabolic Harnack inequality implies the Harnack inequality, the idea behind this implication is that the reproduction at each time of the same harmonic function is a solution of the heat equation. For long it was conjectured that Harnack inequality should actually be weaker than its parabolic version but it is only recently that a counter-example has been proposed.
In a sense the parabolic Harnack inequality reflects the comparison between the heat kernel of the given diffusion with the one of the Brownian motion and the Harnack inequality reflects the comparison between the given operator L with the Laplacian Δ in terms of Green functions.

4. ANOMALOUS DIFFUSION

4.1 Anomalous Diffusion

A diffusion said to be anomalous when one of the criterion 3.1.1, 3.1.2 3.1.3 or 3.6.1 given in previous chapter is not satisfied. Actually, among physicists the breaking of the Fick law is the most popular criterion of anomaly. More precisely when the square displacement of the diffusion X_t grows like a power of the time.

$$\mathbb{E}[X_t^2] \sim t^\nu \tag{4.1}$$

The diffusion is said to be

- normal for $\nu = 1$
- sub diffusive for $\nu < 1$
- super diffusive for $\nu > 1$

The study of anomalous diffusions is an active field of research and each year an important numbers of articles from applied to theoretical (rigorous and heuristic) sciences appear on the subject. It would be foolish to give a complete panorama of the subject here, several good surveys are available: S. Havlin and D. Ben-Avraham [HBA87]; J.-P. Bouchaud and A. Georges [BG90] and M.B. Isichenko [Isi92]. An interesting review of recent articles can be found in the XI Max Born Symposium (1998) [PSW99].

For recent articles in physics, see also [CHS97],[PM96],[GOYK96],[BRC96],
[BF96],[vBD95],[KLLQ98],[Zan98], [VZP98],[VJO98],[YI98],[SD96], [Asl96],
[WUS96],[CMMGV99],[Sai00], [YR99],[DJ96],[Art97], [IIA98],[Leb98],
[Tom98],[WT99].

The sequel will focus mainly on sub diffusive behavior.

4.2 Some Models

The purpose of this section is to present some models of anomalous diffusion. However those models do not have a direct and clear link with the diffusions in multi scale media considered in this work. They are here just to show that other types of causes might generate the anomaly of a diffusion (and the panorama is far from being complete). That's why it is strongly advised to avoid this section in a first lecture in order to not loose the thought path of this thesis.

4.2.1 Trapping Models

Trapping models are an example of anomalous diffusion due to long waiting times. Consider a random walk on a regular lattice ($\mathbb{Z}^d$ for instance), such that the particle has to wait a time $\tau(x)$ on each site x before performing the next jump (see [BG90]) on a neighboring site (the size of the jump is of size 1 here but it can chosen at random). This waiting time is a random variable independently chosen at each new jump according to the same distribution $\psi(\tau)$.

Given $N(t)$, the number of steps performed by the walker during the time t; the mean (with respect to the chosen path) square displacement of the walker is

$$\mathbb{E}_{\text{path}}[X_t^2] = d\, N(t) \tag{4.2}$$

Here t is chosen to be the sum of the N waiting times encountered

$$t = \sum_{n=1}^{N} \tau(X_n) \tag{4.3}$$

When the expectation of the waiting times is finite: $\mathbb{E}[\tau] < \infty$ then the diffusions follows the Fick law

$$\mathbb{E}[X_t^2] \sim d\, \frac{t}{\mathbb{E}[\tau]} \tag{4.4}$$

However when $\psi(\tau)$ is a broad distribution ($\mathbb{E}[\tau] = \infty$) this leads to sub diffusive behavior.
For instance when the tail of ψ behaves like

$$\psi(\tau) = \tau^{\mu}\tau^{-(1+\mu)} \quad (\tau \to \infty) \tag{4.5}$$

with $0 < \mu \leq 1$ then t behaves as

$$t \sim \tau_0 N^{\frac{1}{\mu}} \tag{4.6}$$

and

$$\mathbb{E}[X_t^2] \sim \begin{cases} (\frac{t}{\tau_0})^{\mu} & (0 < \mu < 1) \\ \frac{t}{\tau_0 \ln(\frac{t}{\tau_0})} & (\mu = 1) \end{cases} \tag{4.7}$$

4.2.1.i Comb like structures

Comb like structures constitutes a geometrical example of trapping models and in spite of their apparent simplicity they are reasonable models for disordered media. Imagine on $\mathbb{R}^2$ ((x, y) coordinates) a random walk moving on the $(0, x)$ axis. At each point $x_n = ne_1$ ($n \in \mathbb{Z}$) of this axis is attached a tooth (parallel to the $(0, y)$ axis) with length $L(x_n)$. The teeth length are independent random variable chosen at site according to the same distribution.
The teeth of this of this comb behave as a trap in which the particle stays for some time before continuing its random motion and if $\mathbb{E}[L] = \infty$ then the diffusion shows an anomalous behavior which can be characterized by the relation between the distribution of the waiting time τ and the size of the teeth L.
For infinitely deep teeth ($L = \infty$) the waiting time distribution $\psi(\tau)$ is simply the waiting time distribution of the first return time at the origin (the entrance of a tooth) which decays as $\tau^{-\frac{3}{2}}$ for a $1d$ Brownian motion. And according to the discussion on trapping models 4.2.1 this broad distribution ($\mu = \frac{1}{2}$ see 4.5) the comb like geometry induces an anomalous diffusion along the $(0, x)$ axis with

$$\mathbb{E}[X_t^2 \sim t^{\frac{1}{2}}] \tag{4.8}$$

4.2.1.ii Sub diffusive behavior of random walk on a random cluster

In [Kes86] H. Kesten consider a random walk X_n on the "incipient infinite cluster" of two-dimensional bond percolation at criticality and proves that for some $\epsilon > 0$, the family $\{\frac{1}{n^{\frac{1}{2}-\epsilon}} X_n\}$ is tight. This shows that X_n has sub diffusive behavior. The result is obtained by looking at the incipient cluster as a comb structure: the embedded random walk is analyzed on a sub graph of the percolating cluster called the backbone. Attached to the backbone are "dandling ends" (the teeth of the comb) which are of no help for X_n to get far out. If the random walk enters a dandling end it has to return to the backbone in order to go to infinity. The time spent in the dandling ends is responsible for $|X_n|$ growing slower than $n^{\frac{1}{2}}$.

4.2.1.iii A. Sznitman results

Imagine the following trapping model. A Brownian motion is evolving in $\mathbb{R}^d$ with random compact traps (with non-void interior, closed balls for instance with strictly positive radius) distributed according to a Poissonian law below criticality. Imagine that the waiting time associated to each trap is a.s. infinite. Then the Brownian motion will a.s. be trapped in a finite time where the diffusion will be stuck forever (the Brownian motion is killed). For the moment it is not very interesting indeed, but imagine now that you observe the Brownian motion conditioned on the fact that it hasn't been killed. Then the striking, beautiful and rigorous result of A.-S. Sznitman (see [Szn99]) is that this conditioned diffusion evolving in the quenched disorder has a ballistic behavior.
The typical image obtained is the following: At time t the particles (associated with the diffusion) who have survived have traveled a distance $\frac{t}{(\ln t)^2}$ to find clearings of radius $(\ln t)^{\frac{1}{3}}$ without any obstacle (created by the Poisson process). Those clearings are very small compared to the distance traveled and far from each other (pinning effect). Moreover the survival clearings change quickly with the time (intermittence).

4.2.2 A one dimensional model

4.2.2.i F. Solomon's results

F. Solomon, in [Sol75], consider a random walk X_n on $\mathbb{Z}$, such that

$$\mathbb{P}[X_{n+1} = X_n + 1 | X_n] = \alpha_{X_n} \tag{4.9}$$

$$\mathbb{P}[X_{n+1} = X_n - 1 | X_n] = 1 - \alpha_{X_n} \tag{4.10}$$

Where $\{\alpha_x,\ x \in \mathbb{Z}\}$ is a sequence of independent, identically distributed random variables with $0 \le \alpha_x \le 1$ for all x. Then the behavior of the random walk is analyzed on the quenched random environment ($\{\alpha_x\}$ is fixed). The relevant parameter for the behavior of the walk is

$$\sigma = \frac{1-\alpha}{\alpha} \tag{4.11}$$

indeed

- for $\mathbb{E}[\ln \sigma] < 0$, $\lim X_n = \infty$ a.s. (with respect to the probability measure on the environment)
- for $\mathbb{E}[\ln \sigma] > 0$, $\lim X_n = -\infty$ a.s.
- for $\mathbb{E}[\ln \sigma] = 0$, $\{X_n\}$ is recurrent. Moreover $\liminf X_n = -\infty$ and $\limsup X_n = \infty$

Moreover

- $\mathbb{E}[\sigma] < 1$, implies $\lim \frac{X_n}{n} = \frac{1+\mathbb{E}[\sigma]}{1-\mathbb{E}[\sigma]}$ a.s.
- $\mathbb{E}[\sigma^{-1}] < 1$, implies $\lim \frac{X_n}{n} = -\frac{1+\mathbb{E}[\sigma^{-1}]}{1-\mathbb{E}[\sigma^{-1}]}$ a.s.
- $\mathbb{E}[\sigma]^{-1} \le 1 \le \mathbb{E}[\sigma^{-1}]$, implies $\lim \frac{X_n}{n} = 0$ a.s.

An interesting result of Solomon is the slow approach to infinity. In the case $\lim X_n = \infty$ a.s. but $\lim \frac{X_n}{n} = 0$ a.s. the problem is simplified by considering the case where σ_x can take only two possible value $\sigma_x = 0$ or θ (with $0 < \theta < \infty$ fixed). More precisely $\alpha_x = 1$ with probability γ and $\alpha_x = \frac{1}{1+\theta}$ with probability $1-\gamma$. The points x such that $\sigma_x = 0$ are barriers reflecting to the right. The principal advantage of this scheme is that the random walk can be decomposed into independent excursions from one barrier to the next.
Then

- If $\gamma\theta = 1$ then $\frac{\ln n}{n \ln \theta} X_n$ converges in probability to $\frac{1}{2}$
- If $\gamma\theta > 1$ then along a subsequence $\frac{X_n}{n^\rho}$ (where $\rho = -\frac{\ln \gamma}{\ln \theta} < 1$) converges in distribution to a non degenerate random variable.

4.2.2.ii Y.G.Sinai's results

Y. G. Sinai, in [Sin82], consider the previous model analyzed by F. Solomon under the assumption $\alpha_x, 1 - \alpha_x \geq \text{const} > 0$ and

$$\mathbb{E}[\ln \sigma] = 0 \tag{4.12}$$

It is shown that under these conditions, the random walk has an ultra slow behavior and X_n takes on value of order $(\ln n)^2$. Moreover as $n \to \infty$, the probability distribution for $\frac{X_n}{(\ln n)^2}$ becomes concentrated in an arbitrarily small neighborhood of some point depending on the realization $\{\alpha_x\}$. More precisely for given $\epsilon, \delta > 0$; for all sufficiently large n there exist a set A_n in the space of configurations of the medium $\{\alpha_x\}$ and a point $v_n = v_n(\xi)$ for each $\xi \in A_n$ such that $\mathbb{P}(A_n) \geq 1 - \alpha$, and for $\xi \in A_n$

$$\mathbb{P}\big(|\frac{X_n}{(\ln n)^2} - v_n| < \delta\big) \to 1 \tag{4.13}$$

as $n \to \infty$ uniformly in $\xi \in A_n$. As $n \to \infty$ the probability distributions for v_n converge weakly to some limit distribution.
This "ultra slow" diffusion is explained by the fact that randomness generates long barriers of α_x pointing in the direction opposed to the progress of the diffusion and very difficult to cross. Moreover the longer the distance that the diffusion wants to go is, the longer are the barriers encountered by the diffusion on his way.

4.3 Fractal Models

Fractals form a tool used by physicists to model the geometrical structure of disordered media. The most appealing property of fractals is their self-similarity, or scaling, meaning that some parts of a whole are similar after scaling, to the whole. For fractals involving random element, one speaks about a statistical self-similarity, meaning equivalent, after the proper rescaling, statistical distributions characterizing the geometry of a part and of the whole fractal an example of random fractal is the infinite cluster near the percolation threshold.
It is important ([Isi92]) to note that virtually no physical object in real space qualifies for the formal definition of a fractal involving a nontrivial Hausdorff dimension, because each physical model has certain limits of applicability expressed in the length scales involved. Instead, physical fractals are defined as geometrical objects having sufficiently wide scaling range $[l_{\min}, l_{\max}]$ specifying the length scales of self-similar behavior. As soon as the ratio $l_{\max}/l_{\min}$ becomes a large parameter, one can speak about a fractal.
Thus physicists have been interested by the problems such as random walks, diffusions, heat propagation or waves propagation in fractals in order build models of transport problems in geometrically disordered media [RT83].

This interest grew up with the following observation: Whereas in an Euclidean space the relevant parameter entering in the scaling law between the energy ϵ and the density of states $\mathcal{N}(\epsilon)$ on the substrate is the dimension of the space d.

$$\mathcal{N}(\epsilon) \sim \epsilon^{\frac{d}{2}-1}$$

On a fractal substrate the relevant parameter is not its Hausdorff dimension d_f but a constant d_s called fracton or spectral dimension.

$$\mathcal{N}(\epsilon) \sim \epsilon^{\frac{d_s}{2}-1}$$

This constant is different from d_f and its nature is analytical. This is translated the mathematical point of view by the scaling law between the Weyl distribution function associated to the Laplace operator defined on the fractal. For instance the Laplace operator defined on the L_2 functions of the Sierpinski pyramid is self-adjoint, it has a discrete spectra $\{\lambda_k\}$

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$$

and Kozlov, in [Koz93], shows (we write $N(\lambda) = card\{\lambda_k : \lambda_k \le \lambda\}$) that there exist $C_1, C_2 > 0$ such that.

$$C_1 \lambda^{\frac{d_s}{2}} \le N(\lambda) \le C_2 \lambda^{\frac{d_s}{2}}$$

The fracton also enters in the power relation

$$S(N) \sim N^{\frac{d_s}{2}}$$

between the number of distinct sites $S(N)$ visited by a random walk on the fractal S(N) et and the number of steps N.

Moreover from an analytical point of view, d_s also plays the role of the relevant parameter which enters in the study of the heat kernel or the Sobolev Inequality (which is the dimension of the space if the former is Euclidean).
Indeed the Sobolev inequality which in R^d for $d > 2$, $p = \frac{2d}{d-2}$, f in $H^1(R^d)$ is

$$\|f\|_p \le c_1 \|\nabla f\|_2$$

In the Sierpinski sponge [BB97] this inequality is valid with $p = \frac{2d_s}{d_s-2}$

The Sierpinski Gasket The study of those properties have lead to numerous studies in mathematics. The first constructions of a diffusion have been given on the Sierpinski Gasket by Goldstein, Kusuoka and Barlow-Perkins ([Kus87], [Gol87] and [BP88]). The pathwize construction of a diffusion on the Sierpinski carpet have been given by Barlow-Bass [BB89]; and the Dirichlet form construction by Kusuoka Zhou [KZ92]. The main difference between those two fractals is that the first finitely ramified (a removal of only a finite number of points is required to disconnect a subset of the fractal) and the second not (this makes analysis much more difficult); for instance although the unicity of the diffusion on finitely ramified fractals has been obtained by C. Sabot, it remains an open problem for the Sierpinski carpet. Although these objects have strong symmetries and are perfectly self similar the mathematical tools used are far from being trivial, and they constitutes examples on which a precise analysis can be undertaken in order to infer properties on more general fractals.

Actually physicists have used numerous parameters to characterize fractals [HBA87]. In addition to the Hausdorff dimension and the spectral dimension, the ramification can be finite or infinite, the lacunarity describes the degree of homogeneity of the fractal, the chemical exponent $\tilde{\nu}$ describes the scaling law between the number of links connecting two points on a fractal along the chemical path (minimal path) between them.

One of the important properties of diffusions or random walks on the fractals is their sub-diffusive behavior: Thus for a random walk on the Sierpinski gasket the mean square displacement $< R_N^2 >$

after N steps behaves as $N^{\frac{d_w}{2}}$ where $d_w > 2$ is called the dimension of the walk (because for the Sierpinski carpet d_w is really the fractal dimensionality of the path of the random walker).
Thus in [BB97] the Brownian motion X_t on the Sierpinski sponge verifies

$$c_1 t^{\frac{2}{d_w}} \leq E_x|X_t - x|^2 \leq c_2 t^{\frac{2}{d_w}}$$

And its transition probabilities densities are controlled by :

$$C_1 t^{\frac{-d_f}{d_w}} \exp(-C_2(\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w-1}}) \leq p(t,x,y) \leq C_2 t^{\frac{-d_f}{d_w}} \exp(-C_3(\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w-1}})$$

Random Fractals Random fractals constitute the next step in degree of complexity in the field of constructions of Brownian motion on fractals.
Thus M.B. Hambly, in [Ham97], construct a Brownian motion on a random recursive Sierpinski gasket; in which the symmetries and self-similarity are preserved only in a statistical sense but the object is not spatially homogeneous, the object is seen as a Galton-Watson process. The spatial inhomogeneity in the structure leads to oscillations and logarithmic corrections for the heat kernel.
B.M. Hambly, T. Kumagai, S. Kusuoka and X.Y. Zhou, in [HKKZ98], consider construct a Brownian motion on homogeneous random Sierpinski carpets (those objects have spatial symmetry but their self similarity is statistical), for fixed environment the process is constructed and anomalous control on the transition density is obtained. Here a coupling argument plays the key role to obtain an uniform Harnack inequality. Just as for the random gasket, there are greater oscillations in the heat kernel than that observed in the exactly self-similar case. It is interesting to note that the natural distance involved is the chemical path (it interesting to compare this with the notion of intrinsic metric associated to a Dirichlet form given in the first chapter).
H. Osada, in [Osa98], announce the construction of a diffusion process on a fractal "bubble" like media, the process lives on the surface of the bubbles and its speed depends on the size of the bubble so that the limit process has a Gaussian behavior.
Whereas Barlow-Bass have obtained the limit process through a pathwize construction and tightness, in all those constructions in random fractals the existence of the limit process is obtained by the existence of a limiting regular Dirichlet form on the fractal (Kusuoka-Zhou [KZ92] approach) which is also a natural method for constructing processes on finitely ramified fractals.

4.4 The Sierpinski carpet

The Sierpinski carpet is the fractal subset of $\mathbb{R}^2$ defined as follows. Divide the unit square F_0 into nine identical squares, each with side of length $\frac{1}{3}$ and remove the central one to obtain F_1. Divide each of the remaining eight squares into nine identical squares, each with sides of length $\frac{1}{9}$ and remove the central one to obtain F_2 (see figure 4.1). Call F_n the set which is left when n stages of the construction have been done. Sierpinski Carpet F is the fractal subset which remains after continuing this process indefinitely ($n \to \infty$); it is a closed set with zero Lebesgue measure and Hausdorff dimension $\frac{\ln 8}{\ln 3}$.

M.T. Barlow and R.F. Bass, in [BB89], have given a construction of a Brownian Motion on the Sierpinski carpet; this important paper was the first example of a rigorous construction of a Brownian Motion on an infinitely ramified fractal. The properties of this process are analyzed in [BB90a], [BB90b], [BB92] and [BB93a]. In [BB97], they extend their work to the Sierpinski Sponge thanks to a beautiful coupling argument that allows them to prove the crucial uniform Harnack inequality for $d \geq 3$.
S. Kusuoka and X.Y. Zhou, in [KZ92], adopt a different approach, they construct a Brownian motion on a graphical approximation of the Sierpinski carpet through the Dirichlet form (their proof is valid for superior dimensions) which play the key role for proving the self similarity of the law of the limit process trough a deterministic time change.

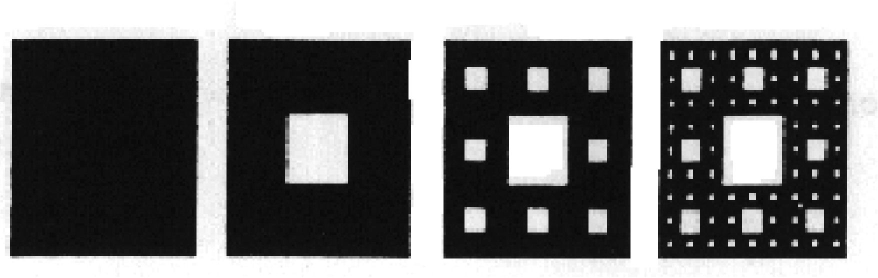

Fig. 4.1: *Construction of the Sierpinski carpet*

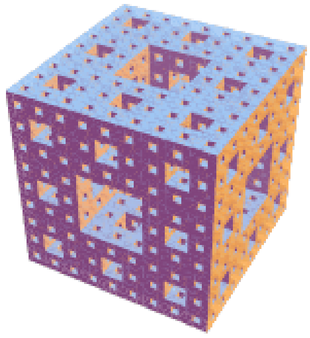

Fig. 4.2: *The Sierpinski sponge*

An alternative approach is the one of H. Osada [Osa90], [Osa95], [Osa99]; this point of view is interesting because the process is constructed and characterized thanks to isoperimetric and analytical inequalities.
In the sequel the idea of the proofs used by Barlow-Bass will be given as an introduction to the subject (mainly taken from the above articles of Barlow-Bass and the survey written by R. F. Bass [Basar]). Basically there are two layers in those proofs, the first layer directly use the very strong self similarity and strong symmetries of the carpet. The second layer is more abstract and general (its development has started with the papers about Brownian motion on the Sierpinski Gasket). Although in the above papers the first layer act as the foundation of the second one, in the sequel, the second layer shall be focused on since it constitutes the part which might be generalized to other fractals. Although Barlow-Bass have extended their proofs to the "generalized Sierpinski carpet" [BB97](one can have $d \geq 3$ and a symmetric "pixel" pattern taken out from the cube at each stage of the self-similar construction) the sequel will focus on the standard Sierpinski carpet for simplicity ($d = 2$).

4.4.1 Construction of the process

It is not hard to constrain the Brownian motion to be in the set F_n: more precisely define $\partial_a F_n$ ("a" for absorbing) to be $F_n \cap \{(x, y) : x = 1 \text{ or } y = 1\}$. Let $\partial_r F_n$ ("r" for reflecting) be $\partial F_n - \partial_a F_n$ (∂F is the boundary of F).
Let $W_n(t)$ be a Brownian motion on F_n with normal reflection on $\partial_r F_n$ and absorption on $\partial_a F_n$. Thus the process W_n is reflecting on the left and lower boundaries of F_n and on all the n scales of square obstacles inside it and is absorbed on the upper and right boundary of F_n.
But as more and more obstructions are introduced, the mean displacement of the Brownian motion gets smaller and smaller; it the limit one has only a process that moves not at all. If however, one performs a renormalization at each stage, that is, if at the nth stage one speeds up the process deterministically and uniformly by an appropriate amount, one gets a nondegenerate limit, a random process that is called Brownian motion on the Sierpinski carpet.

More precisely for any process X, let

$$\tau = \tau(X) = \inf\{t \,:\, X_t \in \partial_a F_0\} \tag{4.14}$$

and

$$\alpha_n = \sup_{x \in F_n} \mathbb{E}_x[\tau(W_n)] \tag{4.15}$$

Set

$$X_n(t) = W_n(\alpha_n t) \tag{4.16}$$

In other words W_n is speeded up so that the largest expected time for X_n to exit F_n is less than or equal to 1. The difficulty is that there might be points y for which the time to exit is much less than 1, that is, the process moves almost instantaneously to $\partial_a F_n$. To show that this cannot happen, let

$$\beta_n = \inf_{x \in F_n \cap [0, \frac{1}{2}]^2} \mathbb{E}_x[\tau(W_n)] \tag{4.17}$$

And it is shown that there exists c_1, c_2, c_3 independent of n such that

$$\alpha_n \leq c_1 \alpha_{n-1} \leq c_2 \beta_{n-1} \leq c_3 \beta_n \leq c_3 \alpha_n \tag{4.18}$$

4.4.2 The Sierpinski pre-Carpet

If one doesn't want to kill the process and consider it on a non compact subset of $\mathbb{R}^d$, an alternative point of view is to construct the Brownian Motion on the object called by H. Osada the "pre-Sierpinski carpet" ([Osa90],[BB97]). The pre carpet is the set $\tilde{F_0}$ defined by

$$\tilde{F_0} = \cup_{n=0}^{\infty} 3^n F_n \tag{4.19}$$

where $3^n F_n$ is the subset of $\mathbb{R}_+^2$ consisting of points $3^n x$ with $x \in F_n$.
Thus the pre-Carpet is subset of $\mathbb{R}_+^2$ consisting of bigger and bigger square holes removed.
Let W_t be the Brownian motion on the pre-carpet $\tilde{F_0}$, with normal reflection on $\partial \tilde{F_0}$, and let $q(t, x, y)$ be its transition density with respect to Lebesgue measure on $\tilde{F_0}$. These transition densities are the fundamental solutions to the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u$ on $\tilde{F_0}$ with Neumann boundary conditions. Then there exist $d_s < d$ and $d_w = 2\frac{d_f}{d_s} > 2$ such that

Theorem 4.4.1. *there exists $c_1, \ldots c_8 \in (0, \infty)$ such that if $x, y \in \tilde{F_0}$ and*

1. *$t \geq \max(1, |x - y|)$, then*

$$\begin{aligned} & c_1 t^{\frac{-d_s}{2}} \exp(-c_2 (\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w - 1}}) \\ & \leq q(t, x, y) \leq c_3 t^{\frac{-d_s}{2}} \exp(-c_4 (\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w - 1}}) \end{aligned} \tag{4.20}$$

2. *if $t \leq 1$, then*

$$c_5 t^{\frac{-d}{2}} \exp(-c_6 \frac{|y-x|^2}{t}) \leq q(t, x, y) \leq c_7 t^{\frac{-d}{2}} \exp(-c_8 \frac{|y-x|^2}{t}) \tag{4.21}$$

3. *If $t \geq 1$, $|x - y| > t$, then*

$$c_5 t^{\frac{-d_s}{2}} \exp(-c_6 \frac{|y-x|^2}{t}) \leq q(t, x, y) \leq c_7 t^{\frac{-d_s}{2}} \exp(-c_8 \frac{|y-x|^2}{t}) \tag{4.22}$$

Those bounds are explained by the following fact: $\tilde{F_0}$ is locally similar to $\mathbb{R}^2$ thus for small time t, q exhibits a Gaussian behavior. When $|x-y|>t$, then the process to move from x to y in time t, stays with high probability close to the shortest path connecting x and y and has no time to feel the fractal structure of $\tilde{F_0}$ (this is standard large-deviation theory for Brownian motion). However for $t \geq \max(1,|x-y|)$, the process feel the global fractal structure of the pre-carpet and exhibit an anomalous behavior.

Parabolic Harnack Inequality It is important to notice that although the pre-carpet satisfies the usual volume doubling condition and an uniform global elliptic Harnack inequality; the standard parabolic Harnack inequality holds locally but not globally (because of the estimates on q). However a different form of parabolic Harnack inequality can be given for the pre-carpet by using the estimates on q and following the argument of [FS86]; in the parabolic Harnack inequality on compact sets containing at least a square of side 1; the natural scaling law between the radius of the compact set r and the time $t=r^2$ (the delay of propagation of the heat to the whole compact set) is modified to $t=r^{d_w}$.

4.4.3 The limit process

Write

$$\tilde{F_n}=\frac{\tilde{F_0}}{3^n} \tag{4.23}$$

$\tilde{F_n}$ can be seen as an extension of F_n outside $[0,1]^2$ by bigger and bigger self-similar square holes structure.
Then one can consider the Brownian motion W_n reflecting on all the boundaries $\partial\tilde{F_n}$ of the pre-carpet $\tilde{F_n}$ (not killed). As before, to obtain a limit process when $n\to\infty$ one has to speed up the reflected Brownian motion on consider the process $X_n=W_n(\alpha_n t)$. Then the limit process X will live on the unbounded Sierpinski carpet:

$$\tilde{F}=\cap_{n=0}^{\infty}\tilde{F_n} \tag{4.24}$$

Important Remark It is important to notice that the hard point is not the tightness of the a speeded up version $W_n(\alpha_n t)$ of the Brownian motion W_n. If α_n is too slow one has always tightness and a degenerate limit process along a subsequent sequence (a process which doesn't move at all). If α_n is too fast, then one has not tightness. So the hard point is to obtain a non-void range of acceleration α_n such that one has tightness and the limit process is non-degenerate. For the Sierpinski-carpet this is possible, because this object has a very strong isotropy due to very strong global symmetries. If this isotropy is broken then the limit object might be degenerate and live on the straight line corresponding to the direction on which W_n is the less slow down by the obstacles.

Uniform Harnack inequality The key tool for the construction, the study of the Brownian motion on the Sierpinski carpet and all the estimates given above is an uniform Harnack Inequality on $\tilde{F_0}$:

Let B an open set in $\mathbb{R}^2$. h is said to be harmonic on $B\cap\tilde{F_0}$ if $\Delta h(x)=0$ for $x\in B\cap \mathrm{int}\tilde{F_0}$, and the normal derivate of h is 0 on $B\cap\tilde{F_0}$ almost everywhere with respect to surface measure on $\partial\tilde{F_0}$.

Theorem 4.4.2. *There exists a constant c_1 (not depending on r) such that if $x\in\tilde{F_0}$, $r>0$, and h is positive and harmonic on $B(x,2r)\cap\tilde{F_0}$, then writing $A=B(x,r)\cap\tilde{F_0}$*

$$\sup_A h(x)\leq c_1\inf_A h(y) \tag{4.25}$$

Since $\tilde{F}_0$ is a Lipschitz domain, for each r a constant $c_1(r)$ does exit. The point of this theorem is that c_1 can be taken independent of r. It is important to notice that this uniform Harnack inequality is an expression of the very strong isotropy of the carpet. That is to say if the obstacles are not squares and the isotropy is broken the Brownian motion might exhibit an anomalous behavior at large times without the necessity of an uniform Harnack inequality. This inequality would become useful if one has to pass to the limit by adding smaller and smaller obstacles in order to obtain a non degenerate limit.

4.4.4 Main results on the limit process

Write μ (a multiple of) the Hausdorff x^{d_f}-measure on $\tilde{F}$.
Then

Theorem 4.4.3. *There exists a no degenerate continuous strong Markov process X_t whose state space is $\tilde{F}$. X_t has transition densities which have the strong Feller property and which are μ-symmetric. The law of the process $(X_t, t \geq 0)$ is locally invariant under local isometries of $\tilde{F}$.*

Write P_t for the semigroup associated with X, and let $(L, \mathcal{D}(L))$ be the infinitesimal generator of P_t; L is called the Laplacian on $\tilde{F}$. The heat equation on $\tilde{F}$ then becomes

$$\frac{\partial u}{\partial t}(x,t) = Lu(x,t), \quad x \in \tilde{F}, \quad t > 0 \tag{4.26}$$

The fundamental solutions to the heat equation are given by the transition densities $p(t,x,y)$ for the process X_t on $\tilde{F}$; then

Theorem 4.4.4. *$p(t,x,y)$ is symmetric and jointly continuous on $(0,\infty) \times \tilde{F} \times \tilde{F}$, and for each x, y the function $p(t,x,y)$ is C^∞ in t. There exist c_1, c_2, c_3, c_4 such that for all $x, y \in \tilde{F}$ and $t > 0$,*

$$c_1 t^{\frac{-d_s}{2}} \exp(-c_2(\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w-1}}) \leq p(t,x,y) \leq c_3 t^{\frac{-d_s}{2}} \exp(-c_4(\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w-1}}) \tag{4.27}$$

4.4.5 A non degenerate limit

4.4.5.i Resolvent convergence

Write X_n for the speeded up Brownian motion living on $\tilde{F}_n$ and killed For f bounded and continuous on $\mathbb{R}^2$ let

$$U_n^\lambda f(x) = \mathbb{E}_x[\int_0^\infty e^{-\lambda t} f(X_n(t))dt] \tag{4.28}$$

By the uniform Harnack inequality and a modulus of continuity estimates for harmonic functions, it is possible to show that $\{U_n^\lambda f(x)\}_{n=1}^\infty$ is equicontinuous on compact sets. By a diagonalization and limit argument, there exists a there exists a subsequence n' such that $\{U_{n'}^{\lambda_i} f\}$ converges uniformly on compacts, to a limit called $U^\lambda f$, for all $\lambda > 0$ and f bounded and continuous.
This resolvent convergence is enough to get a Markov process, indeed, the limit U^λ, satisfies the resolvent identity and $\|U^\lambda\|_\infty \leq \lambda^{-1}$. Thus the Hille-Yosida theorem can be used to guarantee the existence of a limit process associated to the limit resolvent (see the previous chapter).
Although, this resolvent convergence is enough to get a Markov process, is not enough to get a continuous process. For that tightness is used.

4.4.5.ii Tightness estimate

From the equation 4.18 the exit times from F_n are controlled by

$$\mathbb{P}_x(\tau(W_n) \leq s) \leq c_4 + \frac{s}{\alpha_n} \quad x \in F_n \cap [0, \frac{1}{2}]^2 \tag{4.29}$$

This inequality says that if γ is small enough, there is positive probability, say $\theta = \theta(\gamma)$, that W_n does not exit F_n before time $\gamma\alpha_n$.

$$\mathbb{P}_x(\tau(X_n) \leq \gamma) \leq 1 - \theta \tag{4.30}$$

This is enough to give tightness.here is the idea:
Call $D_n(x)$ the square S of side 2.3^{-n} consisting of four squares with side length 3^{-n} and vertices in $3^{-n}\mathbb{Z}^2$ such that x is closest to the center of S. Call

$$\sigma_r^{X^n}(x) = \inf\{t \, : \, X_t^n \notin D_r(x)\} \tag{4.31}$$

Let $\epsilon > 0$. Choose m such that $(1-\theta)^m < \epsilon$. The inequality 4.30 and the self-similarity is used to obtain

$$\mathbb{P}_x(\sigma_{2m}^{X_n} < c_1(m)\gamma) \leq 1 - \theta \tag{4.32}$$

To exit F_n, X_n must cross at least m squares of the form $D_{2m}(y)$. So by the strong Markov property,

$$\mathbb{P}_x(\tau < c_1(m)\gamma) \leq (1-\theta)^m \leq \epsilon \tag{4.33}$$

and another scaling leads to

$$\mathbb{P}_x(\sigma_r^{X_n}(x) < c_2(r,m)\gamma) \leq \epsilon \tag{4.34}$$

which is the tightness estimate in the space of cad lag functions from $\mathbb{R}_+$ to F_0 (see [EK86] proposition 3.8.3, Lemma 3.8.1 and theorem 3.7.2). Let $\mathbb{P}_x^n$ denote the law of $X_n(t)$ started at x killed when exiting F_n. Since X is P_x^n-a.s. continuous, it follows from [EK86], theorem 10.2, that if Q is any limit point of $\{P_x^n, \, n \geq 1\}$ then X is Q-a.s. continuous. Thus $\{P_x^n, \, n \geq 1\}$ is tight in the space of continuous functions from R_+ to F_0.

Since $\{P_{n'}^x\}$ is tight, for each x, there exist convergent subsequences. Any limit point P_x satisfies $\mathbb{E}_x[\int_0^\infty e^{-\lambda t} f(X_t)dt] = U^\lambda f(x)$ for f bounded and continuous, from which one deduces that $\mathbb{P}_x^{n'}$ converges. If one calls the limit $\mathbb{P}_x$ and lets X_t be the canonical process on F, one then can show that $(\mathbb{P}_x, X_t)$ has the strong Markov property as well as the other required properties. It is then straightforward to extend $(\mathbb{P}_x, X_t)$ to a process on $\tilde{F}$. Since the expectation of the exit time of this process for F is less or equal to one, the process is not degenerate and since each X_n is invariant under isometries of F_n, then so is X_t (so X_t doesn't live on a straight line). Moreover by the tightness estimate, the paths of X_t are continuous under each P_x.

4.4.5.iii Upper and lower bounds

In the sequel the idea of the proof of the bounds for the reflected Brownian motion on the pre-Carpet in the theorem 4.4.1 will be given in the case $t \geq \max(1, |x-y|)$.

The upper bound Write

$$S = \inf\{t \geq 0 \, : \, |W_t - W_0| > \frac{1}{2}|x-y|\} \tag{4.35}$$

then one has by the continuity of q

$$q(t,x,y) \leq 2 \sup_{x'} \mathbb{P}_{x'}(S < \frac{t}{2}) \sup_{z} q(\frac{t}{2}, z, z) \tag{4.36}$$

Thus to prove the upper bound it is sufficient to obtain an estimate on the exit time S of the form

$$\mathbb{P}_x(S < \frac{t}{2}) \leq \exp(-c_4(\frac{|y-x|^{d_w}}{t})^{\frac{1}{d_w-1}}) \tag{4.37}$$

and show that for $x \in \tilde{F}_0$ and $t \geq 1$

$$q(t,x,y) \leq c_1 t^{-\frac{d_s}{2}} \tag{4.38}$$

The inequality 4.38 for $x = y$ is deduced from the inequality

$$q(t,x,y) \leq c_1 t^{-\frac{d}{2}}$$

for $0 < t \leq 1$ (obtained by dividing the space into cells, proving this local property for the process living into each cell and using estimate on the exit times of each cell and the strong Markov property) and extended to $x \neq y$ by using

$$q(t,x,y) \leq q(t,x,x)^{\frac{1}{2}} q(t,y,y)^{\frac{1}{2}}$$

The lower bound Using a standard chaining argument the lower bound can be proved once one has established the estimate

$$q(t,x,y) \geq c_9 t^{-\frac{d_s}{2}}, \quad |x-y| \leq c_{10} t^{\frac{1}{d_w}}, \ t \geq 1 \tag{4.39}$$

This estimate is deduced from the upper bound by the following way:
Write

$$A = \{y \ : \ q(\frac{t}{2}, x, y) > c_{19} t^{-\frac{d_s}{2}}\} \tag{4.40}$$

Then since

$$q(t,x,y) \geq c_{19} t^{-\frac{d_s}{2}} \mathbb{P}_y(W_{\frac{t}{2}} \in A) \tag{4.41}$$

it is sufficient to show that $\mathbb{P}_y(W_{\frac{t}{2}} \in A)$ is bounded below by a constant. To achieve this call T_C the coupling time of a Brownian motion W^x starting from x with an other W^y started from y. Then one has

$$\mathbb{P}_y(W_{\frac{t}{2}} \in A) \geq \mathbb{P}_x(W_{\frac{t}{2}} \in A) - \mathbb{P}(T_C > \frac{t}{2}) \tag{4.42}$$

Thus it is sufficient to show that $\mathbb{P}_x(W_{\frac{t}{2}} \in A)$ is bounded below by a constant and to control $\mathbb{P}(T_C > \frac{t}{2})$; the latter control on the coupling time is the hard point using the strong symmetry and isotropy of the carpet. The estimate on $\mathbb{P}_x(W_{\frac{t}{2}} \in A)$ can be obtained by proving

$$q(t,x,x) \geq c_{13} t^{-\frac{d_s}{2}}, \quad y \in \tilde{F}_0, \ t \geq 1 \tag{4.43}$$

And this is obtained from the upper bound by using the following inequality

$$q(t,x,x)\mu_0(D_r(x)) \geq [\mathbb{P}_x(W_{\frac{t}{2}} \in D_r(x))]^2 \tag{4.44}$$

where μ_0 is the Lebesgue measure on $\tilde{F}_0$ and $D_r(x)$ is a square set surrounding x whose side is optimized so that

$$\mathbb{P}_x(W_{\frac{t}{2}} \in D_r(x)) \geq \frac{1}{2} \quad \text{and} \quad \mu_0(D_r(x)) \leq c_{17} t^{\frac{d_s}{2}} \tag{4.45}$$

4.4.5.iv The spectral dimension

It is interesting to notice that the spectral dimension is defined by the behavior of th electrical resistance of the pre-Carpet. More precisely:
Define the resistance constant R_n by

$$R_n^{-1} = \inf\{\int_{3^n F_n} |\nabla f|^2 dx \, f = 0 \,\text{on}\, x_1 = 0,\ f = 1 \,\text{on}\, x_1 = 3^n\} \tag{4.46}$$

Thus R^n is the resistance between two opposite faces of the set $3^n F_n$. Then it is shown that there exists a constant

$$\rho_F > m_F^{\frac{2}{d}-1} \tag{4.47}$$

(here $m_F = 3^2 - 1 = 8$ is equal to the number of sub-squares generated minus the number of square taken out at each iteration of the construction process of the Sierpinski carpet) and constants c_1, c_2 such that

$$c_1 \rho_F^n \leq R_n \leq c_2 \rho_F^n \tag{4.48}$$

Then the spectral dimension is defined by

$$d_s = 2\frac{\ln m_F}{\ln m_F + \ln \rho_F} < d \tag{4.49}$$

and the dimension of the walk by

$$d_w = 2\frac{d_f}{d_w} \tag{4.50}$$

The latter relation between d_w and d_s might seem intriguing, it will be explained in the next section.

4.5 Fractional Diffusions

M.T. Barlow, in [Bar98](section 3), gives a conceptual framework for the study of diffusions on fractal media through the definition and characterization of fractional diffusions on fractional metric spaces. This point view is interesting because it allows seeing clearly the relationship between an anomalous control on the probability densities and the mean time to exit a ball. It also explains the relation between the spectral dimension d_s, the fractal dimension d_f and the "walk" dimension d_w (which is not always the dimension of the paths of the process).

4.5.1 Fractional metric space

A metric space (F, ρ) has the midpoint property if for each $x, y \in F$ there exists $z \in F$ such that

$$\rho(x, z) = \rho(z, y) = \frac{1}{2}\rho(x, y) \tag{4.51}$$

Definition 4.5.1. Let (F, ρ) be a complete metric space, and μ be a Borel measure on $(F, \mathcal{B}(F))$. (F, ρ, μ) is called a fractional metric space if (F, ρ) has the midpoint property and there exist $d_f > 0$, and constants c_1, c_2 such that if $r_0 = \sup\{\rho(x, y) : x, y \in F\} \in (0, \infty]$ is the diameter of F then

$$c_1 r^{d_f} \leq \mu(B(x, r)) \leq c_2 r^{d_f} \quad \text{for} \quad x \in F,\ 0 < r \leq r_0 \tag{4.52}$$

Here $B(x, r) = \{y \in F : \rho(x, y) < r\}$

From this definition it follows that

1. d_f is the Hausdorff dimension of F and the packing dimension of F
2. F is locally compact.
3. $d_f \geq 1$

4.5.2 Fractional diffusion

Definition 4.5.2. Let (F, ρ, μ) be a fractional metric space. A Markov process $X = (\mathbb{P}_x, x \in F, X_t, t \geq 0)$ is a fractional diffusion if

1. X is a conservative Feller diffusion with state space F.
2. X is μ-symmetric
3. X has a symmetric transition density $p(t, x, y) = p(t, y, x), t > 0, x, y \in F$, which satisfies, the Chapman-Kolmogorov equations and is, for each $t > 0$, jointly continuous.
4. There exist constant $\alpha, \beta, \gamma, c_1 - c_4, t_0 = r_0^\beta$, such that

$$\begin{aligned} c_1 t^{-\alpha} \exp\big(-c_2 \rho(x,y)^{\beta\gamma} t^{-\gamma}\big) &\leq p(t,x,y) \\ &\leq c_3 t^{-\alpha} \exp\big(-c_4 \rho(x,y)^{\beta\gamma} t^{-\gamma}\big),\ x, y \in F, 0 < t \leq t_0 \end{aligned} \tag{4.53}$$

This fractional diffusion will be written $FD(d_f, \alpha, \beta, \gamma)$

The Brownian motion on the Sierpinski carpet is a fractional diffusion with $\alpha = d_f(SC)/d_w(SC) = d_s(SC)/2$, $\beta = d_w(SC)$ and $\gamma = 1/(\beta - 1)$
An important property of a $FD(d_f, \alpha, \beta, \gamma)$ diffusion is that

$$\alpha = \frac{d_f}{\beta} \tag{4.54}$$

This equation is imposed by the fact that $p(t, x, .)$ is a probability density for all t and the growing rate of the 4.52 of the measure μ. This explains why the general relation

$$d_s = 2\frac{d_f}{d_w} \tag{4.55}$$

is satisfied for diffusions on fractals.

4.5.3 Connections

The estimates 4.53 have the following consequences on the large deviation properties of the $FD(d_f, d_f/\beta, \beta, \gamma)$ process X and its mean time to exit a ball..

1. For $t \in (0, t_0]$, $r > 0$

$$\mathbb{P}_x(\rho(x, X_t) > r) \leq c_1 \exp(-c_2 r^{\beta\gamma} t^{-\gamma}) \tag{4.56}$$

2. There exists $c_3 > 0$ such that

$$c_4 \exp(-c_5 r^{\beta\gamma} t^{-\gamma}) \leq \mathbb{P}_x(\rho(x, X_t) > r) \quad \text{for } r < c_3 r_0,\ t < r^\beta \tag{4.57}$$

3. For $x \in F$, $0 < r < c_3 r_0$, if $\tau(x, r) = \inf\{s > 0 : X_s \notin B(x, r)\}$ then

$$c_6 r^\beta \leq \mathbb{E}_x[\tau(x, r)] \leq c_7 r^\beta \tag{4.58}$$

Now the interesting part is that sufficient conditions for a process to be a fractional diffusion are isolated in the following theorem.

Theorem 4.5.1. *Let (F, ρ, μ) be a fractional metric space of dimension d_f. Let $(Y_t, t \geq 0, \mathbb{P}_x, x \in F)$ be a μ-symmetric diffusion on F which has a transition density $q(t, x, y)$ with respect to μ which is jointly continuous in x, y for each $t > 0$. Suppose that there exists a constant $\beta > 0$, such that*

$$q(t, x, y) \leq c_1 t^{-\frac{d_f}{\beta}} \quad \text{for all } x, y \in F, t \in (t, t_0] \tag{4.59}$$

$$q(t, x, y) \geq c_2 t^{-\frac{d_f}{\beta}} \quad \text{if } \rho(x, y) \leq c_3 t^{\frac{1}{\beta}}, t \in (t, t_0] \tag{4.60}$$

$$c_4 r^\beta \leq \mathbb{E}_x[\tau(x, r)] \leq c_5 r^\beta \quad \text{for } x \in F, 0 < r < c_6 r_0 \tag{4.61}$$

where $\tau(x, r) = \inf\{s > 0 : Y_s \notin B(x, r)\}$. Then $\beta > 1$ and Y is a fractional diffusion with parameters d_f, d_f/β, β and $1/(\beta - 1)$

Thus to obtain a fractional diffusion, it is sufficient to have a sharp pointwise estimate on the mean exit times and a control of the heat kernel near the diagonal. From the latter theorem one obtains that if X is a $FD(d_f, d_f/\beta, \beta, \gamma)$ then necessarily $\beta > 1$ and

$$\gamma = (\beta - 1)^{-1} \tag{4.62}$$

This means that the behavior a fractional diffusion is determined by only two parameters: the fractal dimension of the medium d_f and β with enters in the scaling between the mean time r^β to exit a ball of radius r. Actually

$$\beta = d_w \tag{4.63}$$

And β will now be written d_w.

4.5.4 Further properties

The estimates 4.53 on the probability densities allow to prove basic analytic and probabilistic properties of fractional diffusions.

- For $x \in F$, $t \geq 0$ and $p > 0$

$$\mathbb{E}_x[\rho(X_t, x)^p] \sim t^{\frac{p}{d_w}} \tag{4.64}$$

- (Modulus of continuity). Let $\varphi(t) = t^{\frac{1}{d_w}} (\ln(\frac{1}{t}))^{\frac{d_w - 1}{d_w}}$. Then a.s.

$$c_1 \leq \lim_{\delta \downarrow 0} \sup_{0 \leq s < t \leq 1; |t-s| < \delta} \frac{\rho(X_s, X_t)}{\varphi(t - s)} \leq c_2 \tag{4.65}$$

- (Law of the iterated logarithm). Let $\psi(t) = t^{1/d_w} (\ln \ln(1/t))^{(d_w - 1)/d_w}$. There exist c_1, c_2 and constants $c(x) \in [c_1, c_2]$ such that

$$\lim_{t \downarrow 0} \frac{\rho(X_t, X_0)}{\psi(t)} = c(x) \quad \mathbb{P}_x - a.s. \tag{4.66}$$

- (Dimension of range)

$$\dim_H \big(\{X_t : 0 \leq t \leq 1\}\big) = d_f \wedge d_w \tag{4.67}$$

  This result helps to explain the terminology "walk dimension" for d_w. Provided the space the diffusion X moves in is large enough, the dimension of the range of the process is d_w.

4.5.5 Potential theory of Fractional Diffusions

With the strong estimates on the transition densities it is possible to develop a potential theory for fractional diffusions and obtain a precise description of the diffusion and its associated generator. It is important to notice that once one has satisfied the conditions of the theorem 4.5.1, then all the other properties fall as in a domino game; even if sharp estimates are not obtained for the probability densities some part of the proof might be useful. Conversely for a particular medium an observation of those properties might give an idea whereas sharp estimates may be obtained on p (for instance, the Harnack inequality which express an isotropy of the diffusion is an interesting property).

- 1. If $d_s < 2$ then for each $x, y \in F$

$$\mathbb{P}_x(X \text{ hits } y) = 1 \tag{4.68}$$

  2. If $d_s \geq 2$ then points are polar for X
  3. If $d_s \leq 2$ then X is set-recurrent: for $\epsilon > 0$

$$\mathbb{P}_y(\{t : X_t \in B(y, \epsilon) \text{ is not empty and unbounded}\}) = 1 \tag{4.69}$$

  4. If $d_s > 2$ and $r_0 = \infty$ then X is transient.

- (Polar and non-polar sets). Let A be a Borel set in F.

  1. $\mathbb{P}_x(T_A < \infty) > 0$ if $\dim_H(A) > d_f - d_w$
  2. A is polar for X if $\dim_H(A) < d_f - d_w$

- X has k-multiple points if and only if $d_s < 2k/(k-1)$
- If $d_s < 2$ then X has jointly measurable local times $(L_t^x, x \in F, t \geq 0)$ which satisfy the density occupation formula with respect to μ

$$\int_0^t f(X_s)ds = \int_F f(a)L_t^a \mu(da), \quad f \text{ bounded and measurable} \tag{4.70}$$

- write

$$\tau_A = T_{A^c} = \inf\{t \geq 0 : X_t \in A^c\} \tag{4.71}$$

  Then for $d_s < 2$ and $r_0 = \infty$, there exist constant $c_1 > 1, c_2$ such that if $x, y \in F$, $r = \rho(x, y)$, $t_0 = r^{d_w}$ then

$$\mathbb{P}_x(T_y < t_0 < \tau(x, c_1 r)) \geq c_2 \tag{4.72}$$

- (Harnack inequality) If $d_s < 2$, $r_0 = \infty$; there exist constants $c_1 > 1, c_2 > 0$ such that if $x_0 \in F$, and $h \geq 0$ is harmonic in $B(x_0, c_1 r)$ with respect to the generator associated to X then

$$h(x) \geq c_2 h(y), \quad x, y \in B(x_0, c_1 r) \tag{4.73}$$

- (Spectral property) If $r_0 < \infty$ then there exist continuous functions φ_i and λ_i with $0 \leq \lambda_0 \leq \lambda_1 \leq \ldots$ such that for each $t > 0$

$$p(t, x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \tag{4.74}$$

  and the sum is uniformly convergent in $F \times F$
  Moreover if $N(\lambda) = \text{card}\{\lambda_k : \lambda_k \leq \lambda\})$ that there exist $c_1, c_2 > 0$ such that:

$$c_1 \lambda^{\frac{d_s}{2}} \leq N(\lambda) \leq c_2 \lambda^{\frac{d_s}{2}} \tag{4.75}$$

  which explains the term spectral dimension for d_s.

4.6 A note on turbulence

4.6.1 Turbulence and anomalous diffusion

It heuristically known that diffusion in a fully developed turbulence in incompressible flows will be super-diffusive (between given time scales). It is interesting to note that the expected key to this anomaly is a large number of scales of mixing length and convection rolls. Thus although Brownian motion in Fractals and diffusion in turbulent incompressible flows give rise to different anomalies, the heart of those problems might be close and the understanding of one phenomenon might be beneficial to the other.

Several papers are available on this subject mainly in physics literature (they describe the turbulent convection phenomenon and the flow is seen as fractal convection cells): see M. Avellaneda and A. Majda [AM90]; J. Glimm and Al. [FGLP90], [FGL$^+$91], [GLPP92], J. Glimm and Q. Zhang [GZ92], Q. Zhang [Zha92], M.B. Isichenko and J. Kalda [IK91]. For mathematical literature see A. Fannjiang and G.C. Papanicolaou [FP94],[FP96]; M. Avellaneda [Ave96]; and A. Fannjiang [Fan99] (note that the latter papers appeared in a mathematical literature and homogenization is expected to play a key role).

4.6.2 Navier-Stokes equations

The Naiver-Stokes equations (assume $d = 3$) for an unknown velocity vector $u(x,t) = \big(u_i(x,t)\big)_{1\leq i\leq d}$ are given by

- fundamental Newton law of the dynamic:

$$\frac{\partial}{\partial t}u_i = \nu\Delta u_i - u.\nabla u_i - \frac{1}{\rho}\frac{\partial}{\partial x_i}p + f_i(x,t) \tag{4.76}$$

- Incompressibility

$$\nabla.u = 0 \tag{4.77}$$

- Equation of the internal energy

$$\frac{\partial}{\partial t}e = -u.\nabla e + 2\nu\bar{D} : \bar{D} + \frac{K}{\rho}\Delta T \tag{4.78}$$

- State equation

$$e = e(T) \tag{4.79}$$

$f_i(x,t)$ are the components of a given externally applied force.
ν is the viscosity (a positive coefficient) it has the dimension of "lenght^2 $\times$ time".
$p(x,t)$ is the pressure.
$\bar{D}$ is the rate of deformation $d\times d$ tensor given by ∇u. e is the specific internal energy, it has the dimensions of "$\frac{\text{energy}}{\text{mass}}$".
In the equation 4.78 the term $2\nu\bar{D} : \bar{D}$ is responsible for the transformation of kinetic energy into internal energy (heat), more precisely it is the quantity of kinetic energy transformed into heat per unit time in a unit mass of fluid. This term is usually written ϵ in the literature and called the energy dissipation rate.

4.6.3 A brief reminder on the history

This subsection is taken almost verbatim from [Her95] which is a good introduction to turbulence (this recent PhD thesis is clear and cover the theory from an experimental point of view) see also [MK99] for a recent survey on the subject.

4.6.3.i Turbulence

In many industrial processes turbulence is a common phenomenon. However, in spite of its familiar appearance and research for many years, there is still no good description of turbulence. There is not even a good definition, and researchers try to describe it in one sentence: "Turbulence is the disordered behavior of fluid in space and time "
in [LL84], turbulence is defined as the region where the Navier-Stokes equations become unstable with respect to small perturbations.
In [Les93], M. Lesieur start with an enumeration of features that a flow has to satisfy in order to be classified as turbulent. So, a turbulent flow is or exhibits:

- Irregularity or randomness: the turbulent flow is unpredictable.
- Diffusivity, which causes rapid mixing and increased rates of momentum, heat and mass transfer.
- High Reynolds number: turbulent flows always occur at high Reynolds numbers.
- Three-dimensional vorticity fluctuations: turbulence is rotational and three-dimensional.
- Dissipation: turbulent flows are always dissipative.
- Continuum: turbulence is a continuous phenomenon.
- Flows: turbulence is a feature of fluid flows and not of fluids.

Customary, turbulent flows are classified in terms of the Reynolds number, which is a dimensionless characteristic parameter of the flow. If the Reynolds number is not too large the flow will be laminar, i.e. the variations in the flow are predictable in both space and time. As the Reynolds number increases, the flow becomes unstable, and at some large enough value of the Reynolds number it becomes fully turbulent.
Turbulence may well be the last unsolved problem of classical physics. There is still no theory of fully developed turbulence that is universally valid. Cascades of energy and momentum are key concepts of turbulence and describe the generation of motion on small scales out of large-scale motion. A well-known phenomenological theory for the statistics of small-scale motion was formulated in 1941 by Kolmogorov, based on the idea of cascades. Only in recent years it has become clear that this theory needs essential corrections.
Hence, turbulence is often referred to as the unsolved problem of classical physics and is often the item of discussions at conferences. Lesieur [Les93] divides the scientists involved with turbulence into two groups with opposing points of view: The first group, the statistical and oldest one, tries to model the flow in averaged quantities. This group follows Kolmogorov and believes in the phenomenology of cascades and (strongly) denies the possibility of any coherence or order in turbulence. The second group believes in the coherence among chaos and considers turbulence from a purely deterministic point of view by studying either the behavior of dynamical systems or the stability of flows in various situations.

4.6.3.ii Energy cascade modeling

In case of cascade modeling, Richardson [Ric22] was the first who put forward some ideas on the theory of fully developed turbulence. He assumed a hierarchy of turbulent disturbances on different scales. 'Eddies' of a certain scale would be the result of the instability of larger 'eddies' at a larger scale. In his scenario, Richardson assumed a cascade process of eddies breaking down. In this cascade process there is a transmission of energy of the flow motion of smaller and smaller eddies down to the smallest scale, where the fragmentation process is stopped by dissipation. expressed this idea in the following rhyme ([Her95]), based on Jonathan Swift's fleas sonnet.

> Big whorls have little whorls
> Which feed on their velocity
> And little whorls have lesser whorls
> And so on to viscosity
> (In the molecular sense)

Kolmogorov further developed and formulated the ideas of Richardson in his papers in 1941 [Kol41b], [Kol41a]. He assumed an inertial range in which energy was transported from large eddies to smaller eddies. This range of scales is bounded from above by the size of the eddies at which energy is injected and from below by the size of the eddies where flow kinetic energy is dissipated to heat. Kolmogorov assumed a uniform energy distribution over all eddies.

Since then many researchers proposed ideas and models to describe the statistical (cascade) behavior of fully developed turbulence. Landau was the first to point out the presence of intermittency which leads to a contradiction with the Kolmogorov theory of 1941. Landau stated that the statistical laws of small eddies have to depend not only on the mean energy dissipation but also on the fluctuations of this energy: the Kolmogorov model did not take into account intermittency. By intermittency Landau meant that turbulence is not uniformly distributed in space and there are regions with less intense and regions with more intense turbulence. In 1962 Kolmogorov [Kol62] and Oboukhov [Obo62] derived the so called log-normal model. They assumed that the logarithm of the energy distribution in the inertial range, was Gaussian.

4.6.3.iii Fractal Models

Many measurements were contradicting these assumptions and in 1964 Novikov and Stewart [NS64] proposed a model called absolute curdling. In this model it was assumed that the energy of an eddy is distributed over curds of smaller eddies. In 1974 Mandelbrot [Man76] came with an expansion on the model, his weighted curdling model, where it was assumed that a weighted amount of the energy of a large (mother) eddy is distributed over the curds of smaller (daughter) eddies. The absolute curdling model was reformulated by Frisch, stressing its dynamical and fractal aspects, as the β-model. In this model the daughter eddies occupy a constant fraction of its mother's space. The multifractal model (random β-model) was also introduced by Frisch and Parisi in 1983 [FP85], as a variation on the weighted curdling model and a possible interpretation of the inertial range. Now it was assumed that a random fraction of the mother eddy's space is occupied by its daughter eddies.

In the last years many realizations, interpretations, and variants of the multifractal model have been developed. In these models the structure function is a significant quantity. It is the equivalent of the correlation function of fluctuating velocities in two different spatial points. Structure functions can be measured and their scaling behavior appears to be useful to compare the predictions of these different models with experimental results. Measurements have shown that the scaling exponents of the structure function depends on the order of the structure function in a nonlinear way. The multifractal model seems to give a good description of the turbulence cascade.

Thus fractal models are an attempt to capture intermittency in a geometric framework. The key idea is that self-similar cascades do not need to be space filling. Such a process is characterized by a fractal dimension, even by a continuous dimension function; they provide improved agreement with the observed scale dependent dispersivities in the field data and the observed geological heterogeneities on all length scales.
Finally, from an experimental point of view the small-scale structure of turbulence shows a clear anomalous scaling behavior and multifractal models are useful in attempts to try to understand and describe the scaling behavior of turbulence. However, no universal model exists yet and turbulence still remains an unsolved problem.

4.6.4 A brief reminder on the theory of fully developed turbulence

This subsection is almost verbatim taken from the course of Landau-Lifchitz on Fluid mechanics [LL84] (p. 129). This text has been reproduced here to act as basis of comparison for the physical interpretation of the supper diffusive model given in this thesis in the chapter 7. Of course it does not contain the most recent progresses on the subject (for this see [Her95] and [TuS91] for instance) however it will be sufficient to understand the physical interpretation given in the chapter 7.

4.6.4.i Nature of the irregular motion

The fluid is evolving in a medium with principal (exterior) length scale l. The characteristic variation of the velocity is written δv, then the Reynolds number is

$$Re \sim \frac{\delta u \, l}{\nu} \tag{4.80}$$

Turbulent flow at fairly large Reynolds number is characterized by the presence of an extremely irregular disordered variation of the velocity with time at each point. This is called "fully developed turbulence". The velocity continually fluctuates about some mean value. A similar irregular variation of the velocity exists between points in the flow at a given distance.
Introduce the concept of the mean velocity, obtained by averaging over long time intervals the actual velocity at each point. By such averaging, the irregular variation of the velocity is smoothed out, and the mean velocity varies smoothly from point to point. In what follows, denote the mean velocity by u, the difference $v' = v - u$ between the true velocity and the mean velocity varies irregularly in the manner characteristic of turbulence; call it the fluctuating part of the velocity.

Consider in more details the nature of this irregular motion that is superposed on the mean flow. This motion may in turn be qualitatively regarded as the superposition of "turbulent eddies" of different sizes; by the size of an eddy it is meant the order of magnitude of the distances over which the velocity varies appreciably. As the Reynolds number increases, large eddies appear first; the smaller the eddies, the later they appear. For very large Reynolds numbers, eddies of every size from the largest to the smallest are present. An important part in any turbulent flow is played by the largest eddies, whose size (the fundamental or external scale of turbulence) is of the order of the dimensions of the region in which the flow takes place; in what follows denote l this order of magnitude for any given turbulent flow. These large eddies have the largest amplitudes. The velocity in them is comparable with the variation of the mean velocity over the distance l; denote by δu the order of magnitude of this variation (the order of magnitude, not of the mean velocity itself, but of its variation, since it is this variation δu which characterizes the velocity of the turbulent flow). The mean velocity itself can have any magnitude, depending on the frame of reference used (it seems that in fact the size of the largest eddies is actually somewhat less than l, and their velocity is somewhat less than δu). The frequencies corresponding to these eddies are of order of u/l, the ratio of the mean velocity u (and not its variation δu) to the dimension l. For the frequency determines the period with which the flow pattern is repeated when observed in some fixed frame of reference. Relative to such a frame, however, the whole pattern moves with the fluid at a velocity of order u.

The small eddies, on the other hand, which correspond to large frequencies, participate in the turbulent flow with much smaller amplitudes. They may be regarded as a fine detailed structure superposed on the fundamental large turbulent eddies. Only a comparatively small part of the total kinetic energy of the fluid resides in the small eddies.

From the picture of turbulent flow given above, one can draw a conclusion regarding the manner of variation of the fluctuating velocity from point to point at any given instant. Over large distances (comparable with l), the variation of the fluctuating velocity is given by the variation in the velocity of the large eddies, and is therefore comparable with δu. Over small distances (compared with l), it is determined by the small eddies, and is therefore small compared with δu (but large compared with the variation of the velocity with time at any given point). Over short time intervals (compared with $t \sim l/u$), the velocity does not vary appreciably; over long intervals, it varies by a quantity of the order of δu.

4.6.4.ii Energy dissipation

The length l appears as a characteristic dimension in the Reynolds number Re, which determines the properties of a given flow. Besides this Reynolds number, one can introduce the qualitative concept of the Reynolds number for turbulent eddies of various size. If λ is the order of magnitude of the size of a given eddy, and v_λ the order of magnitude of its velocity, then the corresponding Reynolds number is defined as $Re_\lambda \sim v_\lambda \lambda/\nu$. This number decreases with the size of the eddy.

For large Reynolds number Re, the Reynolds numbers Re_λ of the large eddies are also large. Large Reynolds numbers, however, are equivalent to small viscosities. One therefore concludes that, for large eddies which are the basis of any turbulent flow, the viscosity is unimportant. It follows from this that there is no appreciable dissipation of energy in the large eddies.

The viscosity of the fluid becomes important only for the smallest eddies, whose Reynolds number is comparable with unity. Denote the size of the eddies by λ_0,which shall be determined later in this section. It is in these small eddies, which are unimportant as regards the general pattern of a turbulent flow, that the energy dissipation occurs.
Thus one is lead to the following conception of energy dissipation in turbulent flow (L. Richardson 1922). The energy passes from the large eddies to smaller ones, practically no dissipation occuring in this process. One says that there is a continuous flow of energy from large to small eddies,i.e. from small to large frequencies. This flow of energy is dissipated in the smallest eddies, where the kinetic energy is transformed into heat. For a steady state to be maintened, it is of course necessary that external energy sources should be present which continually supply energy to large eddies.

Since the viscosity of the fluid is important only for the smallest eddies, one may say that none of the quantities pertaining to eddies of size $\lambda >> \lambda_0$ can depend on ν (more exactly, these quantities cannot be changed if ν varies but the other condition of the motion are unchanged). These circumstances reduces the number of quantities which determine the properties of turbulent flow, and the result is that similarity arguments, involving the dimensions of the available quantities, become very important in the investigation of turbulence.

Apply these arguments to determine the order of magnitude of the energy dissipation in turbulent flow. Let ϵ be the mean dissipation of energy per unit time per unit mass of the fluid. It has been shown that this energy is derived from the large eddies, whence it is gradually transferred to smaller eddies until it is dissipated in eddies of size $\sim \lambda_0$. Hence, although the dissipation is ultimately due to the viscosity, the order of magnitude of ϵ can be determined only by those quantities which characterize the large eddies. These are the fluid density ρ, the dimension l and the velocity δu. From these three quantities one can form only one having the dimensions of ϵ, namely $\frac{\text{"energy"}}{\text{"mass"} \times \text{"time"}}$. Thus one find

$$\epsilon \sim \frac{(\delta u)^3}{l} \tag{4.81}$$

and this determines the order of magnitude of the energy dissipation in turbulent flow.

In some respects a fluid in a turbulent motion may be qualitatively described as having a "turbulent viscosity" ν_{turb} which differs from the true kinematic viscosity ν. Since ν_{turb} characterizes the properties of the turbulent flow, its order of magnitude must be determined by ρ, δu and l. The only quantity that can be formed from these and has the dimensions of kinematic viscosity $l\delta u$, and therefore

$$\nu_{turb} \sim l\,\delta u \tag{4.82}$$

The ratio of the turbulent viscosity to the ordinary viscosity is consequently

$$\frac{\nu_{turb}}{\nu} \sim Re \tag{4.83}$$

i.e. it increases with the Reynolds number (in reality, however, a fairly large numerical coefficient should be included. This is because, as mentioned above, l and δu may differ quite considerably from the actual scale and velocity of the turbulent flow. The ratio ν_{turb}/ν may be more accurately written $\nu_{turb}/\nu \sim Re/Re_{cr}$, which formula takes into account the fact that ν_{turb} and ν must in reality be comparable in magnitude not for $Re \sim 1$, but for $Re \sim Re_{cr}$).

4.6.4.iii Kolmogorov-Obukhov's law

Now determine the order of magnitude v_λ of the turbulent velocity variation over distances of the order of λ. It must be determined only by ϵ and, of course, the distance λ itself. From these two quantities, one can form only one having the dimensions of velocity, namely $(\epsilon\lambda)^{\frac{1}{3}}$. Hence one can say that the relation

$$v_\lambda \propto (\epsilon\lambda)^{\frac{1}{3}} \tag{4.84}$$

must hold. Thus one found that the velocity variation over a small distance is proportional to the cube root of the distance (Kolmogorov and Obukhov's law). The quantity v_λ may also be regarded as the velocity of turbulent eddies whose size is of order of λ: the variation of the mean velocity over small distances is small compared with the variation of the fluctuating velocity over those distances, and may be neglected.

The relation 4.84 may be obtained in another way by expressing a constant quantity, the dissipation ϵ, in terms of quantities characterizing the eddies of size λ; ϵ must be proportional to the squared gradient of velocity v_λ and to the appropriate turbulent viscosity coefficient $\nu_{turb,\lambda} \propto v_\lambda\lambda$:

$$\epsilon \propto \nu_{turb,\lambda}(v_\lambda/\lambda)^2 \propto v_\lambda^3/\lambda \tag{4.85}$$

whence one obtain 4.84.

Now put the problem somewhat differently, and determine the order of magnitude v_τ of the velocity variation at a given point over a time interval τ which is short compared with the time $t \sim l/u$ characterizing the flow as a whole. To do this, notice that, since there is a net mean flow, any given portion of the fluid is displaced, during the interval τ, over a distance of order τu, u being the mean velocity. Hence the portion of fluid which is at a given point at time τ will have been at a distance τu from that point at the initial instant. One can therefore obtain the required quantity v_τ by direct substitution of τu for λ in 4.84:

$$v_\tau \propto (\epsilon\tau u)^{\frac{1}{3}} \tag{4.86}$$

The quantity v_τ must be distinguished from v'_τ, the variation in velocity of a portion of fluid as it moves about. This variation can evidently depend only on ϵ, which determines the local properties of the turbulence, and of course on τ itself. Forming the only combination of ϵ and τ that has the dimensions of velocity, one obtains

$$v'_\tau \propto (\epsilon\tau)^{\frac{1}{2}} \tag{4.87}$$

Unlike the velocity variation at a given point, it is proportional to the square root of τ, not to the cube root. It is easy to see that, for τ small compared with t, v'_τ is always less than v_τ (the inequality $v'_\tau << v\tau$ has in essence been assumed in the derivation of 4.86).

Using the expression 4.81 for ϵ, one can rewrite 4.85 and 4.86 as

$$\begin{cases} v_\lambda \propto \delta u(\frac{\lambda}{l})^{\frac{1}{3}} \\ v_\tau \propto \delta u(\frac{\tau}{t})^{\frac{1}{3}} \end{cases} \tag{4.88}$$

This form shows clearly this similarity property of local turbulence: the small-scale characteristics of different turbulent flows are the same apart from the scale of measurement of lengths and velocities (or, equivalently, lengths and times).
Now find at what distances the fluid viscosity begins to be important. These distances λ_0 also determine the order of magnitude of the size of the smallest eddies in the turbulent flow (called the "internal scale" of the turbulence, in contradiction to the "external scale" l). To determine λ_0, form the local Reynolds number $Re_\lambda \sim v_\lambda \lambda/\nu \sim \delta u.\lambda^{\frac{4}{3}}/\nu l^{\frac{1}{3}} \sim Re(\lambda/l)^{\frac{4}{3}}$, with the Reynolds number $Re \sim l\delta u/\nu$ for the flow as a whole. The order of magnitude of λ_0 is that for which $Re_{\lambda_0} \sim 1$. Hence one find

$$\lambda_0 \sim l/Re^{\frac{3}{4}} \tag{4.89}$$

The same expression can be obtained by forming from ϵ and ν the only combination having the dimensions of length, namely

$$\lambda_0 \sim (\frac{\nu^3}{\epsilon})^{\frac{1}{4}} \tag{4.90}$$

Thus the internal scale of the turbulence decreases rapidly with increasing Re. For the corresponding velocity one have

$$v_{\lambda_0} \sim \frac{\delta u}{R^{\frac{1}{4}}} \tag{4.91}$$

this also decreases when Re increases (formulae 4.89-4.91 give the manner of variation of the relevant quantities with Re. Quantitatively, it would be more correct to replace Re in terms of Re/Re_{cr}).

The range scale $\lambda \sim l$ is called the energy range; the majority of the kinetic energy of the fluid is concentrated there. Values $\lambda \leq \lambda_0$ form the dissipation range, where the kinetic energy is dissipated. For very large values of Re, these two ranges are quite far apart, and between them lies the inertial range, in which $\lambda_0 << \lambda << l$; the results derived in this section are valid there.

Kolmogorov and Obukhov's law can be expressed in an equivalent spatial spectrum form. Replace the scales λ by corresponding wave numbers $k \sim 1/\lambda$ of the eddies; let $E(k)dk$ be the kinetic energy per unit mass of fluid in eddies with k values in the range dk; let $E(k)dk$ be the kinetic energy per unit mass of fluid in eddies with k values in the range dk. The function $E(k)$ has the dimensions (lenght)^3/(time)^2; the combinations of ϵ and k having these dimensions gives

$$E(k) \propto \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{4.92}$$

The equivalence of this expression and 4.84 is easily seen by noting that v_λ^2 gives the order of magnitude of the total energy in eddies with all scales of the order of λ or less. The same result is reached by integration of 4.92:

$$\int_k^\infty E(k)dk \propto \frac{\epsilon^{\frac{2}{3}}}{k^{\frac{2}{3}}} \sim (\epsilon\lambda)^{\frac{2}{3}} \sim v_\lambda^2 \tag{4.93}$$

Together with the spatial scales of the turbulent eddies, one can consider their time characteristics (frequencies). The lower end of the of the frequency spectrum of turbulent motion is at frequencies $\sim u/l$. The upper end is

$$\omega_0 \sim u/\lambda_0 \sim uRe^{\frac{3}{4}}/l \tag{4.94}$$

corresponding to the internal scale of turbulence. The inertial range corresponds to frequencies

$$u/l << \omega(\sim u/l)Re^{\frac{3}{4}} \tag{4.95}$$

The inequality $\omega >> u/l$ signifies that as regards the local properties of turbulence the unperturbed flow may be treated as steady. The energy distribution in the frequency spectrum in the inertial range is found from 4.92 by substituting $k \sim \omega/u$:

$$E(\omega) \propto (u\epsilon)^{\frac{2}{3}}\omega^{-\frac{5}{3}} \tag{4.96}$$

where $E(\omega)d\omega$ is the energy in the frequency range $d\omega$. The frequency ω gives the time repetition period in the region of space concerned, as observed from a fixed frame of reference. It is to be distinguished from the frequency ω' which gives the repetition period in a given portion of fluid moving in space. The energy distribution in this frequency spectrum cannot depend on u, and must be determined only by ϵ and the frequency ω' itself. Again using dimensional arguments, one finds

$$E(\omega') \propto \frac{\epsilon}{\omega'^2} \tag{4.97}$$

This is the same relationship to 4.96 as 4.97 is to 4.96.

Turbulence mixing causes a gradual separation of fluid particles that were originally close together. Consider two particles at a distance λ that is small (in the inertial range). Again, by dimensional arguments, the rate change of this distance with time is

$$\frac{d\lambda}{dt} \propto (\epsilon\lambda)^{\frac{1}{3}} \tag{4.98}$$

Integration of this shows that the time over which two particles initially at a distance λ_1, move apart to a distance $\lambda_2 >> \lambda_1$ is in order of magnitude

$$\tau \sim \frac{\lambda_2^{\frac{4}{3}}}{\epsilon^{\frac{1}{3}}} \tag{4.99}$$

Note that the process is self-accelerating: the rate of separation increases with λ. This occurs because only eddies with scale $\leq \lambda$ contribute to the separation of particles at a distance λ; the large eddies carry both particles and do not cause them to separate (these results can be applied to particles suspended in the fluid, which are passively conveyed by its motion).

Finally, consider the properties of the flow in regions whose dimension λ is small compared with λ_0. In such regions the flow is regular and its velocity varies smoothly. Hence one can expand v_λ in a series of powers of λ and retaining only the first term, obtain $v_\lambda =$ constant $\times\ \lambda$. The order of magnitude of the constant is v_{λ_0}/λ_0, since for $\lambda \sim \lambda_0$ one must have $v_\lambda \sim v_{\lambda_0}$. Thus

$$v_\lambda \sim v_{\lambda_0}\lambda/\lambda_0 \sim \delta u.Re^{\frac{1}{2}}\lambda/l \tag{4.100}$$

This formula may also be obtained by equating two expressions for the energy dissipation ϵ: the expression $(\delta u)^3/l$, which determines ϵ in terms of quantities characterizing the large eddies, and the expression $\nu(v_\lambda/\lambda)^2$, which determines ϵ in terms of the velocity gradient for the eddies in which the energy dissipation actually occurs.

4.6.5 Super diffusion in Turbulence: the history

See [Isi92] for this part. In 1926, Richardson ([Ric26]) analyzed available experimental data on diffusion in air. Those data varied about 12 orders of magnitude. On that basis, Richardson phenomenologically conjectured that the diffusion coefficient D_λ in turbulent air depend on the scale length λ of the measurement. The Richardson law,

$$D_\lambda \propto \lambda^{\frac{4}{3}} \tag{4.101}$$

was related to Kolmogorov-Obukhov turbulence spectrum, $v \propto \lambda^{\frac{1}{3}}$, by Batchelor [Bat52]. The supper diffusion law of the root-mean-square relative displacement $\lambda(t)$ of advected particles

$$\lambda(t) \propto (D_{\lambda(t)}t)^{\frac{1}{2}} \propto t^{\frac{3}{2}} \tag{4.102}$$

was derived by Obukhov [Obu41] from a dimensional analysis similar to the one that led Kolmogorov [Kol41b] to the $\lambda^{\frac{1}{3}}$ velocity spectrum.

5. HOMOGENIZATION

5.1 A reminder about the theory

This section is mainly based on [BLP78], [JKO91] and [Oll94].

5.1.1 The simplest example - Diffusion in a Periodic Potential Field

Let U be a smooth function on $\mathbb{R}^d$, periodic of period one. So $U \in C^\infty(T_1^d)$, where $T_1^d)$ is the d dimensional torus of side 1.
Let $m_U(dx)$ be the measure

$$m_U(dx) = \frac{\exp(-2U)dx}{\int_{T_1^d} \exp(-2U)} \tag{5.1}$$

This measure is the invariant measure associated to the operator

$$L_U = \frac{1}{2}\Delta - \nabla U.\nabla = \frac{1}{2}e^{2U}\nabla(e^{-2U}\nabla) \tag{5.2}$$

which is symmetric with respect to the probability measure m_U on T_1^d. The Stochastic Differential Equation associated to this operator (generator) is:

$$\begin{cases} dy_t = d\omega_t - \nabla U(y_t)dt \\ y_0 = x \end{cases} \tag{5.3}$$

where ω is a standard Brownian motion in $\mathbb{R}^d$
One can wonder what is the behavior of y_t for large t ?
The answer is a Central Limit Theorem given by homogenization.
Define for $\epsilon > 0$

$$y^\epsilon = \epsilon y(\frac{t}{\epsilon^2})$$

Then y^ϵ converges in law to a Brownian with Effective Diffusivity $D(U)$ (for which a formula will be given).
The proof of this fact is interesting because it will allows to introduce the fundamental tools that will be useful in the following chapters.

5.1.2 Spectral Gap

The analytical tool that will allow homogenization to take place is the existence of a spectral gap for the operator L_U defined on the torus T_1^d (in the sense of its closure in $L^2(m_U)$).
Indeed the self-adjoint (with respect to m_U) form 5.1.1 of the operator L_U shows that defined as an evolution operator it is contractive. Since L_U is contractive it has a real negative spectrum.
Define λ_0 as the gap in the spectrum of L,

$$\lambda_0 = \inf\left\{\lambda > 0 \,:\, -\lambda \in \text{spec}(L)\right\}$$

then Poincaré inequality shows that this gap is strictly positive.

$$\lambda_0 = \inf_{\phi \in C^\infty(T_1^d)\,:\,\int_{T_1^d} \phi dm_U = 0} \frac{1}{2} \frac{\int_{T_1^d} (\nabla \phi)^2 dm_U}{\int_{T_1^d} \phi^2 dm_U} > 0$$

Because of the strict positivity of the spectral gap the convergence of the transition density probability associated to the diffusion y_t seen as a diffusion on the torus towards the equilibrium measure is exponential and its speed is controlled by λ_0.
Indeed there exist $C > 0$ such that for any function $f \in L^2(m_U)$ on the torus and $t \geq 1$

$$sup_{x \in T_1^d} \big| \mathbb{E}_x[f(y_t)] - \int_{T_1^d} f dm_U \big| \leq C \exp(-\lambda_0 t) \|f\|_{L^2}^2$$

This exponential speed of convergence shows that, given a bounded function ϕ on the torus such that its mean value with respect to the invariant measure is 0:

$$\int_{T_1^d} \phi dm_U = 0$$

there exists a bounded function ψ on the torus (periodic) unique up to a constant solution of

$$L_U \psi = \phi$$

The idea of the probabilistic proof is to write ψ as the limit of a sequence of functions ψ^α solution of $(L_u - \alpha)\psi = \phi$

$$\psi^\alpha = \int_0^\infty \exp(-\alpha t) \mathbb{E}_x[\phi(y_t)] dt$$

and this sequence remains bounded as α converges towards 0 (because of the spectral gap).

5.1.3 The Cell problem

For l unit vector in $\mathbb{R}^d$, define χ_l by the Poisson equation (on the Torus)

$$L_U \chi_l = -l.\nabla U \tag{5.4}$$

$\chi_l \in C(T_1^d)$ (χ is periodic) is called "the solution of the Cell Problem" and is here smooth and bounded.
The important fact to notice is that a periodic solution to the Poisson equation 5.4 can be found because the average of the drift ∇U with regards to the invariant measure m_U is null.

$$\int_{T_1^d} \nabla U \, dm_U = 0$$

χ_l is defined up to a constant, so it will often be assumed that $\chi_l(0) = 0$ to ensure its unicity. It is also important to notice the χ_l depends linearly on the unit vector l

$$\chi_l = \sum_{i=1}^d \chi_{e_i} l_i \tag{5.5}$$

Where (e_i) is an orthonormal basis of $\mathbb{R}^d$ and (l_i) are the coordinates of l on this basis. This linear dependence allows to define the vector $\chi_.$ and the matrix $\nabla \chi_.$ as intrinsic objects whose components on this basis are

$$(\chi_.)_j = \chi_{e_j} \qquad (\nabla \chi_.)_{i,j} = \partial_i \chi_{e_j} \tag{5.6}$$

This will allow to use the following intrinsic notation:

$$\chi_l = \chi_. . l \tag{5.7}$$

5.1.3.i Utilization of the cell problem

The solution of the cell problem allows to replace the drift of y_t^ϵ and express it as the sum of a martingale and a bounded term. Indeed by Ito calculus (Here $y(0) = 0$)

$$y^\epsilon(t).l = -\epsilon \int_0^{\frac{t}{\epsilon^2}} l.\nabla U(y_s)ds + \epsilon\omega_{\frac{t}{\epsilon^2}}$$

$$\chi_l(y_t) - \chi_l(y_0) = \int_0^t \nabla\chi_l(y_s)\, d\omega_s - \int_0^t l.\nabla U(y_s)ds$$

So that:

$$\begin{aligned} y^\epsilon.l &= \epsilon \int_0^{\frac{t}{\epsilon^2}} (l - \nabla\chi_l)(y_s)d\omega_s + \epsilon\big[\chi_l(y_{\frac{t}{\epsilon^2}}) - \chi_l(0)\big] \\ &= \epsilon M^l_{\frac{t}{\epsilon^2}} + \epsilon(\text{bounded term}) \end{aligned}$$

When ϵ converges towards 0 the ϵ(bounded term) disappears and the martingale $\epsilon M^l_{\frac{t}{\epsilon^2}}$ is characterized by the convergence of its quadratic variation. Indeed

$$\begin{aligned} \big\langle \epsilon M^l_{\frac{t}{\epsilon^2}} \big\rangle &= \epsilon^2 \int_0^{\frac{t}{\epsilon^2}} |l - \nabla\chi_l|^2(y_s)ds \\ &\xrightarrow{\text{a.e.}} tD_l(U) \end{aligned} \tag{5.8}$$

The last limit is obtained by ergodicity with

$$D_l(U) = \int_{T_1^d} |l - \nabla\chi_l|^2 dm_U$$

$D_l(U)$ is a quadratic form in l Define $D(U)$ the effective diffusivity (or homogenized matrix) associated to the potential U by

$$l.D(U).l = D_l = \int_{T_1^d} |l - \nabla\chi_l|^2 dm_U \tag{5.9}$$

5.1.4 Convergence of the quadratic variation, towards equilibrium

In the sub-section 5.4 the limit 5.1.3.i of the quadratic variation has been obtained trough ergodicity; it is interesting to give an alternative and more precise proof of the existence of this limit in order to be able to control the speed of convergence.
The key tool here is still the spectral gap. Indeed, since the mean of the function $|l - \nabla\chi_l|^2 - D_l$ with respect to m_U is zero. A periodic solution Φ_l to the following Poisson equation can be found.

$$L_U\Phi_l = |l - \nabla\chi_l|^2 - D_l \tag{5.10}$$

Here Φ_l is smooth and bounded and to ensure its unicity it will be assumed that $\Phi(0) = 0$
Then by Ito formula

$$\int_s^t |l - \nabla\chi_l|^2(y_z)dz = D_l(t - s) + \Phi_l(y_t) - \Phi_l(y_s) - \int_s^t \nabla\Phi_l(y_z)d\omega_z \tag{5.11}$$

This expression allows to prove that the quadratic variation of the difference martingale associated to the cell problem

$$M^\epsilon_{s,t} = \epsilon M^l_{\frac{t}{\epsilon^2}} - \epsilon M^l_{\frac{s}{\epsilon^2}}$$

converges a.e. towards $(t-s)D_l$
This convergence for all t and s and the Markov property of M_t^ϵ are sufficient to prove that the finite dimensional distributions of the continuous martingale $\epsilon M^l_{\frac{t}{\epsilon^2}}$ converges to those of a Brownian motion with effective diffusivity D_l. And the continuity of the path of the limit process is proven by the following compactness criterion

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\mathbb{P}\big(\sup_{|t-s|\le\delta\; 0<s<t<T}|M^\epsilon_{s,t}|\ge R\big)=0$$

which can be satisfied because the quadratic variation of the martingale $M^\epsilon_{s,t}$ are bounded by Constant.$(t-s)$

Thus y^ϵ converges in law to a Brownian Motion b with Effective Diffusivity $D(U)$ and transition probability density given by:

$$p_t(x,y)=\frac{1}{\sqrt{(2\pi)^d\det D(U)}}\exp\big(-\frac{{}^t(x-y)D^{-1}(U)(x-y)}{2t}\big) \tag{5.12}$$

and mean square displacement in the direction l

$$\mathbb{E}[(b_t.l)^2]={}^t lD(U)l\,t \tag{5.13}$$

5.1.5 The Effective Diffusivity

A lots of efforts have been spent, in several fields of applied sciences (such as composite materials) to characterize the effective matrix associated to an homogenization problem. Here the effective diffusivity will be controlled by known variational principles (see [JKO91], Homogenization of Second Order Elliptic Operators).

5.1.5.i Decrease of the Diffusion

By the Green Formula and the periodicity of the solution to the Cell Problem,

$$\begin{aligned}\int_{T_1^d}|\nabla\chi_l|^2dm_U&=-\int_{T_1^d}\nabla(e^{-2U}\nabla\chi_l)\,\chi_l\,\frac{dx}{\int_{T_1^d}e^{-2U(x)}dx}\\&=-\int_{T_1^d}l.\nabla(e^{-2U})\,\chi_l\,\frac{dx}{\int_{T_1^d}e^{-2U(x)}dx}\\&=\int_{T_1^d}l.\nabla\chi_l\,dm_U\end{aligned} \tag{5.14}$$

thus the formula 5.9 can be written.

$$l.D(U).l=l^2-\int_{T_1^d}|\nabla\chi_l|^2dm_U \tag{5.15}$$

so

$$l.D(U).l\le l^2 \tag{5.16}$$

and an homogenization on a potential field causes a decrease of the diffusivity. If $l.\nabla U$ is not the null function then χ_l which is smooth here, is not a constant function, and the diffusion is strictly decreased by the homogenization.

$$l.\nabla U\not\equiv 0\Rightarrow l.D(U).l<l^2 \tag{5.17}$$

Notice also the effective diffusivity $D(U)$ is symmetric and its matrix can be written

$$D(U) = \int_{T_1^d} {}^t(I_d - \nabla\chi_.).(I_d - \nabla\chi_.)dm_U \tag{5.18}$$

Moreover, since $l.x - \chi_l$ with respect to L_U if $g \in \mathcal{C}^\infty(T_1^d)$ then by the Green formula

$$\int_{T_1^d} (l - \nabla\chi_l).\nabla g dm_U = 0 \tag{5.19}$$

Thus

$$D(U).l = \int_{T_1^d} (l - \nabla\chi_l)dm_U \tag{5.20}$$

5.1.5.ii Upper Bound

By the equation 5.19, if $g \in \mathcal{C}^\infty(T_1^d)$ then

$$\begin{aligned}\int_{T_1^d} |l - \nabla(\chi_l + g)|^2 dm_U &= D_l(U) + \int_{T_1^d} |\nabla g|^2 dm_U + \int_{T_1^d} \nabla g.(l - \nabla\chi_l)dm_U \\ &= D_l(U) + \int_{T_1^d} |\nabla g|^2 dm_U \geq D_l(U)\end{aligned}$$

This gives us a nice variational formula acting as an upper bound, the point where this formulation reaches its minimum $D_l(U)$ is unique up to constant functions and the solution to the cell problem is a minimizer.

$${}^t l D(U) l = \inf_{f \in \mathcal{C}^\infty(T_1^d)} \int_{T_1^d} |l - \nabla f|^2 dm_U \tag{5.21}$$

This formulation is also often used as a definition of the effective diffusivity because it has a sense even if U is not smooth, all we need is a well defined probability measure m_U on the torus.

5.1.5.iii Lower Bound

First notice that, $D(U)$ is elliptic.Indeed, by the equation 5.9

$$\begin{aligned}{}^t l D(U) l &\geq e^{-2\,\mathrm{Osc}(U)} \int_{T_1^d} |l - \nabla\chi_l|^l \, dx \\ &\geq e^{-2\,\mathrm{Osc}(U)} l^2\end{aligned}$$

Thus $D(U)$ has an inverse that will be written $D(U)^{-1}$
In order to prove the lower bound, define the set of smooth divergence free periodic vector field as:

$$Q_{sol}(T_1^d) = \left\{ p \in (\mathcal{C}^\infty(T_1^d))^d | \operatorname{div}(p) = 0 \text{ and } \int_{T_1^d} p(x)dx = 0 \right\}$$

and write m_{-U} the probability measure on the torus associated to $-U$

$$m_{-U}(dx) = \frac{dx \exp(2U)}{\int_{T_1^d} \exp(2U)}$$

Then the Lower Bound of D_l is given by a variational formula for the inverse matrix of the effective diffusivity. Indeed let $\xi \in \mathbb{R}^d$ be a unit vector $|\xi| = 1$, then

$$\frac{{}^t\xi D(U)^{-1}\xi}{\int_{T_1^d} \exp(-2U(x))dx \int_{T_1^d} \exp(2U(x))dx} = \inf_{p \in Q_{sol}} \int_{T_1^d} |\xi - p|^2 dm_{-U} \tag{5.22}$$

Indeed the above problem admits a unique solution p_ξ which can be easily expressed in terms of the solution of the Cell problem.

$$p_\xi = \Big(I_d - \frac{e^{-2U}}{\int_{T_1^d} e^{-2U(x)}dx}(I_d - \nabla\chi_.)D(U)^{-1}\Big)\xi \tag{5.23}$$

Notice that $\mathrm{div}(p_\xi) = 0$ and by the equation 5.20, p_ξ satisfies $\int_{T_1^d} p_\xi(x)dx = 0$. Moreover, if $\nu \in Q_{sol}$, then

$$\begin{aligned}\int_{T_1^d} |\xi - p_\xi - \nu|^2 dm_{-U} =& \frac{{}^t\xi D(U)^{-1}\xi}{\int_{T_1^d} e^{-2U(x)}dx \int_{T_1^d} e^{2U(x)}dx} + \int_{T_1^d} |\nu|^2 dm_{-U} \\ & - 2\int_{T_1^d} (\xi - p_\xi).\nu dm_{-U} \\ =& \frac{{}^t\xi D(U)^{-1}\xi}{\int_{T_1^d} e^{-2U(x)}dx \int_{T_1^d} e^{2U(x)}dx} + \int_{T_1^d} |\nu|^2 dm_{-U} \\ \geq& \frac{{}^t\xi D(U)^{-1}\xi}{\int_{T_1^d} e^{-2U(x)}dx \int_{T_1^d} e^{2U(x)}dx}\end{aligned}$$

For $p = 0$ this variational formulation gives the Voigt-Reiss' Inequality:

$$D(U) \geq \frac{1}{\int_{T_1^d} \exp(-2U(x))dx \int_{T_1^d} \exp(2U(x))dx} \tag{5.24}$$

The lower bound of this inequality is the effective diffusivity in dimension one but for $d \geq 2$ the lower bound is not reached in general.

5.1.6 Tighter bounds and wider class of homogenized matrices

The bounds specified by the Voigt-Reiss inequality are usually to wide and give little information about the homogenized matrix associated to a second order elliptic operator. The problem of tighter bounds has been the subject of intensive research in physics and continuum mechanics, especially in the theory of dispersion of electromagnetic waves on small particles and the theory of elasticity for microscopically non-homogenous media.
For instance in any dimension an example of stratified media shows that the above bounds are precise in any dimension, but they are too general to be sharp for more particular media, such as a two-phase composite medium for which the Hashin-Shtrickman Bounds are more precise. The Chapter 6 of the book of S:M. Kozlov, V.V. Jikov and O.A. Oleinik [JKO91] is a good introduction to the subject; see also [MB97].
It is also important to notice that the variational formulae obtained above are nice, simple and local because the matrix $\frac{I_d \exp(-2U)}{\int_{T_1^d} \exp(-2U)}$ describing the inhomogeneous medium, is real, symmetric, definite, positive; elsewhere (for instance when the inhomogeneous medium is associated to a non-symmetric matrix which can even be complex in conductivity problems in presence of a magnetic field) the effective matrix would be associated to non local variational formulae or to a pair of saddle-point variational principles. Here the articles of G.W. Milton [Mil88];[Mil90]; A. Fannjiang, G.C. Papanicolaou [FP94]; J. R. Norris [Nor97] are a good source of information.

5.1.7 A note on convection enhanced diffusion

If the homogenization takes place on a period free divergence drift then the homogenization is enhanced. An interesting series of papers by A. Fannjiang and G. Papanicolaou is available on the subject (see [FP94] for periodic flows and [FP96] for random flows). In this subsection some basic results will be given.

Let Γ be a smooth skew-symmetric $d \times d$ matrix on $\mathbb{R}^d$, periodic of period T_1^d.
So $\Gamma \in \left(C^\infty(T_1^d)\right)^{d(d-1)/2}$ and is the stream function of an incompressible flow ${}^t\nabla.\Gamma$; this notation designate the horizontal vector

$$\left({}^t\nabla\Gamma\right)_j = \sum_{i=1}^{d} \partial_i \Gamma_{ij} \tag{5.25}$$

Let L_Γ be the operator associated to this flow:

$$L_\Gamma = \frac{1}{2}\Delta + {}^t\nabla\Gamma\nabla = \nabla\left(\frac{I_d}{2} + \Gamma\right)\nabla \tag{5.26}$$

which has for invariant measure the Lebesgue measure. The Stochastic Differential Equation associated to this operator (generator) is:

$$\begin{cases} dy_t = d\omega_t + \nabla\Gamma(y_t)dt \\ y_0 = x \end{cases} \tag{5.27}$$

where ω is a standard Brownian motion in $\mathbb{R}^d$
The behavior of y_t for large t is a Central Limit Theorem given by homogenization:
Define for $\epsilon > 0$

$$y^\epsilon = \epsilon y\left(\frac{t}{\epsilon^2}\right)$$

Then y^ϵ converges in law to a Brownian with Effective Diffusivity $D(\Gamma)$ which is positive, definite and symmetric.
The cell problem associated to the homogenization phenomenon is for $l \in \mathbb{R}^d$ the solution $\chi_l \in C^\infty(T_1^d)$ (normalized with $\chi_l(0) = 0$)

$$L_\Gamma\left(\chi_l - l\right) = 0 \tag{5.28}$$

Effective diffusivity The effective diffusivity is then given by

$${}^t l D(\Gamma) l = \int_{T_1^d} |l - \nabla\chi_l|^2 dx = l^2 + \int_{T_1^d} |\nabla\chi_l|^2 dx \tag{5.29}$$

Note that 5.29 shows clearly the enhancement of the diffusion by an homogenization on a periodic free divergence drift.
The effective diffusivity (which is defined by the behavior of $y_k(t).y_l(t)/t$ for large t, see section 2.4 of [JKO91] and 4. of [Oll94]) is given by:
For $k, l \in \mathbb{R}^d$

$${}^t k.D.l = \int_{T_1^d} {}^t(k - \nabla\chi_k)(l - \nabla\chi_l) dx \tag{5.30}$$

This explains why the effective diffusivity D is symmetric.

Flow effective diffusivity One must be careful to not confuse $D(\Gamma)$ with the "flow effective diffusivity" $\sigma(\Gamma)$ which is a non-symmetric matrix relating the gradient of the heat intensity with the flux [FP94].
More precisely write,

$$F_l(x) = l.x - \chi_l(x) \tag{5.31}$$

and

$$\text{FLUX}_l(x) = \left(\frac{I_d}{2} + \Gamma\right)\nabla F_l \tag{5.32}$$

then ∇F_l represents the gradient of the heat intensity and FLUX_l represents the flux induced by the gradient of heat intensity .
The relation between the mean of the gradient of heat intensity and the mean flux is given by the "flow effective diffusivity "(this relation is used to define $\sigma(\Gamma)$)

$$\sigma(\Gamma)\int_{T_1^d} \nabla F_l(x)dx = \int_{T_1^d} \text{FLUX}_l(x)dx \tag{5.33}$$

Thus for $k, l \in \mathbb{R}^d$

$$^t k.\sigma(\Gamma).l = \int_{T_1^d} {}^t k(\frac{I_d}{2} + \Gamma)(l - \nabla\chi_l) \tag{5.34}$$

Note that the symmetric part of the "flow effective diffusivity" $\sigma_{\text{sym}}(\Gamma)$ gives the effective diffusivity by the following relation:

$$D = 2\sigma_{\text{sym}}(\Gamma) \tag{5.35}$$

The following variational formulations are proven by J.R. Norris in [Nor97]

General variational characterization for all $l, \xi \in R^d$

$$\begin{aligned} &{}^t\big(\xi - \sigma(\Gamma)l\big)D_{sym}^{-1}(\Gamma)\big(\xi - \sigma(\Gamma)l\big) \\ &\qquad = \inf_{f,H}\int_{T_1^d} |\xi - H\nabla - (\frac{I_d}{2} + \Gamma)(l - \nabla f)|^2 dx \end{aligned} \tag{5.36}$$

Where the minimum is taken on $f \in C^\infty(T_1^d)$, H varies in the set of smooth skew-symmetric $d \times d$ periodic matrices in $C^\infty(T_1^d)^{d(d-1)/2}$. $H\nabla$ is the vertical vector

$$\big(H\nabla\big)_i = \sum_{j=1}^{d} \partial_j H_{ij} \tag{5.37}$$

Lower bound For $\xi \in \mathbb{R}^d$

$$^t\xi D(\Gamma)^{-1}\xi = \inf_{f,H}\int_{T_1^d} |\xi - H\nabla(x) + (\frac{1}{2} + \Gamma(x))\nabla f(x)|^2 dx \tag{5.38}$$

Where the minimum is taken on $f \in C^\infty(T_1^d)$, H varies in the set of smooth skew-symmetric $d \times d$ periodic matrices.
Note that homogenization on a free divergence drift enhance the diffusion ($D(\Gamma) \geq 1$).

Upper bound For $l \in \mathbb{R}^d$

$$^t l D(\Gamma) l = 4\inf_{\xi,f,H}\int_{T_1^d} |\xi - H\nabla(x) - (\frac{1}{2} + \Gamma(x))(l - \nabla f(x))|^2 dx \tag{5.39}$$

Where the minimum is taken on $f \in C^\infty(T_1^d)$, H varies in the set of smooth skew-symmetric $d \times d$ periodic matrices. And $\xi \in \mathbb{R}^d$ such that $\xi.l = 0$

5.2 Multi-scale Homogenization

In this section, the method of asymptotic expansion will be introduced as an introduction to multi-scale homogenization. Consider for instance a medium characterized by two scales of inhomogeneities and assume that the ratio between those two scales is big. Now there are three description of such a medium:

- The first one is microscopical: it means that the mathematical equations associated to the medium reflect all the inhomogeneities with their distinctive features.
- The second one is macroscopic: it means that those mathematical equations only reflect an effective medium without any inhomogeneity.
- The third one is mesoscopic: it means that those mathematical equations reflect the large scale of inhomogeneities but the smaller scale is seen trough an effective medium.

The definition of microscopic, mesoscopic and macroscopic scale depends on a particular observation of a physical system and the kind of properties to be analyzed. The mesoscopical scale acts as a link between the extreme scales, and to understand the properties reflected in the macroscopic scale by the microscopic one, this link is essential.

Notice that this kind of description is as old as Statistical Physics (but the mathematical models of multi-scale homogenization that will be discussed here are more recent). In fact, most of the quantities of physical interest, accessible to experience and necessary to applications, are macroscopic: volume, pressure, temperature, heat capacity, viscosity, refraction value, magnetic susceptibility, resistivity, ... To compute them from microscopic properties, one has to link them with statical means on the set of particles, whose individual characteristics are inaccessible and not interesting. The explanation of the macroscopic properties from microscopic components requires the use of probabilistic concepts and methods, even if the elementary laws are perfectly known and the subjacent were deterministic.

Here the mesoscopic description will be obtained from the microscopic one, in the latter the mathematical equations are characterized by "slow" and "fast" variables according associated to the different scales of inhomogeneities and the mesoscopic scale is obtained through an homogenization of the equations on the "fast" variables.

5.2.1 The Method of Asymptotic Expansion

To illustrate the method of asymptotic expansion, a mesoscopic description of the cell problem associated to a periodic potential, characterized by a slow and a fast period, will be given here. In addition to the enormous heuristic importance of this method, it opens up new possibilities for the mathematical justification of various homogenization phenomena.
Let $V, T \in C^\infty(T_1^d)$, $R \in \mathbb{N}/\{0,1\}$ and

$$U(x) = V(Rx) + T(x) \tag{5.40}$$

Let $\chi_l^{U,R}$ be the solution of the cell problem 5.4 associated to U
The method in question is based upon the concepts of asymptotic analysis, the aim is to find an approximation for this solution that take into account the rapid oscillation of the coefficients of the equation.
Thus, it is natural to seek the first approximation in the form:

$$\chi_l^{U,R}(x) = \chi^0(x) + \frac{1}{R}\chi^1(x, Rx) + \eta^1(x) \tag{5.41}$$

where χ_l^0 does not depend on the ratio R between scales and is periodic of period one and the function of two variables $\chi^1(x, y)$ does not depend on R and is periodic of period one in x (slow variable) and in y (fast variable.) Moreover it is assumed that those two functions are two times continuously

differentiable on the torus.
The method of asymptotic expansion allows to find χ^0 and χ^1; here, since V and T are smooth, those two functions will be smooth.
Here the function η^1 is assumed to act as an error term. This method brings up several problems of justification according to the utilization of the approximation one has in mind. For instance, the justification of the asymptotic expansion at order 0 means to show that χ^0 is indeed asymptotically an approximation of $\chi_l^{U,R}(x)$; then one has to show that the function $\frac{1}{R}\chi^1(x,Rx)+\eta^1$ does act as an error term, and tends towards 0 as R tends towards ∞ in a norm which choice depends on the regularity of the coefficients of the equations. In general this norm is equivalent to $\|.\|_{L^2(T_1^d)}$; assuming all coefficients and all functions to be smooth enough then the convergence will be in the uniform topology.
Moreover, as it will be shown, it will be important to notice that to be able to find the proper candidate for χ^0 and prove this convergence, one has to seek the expansion of $\chi_l^{U,R}$ up to order $\frac{1}{R^2}$.

$$\chi_l^{U,R}(x)=\chi^0(x,Rx)+\frac{1}{R}\chi^1(x,Rx)+\frac{1}{R^2}\chi^2(x,Rx)+\eta^2(x) \tag{5.42}$$

Notice also that, the justification of the asymptotic expansion at order 1 means to show that the function $R\big(\frac{1}{R^2}\chi^2(x,Rx)+\eta^2\big)$ does act as an error term, and tends towards 0 as R tends towards ∞. And to find the right candidate for χ^1 it is needed to seek the expansion of $\chi_l^{U,R}$ up to order $\frac{1}{R^3}$.
Usually, what is sought is an approximation of $\nabla\chi_l^{U,R}$; in this case, because of the dependence of χ^1 on the fast variable; one needs to show that $\nabla\eta^1$ tends towards 0 as R tends towards ∞. In general, the norm associated to this convergence is equivalent to $\|\nabla\eta^1\|_{L^2(T_1^d)}$. Moreover, for wide range of boundary value problems (on a domain Ω, notice that here, since the problem takes place on the torus, this difficulty is not present) the speed of convergence of this norm is in $\frac{1}{\sqrt{R}}$ and to be able to improve it one has to add a "boundary corrector term" $C_R(x)$ so that $\nabla\chi_l^{U,R}-\chi^0(x,Rx)+\frac{1}{R}\chi^1(x,Rx)-C_R(x)$ belongs to the Sobolev space $H_0^1(\Omega)$ and the speed of convergence in the norm $\|.\|_{H_0^1(\Omega)}$ becomes in $\frac{1}{R}$
However, those justifications are not always needed. That is to say, sometimes, all that is useful is to know the dependence of χ^0 and χ^1 in the slow and the fast variables and some information about their formulae in order to use them as candidates for a variational formula. In those cases the method allows to find those formulae and this shall be sufficient.

5.2.1.i Computation Rules of the Solution of the Cell Problem

Here, several basic computation rules associated to the solution of the cell problem will be presented. Those rules will be useful to evaluate the components of the asymptotic expansion of $\chi_l^{U,R}$
Let $\chi_l^V(y)$ be the solution of the cell problem 5.4 associated to a periodic potential $V(y)$, then by the computation 5.14

$$\int_{T_1^d}\nabla_y\chi_.^V(y)dm_V(y)=I_d-D(V) \tag{5.43}$$

This implies that for all $(b,h)\in\mathbb{R}^d\times\mathbb{R}^d$

$$\int_{T_1^d}{}^tb.\nabla_y\chi_h^V(y)dm_V(y)={}^tb\big(I_d-D(V)\big)h \tag{5.44}$$

Moreover, for all $f\in\big(C^1(\mathbb{R}^d)\big)^d$

$$\int_{T_1^d}{}^t\nabla_x.\nabla_y\chi_{f(x)}^V(y)dm_V(y)={}^t\nabla_x\big(I_d-D(V)\big)f(x) \tag{5.45}$$

Finally, the Green formula shows that

$$\int_{T_1^d} {}^t\nabla_y V(y).\nabla_x \chi^V_{f(x)}(y) dm_V(y) = \frac{1}{2}{}^t\nabla_x\big(I_d - D(V)\big)f(x) \tag{5.46}$$

5.2.1.ii The Method

As it was mentioned, to be able to find the proper term χ^0 we seek the asymptotic expansion up to order 2.

$$\chi_l^{U,R}(x) = \chi^0(x, Rx) + \frac{1}{R}\chi^1(x, Rx) + \frac{1}{R^2}\chi^2(x, Rx) + \eta^2(x) \tag{5.47}$$

The slow variables will be written x and the fast ones y. Then the operator L_U decomposes onto a sum of three operators acting on those variables:

$$L_U = R^2 L_y + R L_{x,y} + L_x \tag{5.48}$$

With

$$L_x = \frac{1}{2}\Delta_x - \nabla_x T \nabla_x$$

$$L_{x,y} = \nabla_x\nabla_y - \nabla_x T\nabla_y - \nabla_y V\nabla_x$$

$$L_y = \frac{1}{2}\Delta_y - \nabla_y V\nabla_y$$

The basic method is simply to identify on each side of the following equation the terms of the same order n in R^n

$$L_U\chi_l^{U,R} = -Rl.\nabla_y V(y) - l.\nabla_x T(x) \tag{5.49}$$

- Term of order R^2

$$L_y\chi^0(x, y) = 0 \Leftrightarrow \chi^0(x, y) = \chi^0(x) \tag{5.50}$$

  So at it was expected, χ^0 does not depend on the fast variable.

- Term of order R

$$\begin{aligned} L_y\chi^1(x, y) + L_{x,y}\chi^0(x) &= -l.\nabla_y V(y) \\ &\Leftrightarrow L_y\chi^1(x, y) = -(l - \nabla_x\chi^0(x)).\nabla_y V(y) \end{aligned}$$

  So

$$\chi^1(x, y) = \chi^V_.(y).\big(l - \nabla_x\chi^0(x)\big) + \chi^1_0(x) \tag{5.51}$$

  Notice, that the equation associated to the term of order 0 does not give us the evaluation of χ^0 but a mesoscopic relation between χ^1 and χ^0. This relation is interesting in itself because it suggests that at the order 0, the matrix $I_d - \nabla\chi^{U,R}_.$ acts like $\big(I_d - \nabla\chi^V_.(Rx)\big)\big(I_d - \nabla\chi^0_.(x)\big)$

- Term of order R^0

$$L_y\chi^2(x, y) + L_{x,y}\chi^1(x, y) + L_x\chi^0(x) = -l.\nabla_x T(x)$$

This Poisson equation relative to the operator L_y has a periodic solution $\chi^2(x,y)$ if and only if its mean with respect to the measure $dm_V(y)$ is equal to 0, thus the following equation must be fulfilled in order to ensure the existence of $\chi^2(x,y)$:

$$\int_{T_1^d} L_{x,y}\chi^1(x,y)dm_V(y) + L_x\chi^0(x) = -l.\nabla_x T(x)$$

Now, by the computation rules 5.2.1.i

$$\begin{aligned}
&\int_{T_1^d} L_{x,y}\chi^1(x,y)dm_V(y)\\
&= \int_{T_1^d}\Big(\nabla_x\nabla_y - \nabla_x T\nabla_y - \nabla_y V\nabla_x\Big)\Big(\chi^V_.(y)\big(l-\nabla_x\chi^0(x)\big)\Big)dm_V(y)\\
&= \big(\nabla_x - \nabla_x T(x) - \frac{\nabla_x}{2}\big)(I_d - D(V))\big(l-\nabla_x\chi^0(x)\big)\\
&= -\big(\frac{\nabla_x}{2} - \nabla_x T(x)\big)(I_d - D(V))\big(\nabla_x\chi^0(x) - l\big)
\end{aligned}$$

Thus the condition for the existence of χ^2 fix the value of χ^0

$$\big(\frac{\nabla_x}{2} - \nabla_x T(x)\big)D(V)\big(\nabla_x\chi^0(x) - l\big) = 0 \tag{5.52}$$

Notice that, although χ^0 is the solution of a cell problem associated to the slow potential T; the influence of the fast potential on χ^0 is felt trough a mean behavior characterized by its associated effective diffusivity.

These results are sufficient to understand the following chapters, however in what follows, the terms of $\chi_l^{U,R}$ will be computed up to order 3 to satisfy our curiosity.

5.2.1.iii Further terms

Since the general method is always the same (equals the terms of the same order in R^n in 5.49 and satisfy the conditions for the existence of a periodic solution), only the results will be given here. We seek to characterize the remaining terms in the following asymptotic expansion. Those remaining terms will not directly be useful because χ^0, χ^1 and the solutions of the cell problems associated to them contain all the information that we need. However the remaining terms will put into light new intrinsic objects (different from the solutions to the cell problems) associated to this mesoscopic relation between the slow and the fast potential. Those objects would become useful and would be given a name if one would like to answer to more subtile questions about the mesoscopic relation.

$$\chi_l^{U,R}(x) = \chi_l^0(x,Rx) + \frac{1}{R}\chi_l^1(x,Rx) + \frac{1}{R^2}\chi_l^2(x,Rx) + \frac{1}{R^3}\chi_l^3(x,Rx) + \eta^3(x) \tag{5.53}$$

Define the matrices $H^V_{i,j}$ and $B^V_{i,j}$ by

$$L_y H^V_{i,j} = \big(\frac{\partial_i^y}{2} - \partial_i^y V(y)\big)\chi^V_{e_j}(y) \tag{5.54}$$

$$L_y B^V_{i,j} = \partial_i^y\chi^V_{e_j}(y) - \big(I_d - D(V)\big)_{i,j} \tag{5.55}$$

Then $\chi^2(x,y)$ is given by

$$\begin{aligned}
\chi^2(x,y) &= -\chi^V_.(y)\nabla_x\chi_0^1(x) + \nabla_x H^V(y)\nabla_x\chi_0(x)\\
&+ \big(\frac{\nabla_x}{2} - \nabla_x T(x)\big)B^V(y)\nabla_x\chi^0(x) + \nabla_x T(x)B^V(y).l + \chi_0^2(x)
\end{aligned} \tag{5.56}$$

Define the tensors $L^k_{i,j}(x)$, $M^k_{i,j}(x)$ and $N^k_i(x)$ by

$$\big(\frac{\nabla_x}{2} - \nabla_x T(x)\big) D(V) \nabla_x L^k_{i,j}(x) = \big(\frac{\partial^x_k}{2} - \partial^x_k T(x)\big)\partial^x_{ij}\chi^0(x) \tag{5.57}$$

$$\big(\frac{\nabla_x}{2} - \nabla_x T(x)\big) D(V) \nabla_x M^k_{i,j}(x) = \big(\frac{\partial^x_k}{2} - \partial^x_k T(x)\big)\big(\frac{\partial^x_i}{2} - \partial^x_i T(x)\big)\partial^x_j\chi^0(x) \tag{5.58}$$

$$\big(\frac{\nabla_x}{2} - \nabla_x T(x)\big) D(V) \nabla_x N^k_i(x) = \big(\frac{\partial^x_k}{2} - \partial^x_k T(x)\big)\partial^x_i\chi^0(x) \tag{5.59}$$

Then $\chi^1_0(x)$ is given by

$$\begin{aligned}\chi^1_0(x) =& - L^k_{i,j}(x)\Big(\int_{T^d_1} \partial^y_k H^V_{i,j}(y) dm_V(y)\Big) - M^k_{i,j}(x)\Big(\int_{T^d_1} \partial^y_k B^V_{i,j}(y) dm_V(y)\Big)\\ &- N^k_i(x)\Big(\int_{T^d_1} \partial^y_k B^V_{i,j}(y) dm_V(y)\Big) l_j\end{aligned} \tag{5.60}$$

Define the tensors $h_j(y)$, $\tilde{H}^k_{i,j}(y)$,$\tilde{B}^k_{i,j}(y)$, $\hat{H}^k_{i,j}(y)$ and $\hat{B}^k_{i,j}(y)$ by

$$L_y h_j(y) = \chi^V_{e_j}(y) - \int_{T^d_1} \chi^V_{e_j}(y) dm_V(y) \tag{5.61}$$

$$L_y \tilde{H}^k_{i,j}(y) = \big(\frac{\partial^y_k}{2} - \partial^y_k V(y)\big) H^V_{i,j}(y) \tag{5.62}$$

$$L_y \tilde{B}^k_{i,j}(y) = \big(\frac{\partial^y_k}{2} - \partial^y_k V(y)\big) B^V_{i,j}(y) \tag{5.63}$$

$$L_y \hat{H}^k_{i,j}(y) = \partial^y_k H^V_{i,j}(y) - \int_{T^d_1} \partial^y_k H^V_{i,j}(y) dm_V(y) \tag{5.64}$$

$$L_y \hat{B}^k_{i,j}(y) = \partial^y_k B^V_{i,j}(y) - \int_{T^d_1} \partial^y_k B^V_{i,j}(y) dm_V(y) \tag{5.65}$$

Then $\chi^3(x,y)$ is given by

$$\begin{aligned}\chi^3(x,y) =& L_x \partial^x_i \chi^0(x) h_i(y) + H^V_{i,j}(y)\partial^x_{ij}\chi^1_0(x)\\ &- \tilde{H}^k_{i,j}(y)\partial^x_{ij}\chi_0(x) - \tilde{B}^k_{i,j}(y)\Big(\big(\frac{\partial^x_i}{2} - \partial^x_i T(x)\big)\partial^x_j\chi^0(x) + \partial^x_i T(x) l_j\Big)\\ &- \big(\frac{\nabla_x}{2} - \nabla_x T(x)\big) B^V(y) \nabla_x L^k_{i,j}(x)\big(\int_{T^d_1} \partial^y_k H^V_{i,j}(y) dm_V(y)\big)\\ &- \big(\frac{\nabla_x}{2} - \nabla_x T(x)\big) B^V(y) \nabla_x \Big(\big(M^k_{i,j}(x) + N^k_i(x) l_j\big)\big(\int_{T^d_1} \partial^y_k B^V_{i,j}(y) dm_V(y)\big)\Big)\\ &- \hat{H}^k_{i,j}(y)\big(\frac{\partial^x_k}{2} - \partial^k_i T(x)\big)\partial^x_{ij}\chi^0(x)\\ &- \hat{B}^k_{i,j}(y)\big(\frac{\partial^x_k}{2} - \partial^k_i T(x)\big)\Big(\big(\frac{\partial^x_i}{2} - \partial^x_i T(x)\big)\partial^x_j\chi^0(x) + \partial^x_i T(x) l_j\Big) + \chi^3_0(x)\end{aligned} \tag{5.66}$$

And $\eta^3(x)$ is a solution of

$$L_U \eta^3(x) = -\frac{1}{R^2}\Big(L_x \chi^2(x,Rx) + L_{x,y}\chi^3(x,Rx) + \frac{L_x}{R}\chi^3(x,Rx)\Big) \tag{5.67}$$

The expansion will be stopped here, notice that the complexity of the formulae grows quickly with the order in the expansion; this is a general phenomenon.

5.2.1.iv Justification

Let's show that $\eta^1(x) = \chi_l^{U,R} - \chi^0(x) - \frac{1}{R}\chi^1(x, Rx)$ strongly converges towards 0 in $H^1(T_1^d)$:

$$\lim_{R\to+\infty} \int_{T_1^d} |\nabla\eta^1(x)|^2\, dm_U(x) = 0 \tag{5.68}$$

This is an easy task here, since everything is smooth and periodic. Indeed,

$$\eta^1(x) = \frac{1}{R^2}\chi^2(x, Rx) + \eta^2(x)$$

Obviously $\frac{1}{R^2}\chi^2(x, Rx)$ strongly converges towards 0 in $H^1(T_1^d)$ like $\frac{1}{R}$ by the Green Formula and the Poincaré inequality

$$\begin{aligned}\int_{T_1^d} |\nabla\eta^2(x)|^2\, dm_U(x) &= -\int_{T_1^d} L_U\eta^2(x)\big(\eta^2(x) - \int_{T_1^d}\eta^2(y)dm_U(y)\big)\, dm_U(x)\\ &\le C\Big[\int_{T_1^d}\big(L_U\eta^2(x)\big)^2\, dm_U(x)\Big]^{\frac{1}{2}}\Big[\int_{T_1^d}|\nabla\eta^2(x)|^2\, dm_U(x)\Big]^{\frac{1}{2}}\end{aligned}$$

Where C is a constant bounded by $C_d e^{2\,\mathrm{Osc}(V)+2\,\mathrm{Osc}(T)}$
Thus

$$\int_{T_1^d} |\nabla\eta^2(x)|^2\, dm_U(x) \le C^2\int_{T_1^d}\big(L_U\eta^2(x)\big)^2\, dm_U(x)$$

And since

$$L_U\eta^2 = -\frac{1}{R}\Big[L_x\chi^1 + (L_{x,y} + \frac{L_x}{R})\chi^2\Big](x, Rx)$$

whose $L^2(T_1^d)$ norm converges to 0 like $\frac{1}{R}$; this proves the result.
Moreover, notice that the term $\chi_0^1(x)$ in χ^1 is not relevant here since $\frac{1}{R}\chi_0^1(x)$ converges strongly to 0 in $H_0^1(T_1^d)$ (the same phenomenon applies to $\frac{1}{R}\nabla_x\chi^1(x, y)$, the gradient of χ^1 with respect to the slow variable).
This proves that

$$\lim_{R\to+\infty}\int_{T_1^d}\Big|\nabla\chi_l^{U,R}(x) - \nabla_x\chi^0(x) - \nabla_y\chi_{.}^{V}(Rx)\big(l - \nabla_x\chi^0(x)\big)\Big|^2 dm_U(x) = 0 \tag{5.69}$$

When the problem has a boundary, and the coefficients are less regular, several methods have been developed to deal with the justification problem; they will be briefly introduced in the next sub subsection (it is not necessary to read it understand the following chapters).

5.2.1.v Further topics on the justification of asymptotic expansion

Define Ω a bounded open set in $\mathbb{R}^d$. Let f be a function in $L^2(\Omega)$ and $A(x, y)$ a matrix with bounded, measurable, elements, periodic and of period T_1^d in y. Assume that there exists two positive constants $0 < \alpha \le \beta$ such that for all $\xi \in \mathbb{R}^d$

$$\alpha\xi^2 \le {}^t\xi A\xi \le \beta\xi^2 \tag{5.70}$$

The latter condition on the matrix A ensure the existence in $H_0^1(\Omega)$ of the solution of the following equation.

$$\begin{cases}-\nabla.\big(A(x, \frac{x}{\epsilon})\nabla u_\epsilon\big) = f & \text{in } \Omega\\ u_\epsilon = 0 & \text{on } \partial\Omega\end{cases} \tag{5.71}$$

Here ϵ acts as a small parameter reflecting the period of the inhomogeneities. The problem of asymptotic analysis of periodic structures with boundary value is to seek the asymptotic behavior of u_ϵ as ϵ tends towards 0. Moreover the uniform coercivity of the matrix A allows to show that all the solutions u_ϵ are uniformly bounded (for all ϵ)

$$\|u_\epsilon\|_{H_0^1(\Omega)} \leq C(\Omega, \alpha)\|f\|_{L^2(\Omega)} \tag{5.72}$$

Thus by compactness in the weak topology, as ϵ goes to 0 there exists a limit u such that, up to a subsequence u_ϵ converges weakly to u in $H_0^1(\Omega)$ The method of asymptotic expansion allows to guess a good candidate for the limit u; by postulating the following ansatz.

$$u_\epsilon(x) = u_0(x, \frac{x}{\epsilon}) + \epsilon u_1(x, \frac{x}{\epsilon}) + \epsilon^2 u_2(x, \frac{x}{\epsilon}) + \cdots \tag{5.73}$$

By satisfying the equations imposed by the terms of same order in ϵ^n in the equation 5.71 up to order 2. One finds that $u_0(x, y) = u(x)$ does not depend on the fast variable and is characterized by the following equation.

$$\begin{cases} -\nabla.\big(A^*\nabla u\big) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{5.74}$$

Where A^* is an effective, coercive matrix whose formula is given by the cell problem associated to A. Now u is the good candidate and the problem of justification is to prove the convergence of the sequence u_ϵ to u. Many methods are available or have been developed to this end, some of them are briefly given here. Define $A^\epsilon = A(x, \frac{x}{\epsilon})$:

The G-convergence (developed by S. Spagnolo; see [Spa76], [ZKON79], [JKO91])
The sequence of matrices A^ϵ is called G-convergent to the matrix A^* in the domain Ω, if for any $f \in H^{-1}(\Omega)$ the solution u_ϵ of the Dirichlet problem 5.71 converges towards u (defined by the equation 5.74) in $H_0^1(\Omega)$ and $A^\epsilon\nabla u_\epsilon$ converges towards $A^*\nabla u$ in $L^2(\Omega)$.
This notion of convergence is wide in the sense that homogenization is a particular og G-convergence, thus it allows to develop a large range of abstract theorems characterizing the topology associated to this convergence and others that ensure this kind of convergence to take place.
For instance, if those matrices are seen as abstract self-adjoint, uniformly coercive and bounded operators on a separable Hilbert space V with V^* as its dual. Then the abstract energy criterion says that A^ϵ G-converge to A^* if and only if

$$\lim_{\epsilon \longrightarrow 0} \inf_{v \in V} \Big\{\frac{1}{2}(A_\epsilon v, v) - (f, v)\Big\} = \inf_{v \in V} \Big\{\frac{1}{2}(A^* v, v) - (f, v)\Big\} \tag{5.75}$$

Which means that, G-convergence of self adjoint operators is equivalent to pointwise convergence of the quadratic forms associated with the corresponding inverse operators.
This convergence imply the Γ-convergence of the quadratic forms $(A_\epsilon v, v)$ to the form (A^*v, v) (see below)
One of the significant properties of G-convergence is the fact that the G-limit operator depends only on the original sequence of operators but neither on the type of boundary conditions, nor on the domain. Moreover the property of G-convergence is local in the sense that if A_ϵ G-converge towards A in a domain Ω, then A_ϵ G-converge towards A in any sub domain $\Omega_1 \subset$. This local property is sometimes [AB96] used to justify the fact that one does not loose in generality to suppose that inhomogeneities of a problem are periodic (which is stronger that ergodic).

The Γ-convergence The Γ-convergence is a notion of functional convergence which has been introduced by E. De Giorgi; see [Gio75], [JKO91].
The forms $(A_\epsilon v, v)$ Γ-converge to the form (A^*v, v) if and only if the two following properties are satisfied:

- For any $u \in V$ and any sequence $u_\epsilon \in V$ which converges weakly to it, the following inequality is valid

$$\lim_{\epsilon\to 0}(A_\epsilon u_\epsilon, u_\epsilon) \geq (A_0 u, u) \tag{5.76}$$

- for any $u \in V$ there exist a sequence $u_\epsilon \in V$ converging weakly to it and

$$\lim_{\epsilon\to 0}(A_\epsilon u_\epsilon, u_\epsilon) = (A_0 u, u) \tag{5.77}$$

The energy method (developed by L. Tartar; see [Tar77], [Def93], [Mur78])
The main ingredient is a clever choice of test functions in the variational formulation of the equation 5.71

$$\int_\Omega A(\frac{x}{\epsilon})\nabla u_\epsilon(x).\nabla\phi(x)\,dx = \int_\Omega f(x)\phi(x)\,dx \quad \forall\phi \in H_0^1(\Omega) \tag{5.78}$$

The goal of this method is to pass in the limit in the above equation; but the left hand side involves the product of two weakly convergent sequences. However one can do so by replacing the fixed test function ϕ by a chosen sequence ϕ_ϵ (whose formula is given by the solution of the cell problem associated to ${}^tA$) which permits to pass to the limit thanks to a compensated compactness phenomenon ([Tar79],[JKO91]): indeed consider p^ϵ, v^ϵ vector fields in $(L^2(\Omega))^d$ converging in the weak topology to p^0 and v^0. The lack of strong convergence does not allow to pass to the limit in the scalar product $p^\epsilon.v^\epsilon$, however by adding additional properties to those sequences one can "compensate" this lack. For instance if in addition $\operatorname{div} p^\epsilon$ and $\operatorname{curl} v^\epsilon$ are compact sequences in $H^{-1}(\Omega)$ (where $\operatorname{curl} v$ is a skew-symmetric matrix whose elements belong to $H^{-1}(\Omega)$ and are defined by $(\operatorname{curl} v, \varphi)_{ij} = \int_\Omega \big(v_j(x)\partial_i\varphi(x) - v_i(x)\partial_j\varphi(x)\big)dx$ for all $\varphi \in H_0^1(\Omega)$) then $p^\epsilon.v^\epsilon$ remains bounded in $L^1(\Omega)$ and the following weak convergence is established: for all $\varphi \in C_0^\infty(\Omega)$

$$\lim_{\epsilon\to 0}\int_\Omega p^\epsilon(x).v^\epsilon(x)\varphi(x)dx = \int_\Omega p^0(x).v^0(x)\varphi(x)dx$$

This method proves rigorously the convergence of the homogenization process.

The two-scale convergence method (developed by G. Allaire [All92], [All94], and Nguetseng [Ngu90])
A sequence of functions u_ϵ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^2(\Omega\times T_1^d)$ if, for any function $\psi(x,y)$ in $D(\Omega, C^\infty(T_1^d))$ (the space of infinitely smooth and compactly supported functions in Ω with values in the space $C^\infty(T_1^d)$), we have

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(x)\psi(x,\frac{x}{\epsilon})dx = \iint_{\Omega\times T_1^d} u_0(x,y)\psi(x,y)\,dx\,dy \tag{5.79}$$

Here are some properties of this kind of convergence:

- For each bounded sequence u_ϵ in $L^2(\Omega)$ one can extract a subsequence two-scale converging to a limit $u_0(x,y)$ in $L^2(\Omega\times T_1^d)$.
- If a sequence u_ϵ in $L^2(\Omega)$ two-scale converge to $u_0(x,y)$ in $L^2(\Omega\times T_1^d)$ then u_ϵ also weakly converges to $u(x) = \int_{T_1^d} u_0(x,y)dy$ in $L^(\Omega)$.
- If a sequence u_ϵ is bounded in $H^1(\Omega)$ then there exists $u(x) \in H^1(\Omega)$ and $u_1(x,y) \in L^2[\Omega; H^1(T_1^d)]$ such that, up to a subsequence, u_ϵ two-scale converges to $u(x)$, and ∇u_ϵ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$.

This two-scale convergence framework allows to show that the entire sequences u_ϵ of the solutions of 5.71 and ∇u_ϵ converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x,y)$ (where u_1 is the first of order 1 in the asymptotic expansion 5.73). It also allows to obtain corrector results and to show for instance that $u_\epsilon(x) - u(x) - \epsilon u_1(x, \frac{x}{\epsilon})$ strongly converges to 0 in $H^1(\Omega)$

5.2.2 Differential Effective Medium Theory

Polycrystalline metals, porous rocks, colloidal suspensions, epitaxial thin films, rubber, fiber reinforced composites, gels, foams, granular aggregates, sea, ice, shape-memory metals, magnetic materials, electro-rheological fluids, and catalytic materials are all examples of materials where an understanding of the mathematics on the different length scales is a key to interpreting their physical behavior [GGJ+98]; for instance the hydrologists distinguish at least seven different scales ([Zim93]). Thus in many field of physics and engineering some phenomena can not be explained in terms of a model of one scale stochastically homogenous random media. One of the heuristic theory (among others, such as the Self Consistent Approximations, [Bud65], [Hil65], [Wu66]) developed to fill this gap is the Differential Effective Medium Theory (D-EMT). This theory models a two phase composite by incrementally adding inclusions of one phase to a background matrix of the other and then recomputing the new effective background material at each increment ([Bou97], [McL97], [CCL80]). This theory was first proposed by Bruggeman to calculate the conductivity of a two-component composite structure formed by successive substitutions ([Bru35] and [AIP77]) and generalized by Norris ([Nor85]) to materials with more than two phases. This method is also implemented numerically.
Here the utilization of this theory to evaluate the conductivity of a two phase material (matrix with aggregates grains) will be reproduced (see the paper of Garboczi and Berryman [GB99], this material is the concrete) as an introduction to the heuristic DEM Theory though process.

5.2.2.i The heuristic process

The structure is build up by starting from a homogeneous component and using the following iterative process: replace the a small amount of this homogeneous component by the second component, and then regard the resulting "effective" material as the homogeneous component for the succeeding substitution step.
Thus, in the usual D-EMT ([McL97]), when a particle with conductivity σ_p is embedded in a matrix with conductivity σ_{bulk}, the dilute limit is used to generate an approximate equation that can be solved for the effective conductivity: in the dilute limit, the value of c, the volume fraction of aggregates, is small enough so that the aggregate grains do not influence each other. The effective conductivity, σ, is then given exactly (heuristical for a mathematician) by ([Tor91],[SGB95]):

$$\sigma = \sigma_{bulk} + \sigma_{bulk} mc + O(c^2) \tag{5.80}$$

where m is a dimensionless coefficient often called the dilute limit slope or intrinsic conductivity [DG95] that is a function of the shape of the particle, and the ratio $\frac{\sigma_p}{\sigma_{bulk}}$. The higher order terms in the c expansion come from interactions between aggregate particles, and so are negligible in the dilute limit.
The dilute limit is now used to generate a differential equation for the conductivity when an arbitrary amount of aggregates is placed in the matrix. Suppose that a non-dilute volume fraction c of aggregates (of conductivity σ_p) have been placed in the matrix. The effective conductivity of the entire composite system is now σ. This system of matrix (volume fraction $= \phi = 1 - c$) plus aggregates (volume fraction $= c$) is treated as being a homogeneous material. Suppose then that additional aggregates are added by removing a differential volume element, dV, from the homogeneous material, and replacing it by an equivalent volume of aggregates. The new conductivity, $\sigma + d\sigma$, is assumed to be given by the dilute limit

$$\sigma + d\sigma = \sigma + \sigma\, m(\sigma)\frac{dV}{V} \tag{5.81}$$

where V is the total volume and is the same as that in equation 5.80, but with the replacement $\sigma_{bulk} \to \sigma$. This is the key approximation that is made in order to generate the DEM Theory. When the volume element dV was removed, only a fraction ϕ was matrix material so that the actual change in the matrix volume fraction, $d\phi$, is given by

$$d\phi = -\phi \frac{dV}{V} \tag{5.82}$$

Equation 5.81 then reduces to

$$\frac{d\phi}{\phi} = -\frac{d\sigma}{\sigma\, m(\sigma)} \tag{5.83}$$

which can be integrated to yield

$$-\int_{\sigma_{bulk}}^{\sigma} \frac{d\sigma'}{m(\sigma')\sigma'} = \int_{1}^{\phi} \frac{d\phi}{\phi'} = \ln(\phi) \tag{5.84}$$

For spherical aggregates of only one size, with conductivity σ_p, and embedded in a matrix of conductivity σ [Tor91],

$$m(\sigma) = 3\frac{\sigma_p - \sigma}{2\sigma + \sigma_p} \tag{5.85}$$

The integral in equation 5.84 can be done exactly, using equation 5.85, with the result

$$\frac{\sigma - \sigma_p}{\sigma_{bulk} - \sigma_p}\left(\frac{\sigma}{\sigma_{bulk}}\right)^{-\frac{1}{3}} = 1 - c \tag{5.86}$$

This result shows the heuristic efficiency of D-EMT. Notice also that in the generalization of D-EMT to multi-phases materials, grains with n different shapes and conductivity are added to the backbone material, and the equation 5.80 becomes (a sort of "n dimensional heuristic Taylor expansion").

$$\sigma = \sigma_{bulk} + \sigma_{bulk} \sum_{i=1}^{n} m_i\, c_i + \sum_{i=1}^{n} O(c_i^2) \tag{5.87}$$

Where the m_i are the intrinsic conductivity of the different phases and the c_i their volume fraction.

5.2.2.ii Some Applications

Electrical and acoustic properties of fluid-saturated sedimentary rocks For sedimentary rocks, the Archie's law: $\sigma = \sigma_f \phi^m$ (which is an approximate empirical rule) links σ, the dc electrical conductivity, σ_f the fluid conductivity, ϕ the porosity, and $m \simeq 2$ is a constant. By applying the D-EMT iterative picture to rocks, in [SSC81] it has been observed that the form of Archie's law with $m = \frac{3}{2}$ can be exactly reproduced. In a later work, [MC82], it has been shown that the value of the exponent m is in fact dependent on the shape of the substitution unit, with $m = 2$ indicating a general preponderance of randomly oriented platelike solid grains.
In [SC84], calculations based on the differential effective medium of rock microstructure yield prediction of sonic travel times and acoustic attenuation in good agreement with experimental data. In particular, the theory shows that the large frequency peak and its associated velocity dispersion observed in sandstones are characteristic of a composite system containing fluid-filled microcracks.

Elastic properties of a composite material consisting of melt (fluid) and crystals (solid) D-EMT has been used to develop a theoretical model for the elastic properties of a composite material consisting of melt (fluid) and crystals (solid) (see [TS99b] and [TS99a]). Indeed, the quantification of melt properties within a magma reservoir is extremely important in predicting and monitoring of volcanic eruptions. The volcano will erupt when the amount and supply of melt (low density and high viscosity material) in the magma reservoir is large. As the melt cools, crystals are formed, and the density and viscosity of the melt change. Seismic tomography provide information about the wave velocities, and also attenuation and anisotropy of these regions: the size of the magma reservoir(s) can be inferred from seismic methods. That's why a theory to quantify the amount of melt in the magma reservoir has been developed. More precisely DEM theory is used to examine the effect of introducing inclusions of melt into a solid matrix on the elastic constants (and hence shear and compressional velocities), attenuation, and anisotropy of the resulting medium.

Rock elastic properties The DEM theory is used to show that microstructure plays a significant role in determining the effective elastic properties of porous materials such as porous foam composed of glass [BB93b], quite good agreement is obtained with experimental data.

The effective conductivity of concrete The effective conductivity of concrete in its representation as a composite material, with its three phases: matrix, aggregates, and the interfacial transition zone (a thin shell of altered matrix material surrounding each aggregate grain). Assigning each of these phases a different transport parameter, diffusivity or conductivity, results in a complicated composite transport problem. To evaluate the conductivity of concrete, in [GB99] an aggregate particle with a surrounding interfacial transition zone is mapped onto an effective particle of uniform conductivity, which is then treated in usual differential effective medium theory.

5.2.3 Reiterated homogenization and Rigorous D-EMT

5.2.3.i Reiterated Homogenization

The method of Reiterated homogenization was introduced in Bensoussan-Lions-Papanicolaou [BLP78], used, discussed and developed by Avellaneda [Ave87] and Kozlov [Koz95]; by Allaire, Briane [AB96] and Jikov, Kozlov [JK99].
The typical problem solved by reiterated homogenization is the following is the one discussed the in sub subsection 5.2.1.v:

$$\begin{cases} -\nabla.\big(A_\epsilon(x)\nabla u_\epsilon\big) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{5.88}$$

But now the conductivity matrix A_ϵ has n different ordered microstructure length scales $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$ (which depends on a parameter ϵ) and is written

$$A_\epsilon(x) = A(x, \frac{x}{\epsilon_1}, \cdots, \frac{x}{\epsilon_n}) \tag{5.89}$$

where $A(x, y_1, \dots, y_n) \in \big(L^\infty(\Omega)\big)^{(n+1)^2}$ is T_1^d periodic with respect to each variable y_k and respect the coercivity condition 5.70. Each of these scales is microscopic in the sense that for $0 \leq k \leq n$

$$\lim_{\epsilon \to 0} \epsilon_k = 0 \tag{5.90}$$

And the ratio between scales very small: the scales are well separated and can be distinguished from each other.

$$\lim_{\epsilon \to 0} \frac{\epsilon_{k+1}}{\epsilon_k} = 0 \tag{5.91}$$

This is a very strong and yet fundamental assumption for the method used in these works.
The key process in the reiterated homogenization of the operator associated to $A(x, y_1, \dots, y_n)$ is to homogenizes first with respect to the faster variable y_n, considering $x, y_1, \dots, y_{n-1}$ as a parameter to obtain an operator associated with an homogenized matrix $A_{n-1}(x, y_1, \dots, y_{n-1})$ where the dependence on the faster variable has vanished and been replaced by an effective behavior. The

next step is to homogenize with respect to y_{n-1} considering $x, y_1, \dots, y_{n-2}$ as a parameter and this inductive process goes on until one has homogenized on all the fast variables $y_1, \dots, y_n$ to obtain an effective operator associated to an effective matrix $A_0(x)$. The justification of this procedure is obtained thanks to the separations of scales 5.91.
In Bensoussan-Lions-Papanicolaou [BLP78], $\epsilon_k = \epsilon^k$ thus this separation is clear. And this justification is done for A continuous in $y_1, \dots, y_n$ in two steps: First approximate A by A_δ smooth in all the variables and justify the reiterated homogenization for A_δ through asymptotic expansion over the scales ϵ and the energy method (see sub subsection 5.2.1.v). Next control the difference $A - A_\delta$ in L^p norm thanks to a generalization of a analytical theorem ([Mey63]) on the L^p estimate for the gradient of solutions of second order elliptic divergence equations.
In Allaire-Briane [AB96] this justification is done for less regular conditions:

$$A(x, y_1, \dots, y_n) \in \big(L^\infty(\Omega)\big)^{(n+1)^2} \tag{5.92}$$

The main tool is the extension of the notion of two scale convergence 5.2.1.v to the notion of $n+1$ scale convergence. One of the main theorems obtained is the following:
if the scales are microscopic 5.90 and separated 5.91, A is coercive 5.70 and bounded 5.92; if the following mixing condition is satisfied (called $n+1$-scale convergence):
for all functions $\varphi \in L^2\big(\Omega, C(y_1 \in T_1^d, \dots, y_n \in T_1^d)\big)$ (where $C(y_1 \in T_1^d, \dots, y_n \in T_1^d)$ is the space of continuous functions on $(T_1^d)^n$)

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_\Omega A_\epsilon(x) \varphi(x, \frac{x}{\epsilon}, \dots, \frac{x}{\epsilon^n}) dx = \\ &\qquad \int_{\Omega \times (T_1^d)^n} A(x, y_1, \dots, y_n) \varphi(x, y_1, \dots, y_n) dx\, dy_1 \cdots dy_n \end{aligned} \tag{5.93}$$

and if $\lim_{\epsilon \to 0} \|(A_\epsilon)_{ij}\|_{L^2(\Omega)} = \|A_{ij}\|_{L^2(\Omega \times (T_1^d)^n)}$ then, the solution u_ϵ of the equation 5.88 converges weakly to a function u of $H_0^1(\Omega)$ and its gradient ∇u_ϵ $(n+1)$-scale converge to a limit

$$\nabla u(x) + \sum_{k=1}^n \nabla_{y_k} u_k(x, y_1, \dots, y_k)$$

where $(u, u_1, \dots, u_n)$ is the unique solution in the space

$$H_0^1(\Omega) \times \prod_{k=1}^n L^2[\Omega \times (T_1^d)^{k-1}; H_l^1 oc(T_1^d)]$$

of the $(n+1)$-scale homogenized system

$$\begin{cases} -\nabla_{y_n}\Big(A\big(\nabla u(x) + \sum_{j=1}^n \nabla_{y_j} u_j\big)\Big) = 0 \\ -\nabla_{y_k}\Big[\int_{(T_1^d)^{n-k}} A\big(\nabla u(x) + \sum_{j=1}^n \nabla_{y_j} u_j\big) dy_{k+1} \dots dy_n\Big] = 0 \quad 1 \le k \le n-1 \\ -\nabla_x\Big[\int_{(T_1^d)^n} A\big(\nabla u(x) + \sum_{j=1}^n \nabla_{y_j} u_j\big) dy_1 \dots dy_n\Big] = f \end{cases} \tag{5.94}$$

This system of equations reflects the inductive process in the reiterated homogenization of the operator associated to A (u is also the solution of 5.88 with A_ϵ replaced by $A_0(x)$ obtained from A after n successive steps of reiterated homogenization)

5.2.3.ii Multi-scale Dilution of Phases

In [Ave87], M. Avellaneda proposed a mathematical interpretation of the heuristic D-EMT procedure. To this end he uses the techniques of iterated Homogenization (of Bensoussan, Lions and Papanicolaou since this work is anterior to [AB96]) and G-convergence to construct a material which

reflect a D-EMT property (the analytical tool for the study of G-convergence in this paper is the theorem of Meyers-Elcrat [ME75] on the higher integrability of the gradient of H^1 solutions of elliptic systems of partial differential equations).
This is done in the framework of linear elastostatics equations:

$$\operatorname{div} C(x)Du = 0 \tag{5.95}$$

Where $Du(x) = \frac{1}{2}[\nabla u + {}^t\nabla u]$ is the second order tensor of linearized deformations and appears as the symmetric part of the gradient of the displacement $u(x)$ (which is a vector). $C(x) = C_{ijkl}$ is fourth-order tensor field, which links deformations with constraints (symmetric in the permutation of the indices), measurable in $x \in \mathbb{R}^d$ and satisfying for fixed $\tau_1, \tau_2 > 0$, all $x \in \mathbb{R}^d$ and all symmetric $d \times d$ matrix η.

$$\tau_1 \eta : \eta \leq C(x)\eta : \eta \leq \tau_2 \eta : \eta \tag{5.96}$$

(: is the contraction procedure $\eta : \xi = \eta_{ij}\xi_{ij}$)

The DEM material associated to the tensor of elasticity $\Gamma_{j,\epsilon}$ is built by adding m different phases and $m.j$ different periodic scales (its elastic properties are varying in periodic length scales $\epsilon, \epsilon^2, \cdots, \epsilon^m, \cdots, \epsilon^{m.j}$, where ϵ is a small parameter) to a backbone structure associated to the constant tensor field C_0
The phase k appears at scales $k + m.p$ with p varying between 0 and $j-1$. Thus, when the number m of phases remains fixed and j the number of different scales associated to each phase grows large, each phase is present in a "homogeneous" way at all scales. However, it is important to notice that the ratio (here:small scale divided by big scale) between two scales associated to two different phases is at least in ϵ. This means that, although each phase is present at "all" scales, they do not "see" or "interact" with each other; this is true at least in the limit when ϵ goes to 0 and the tensor of elasticity $\Gamma_{j,\epsilon}$ G-converge towards C_j which reflects an effective media issued from the previous one by $m.j$ successive steps of iterated homogenization.
Now, imagine that in the first material $\Gamma_{j,\epsilon}$ the microscopic influence of each phase on the macroscopic structure is very small; this is reflected by an integer parameter n and the mathematical image to have in mind is that when the phase k appears at the scale $k + m.p$ $(0 \leq p \leq j-1)$ it does so with a concentration $\frac{1}{n}\gamma_k(\frac{p}{n})$ where $t \to \gamma_k(t)$ is a continuous function reflecting the fact that the concentration at which each phase appears might change with the phase but also with the scale at which it appears.
Write $\Gamma_{n,j,\epsilon}$ for the tensor field associated to this material, as it was done before, let ϵ goes to 0; you obtain through the $m.j$ successive steps of iterated homogenization an effective tensor $C_{n,j}$.
Now imagine that the number of scales j at which each phase appears in $\Gamma_{n,j,\epsilon}$ goes to infinity and the microscopic influence $\frac{1}{n}$ of each scale goes to 0 but the macroscopic influence of each phase $\frac{j}{n}$ tends towards a constant t. ($\frac{j}{n} \to t$). Then Avellaneda proves that $C_{n,j}$ converges to a tensor $C(t)$ satisfying the ordinary differential equation:

$$\frac{dC(t)}{dt} = \sum_{k=1}^{m} \gamma_k(t) Q_k\big(C(t)\big) \qquad C(0) = C_0 \tag{5.97}$$

where Q_k is an application from the space of all periodic elastic tensor fields to that same space; reflecting the following procedure: Consider a backbone material with constant tensor field C, add to this material the phase k at concentration μ with the periodic scale ϵ (this step is reflected with $\epsilon = 1$ trough the operator $T_{k,\mu}$) then let ϵ goes to 0, you obtain the effective tensor $HT_{k,\mu}(C)$ (the operator H reflects the homogenization step). Avellaneda proves that the new tensor $HT_{k,\mu}(C)$ is differentiable at the concentration 0 and its differential is given by the formula:

$$Q_k(C) = \frac{d}{d\mu}\Big|_{\mu=0} HT_{k,\mu}(C) \tag{5.98}$$

Which reflects the linearized influence on the backbone C of the addition of the phase k at low concentration μ.
The equation 5.97 is to put into relation with the heuristic D-EMT equation

$$C_{k,eff}(\mu) = C + \mu Q_k(C) + o(\mu) \tag{5.99}$$

which gives the effective linear elasticity of the new configuration $C_{k,eff}(\mu)$ after addition of the inclusion phase k (whose geometry and linear elasticity are reflected in the operator Q_k) to the backbone material with linear elasticity C. (this also to put into relation with the equation 5.87)
In resume, this mathematical model, says that the equations appearing in DEM Theory and associated with m different phase inclusions are "homogeneous limit equations" reflecting the following limit image: each phase is present at an infinite number of scales in a homogeneous way. Yet two different phases never interact because they always appear at scales whose ratio is 0. Moreover the macroscopic influence of each phase is totally (but non uniformly) diluted in the infinite number of scales at which it appears (so its influence is 0 at the microscopic level but the total influence=infinite times 0 is finite and non null).

5.2.3.iii Multi-scale control of the Homogenized Matrix

The work of Jikov-Kozlov [JK99] is interesting as the most recent result on the subject. This works develops the ideas which were first appeared in S. Kozlov's paper [Koz95] and follows the paper of Avellaneda [Ave87] since it can also be seen as the asymptotic Justification of DEM theory as limit equations.
Consider the divergent operator,

$$\nabla.\Big(K_N(x)\nabla\Big) \tag{5.100}$$

Where the matrix K_N is equal to

$$K_N(x) = I_d\; a^N(\frac{x}{\epsilon_1})a^N(\frac{x}{\epsilon_2})\cdots a^N(\frac{x}{\epsilon_N}) \tag{5.101}$$

The operator associated to this matrix reflects a multi-scale medium with decreasing scale factors $\epsilon_1, \epsilon_2, \dots \epsilon_N$. Notice also that the macroscopic influence of each microscopic scale is self similar in the sense that K_N has a product form and the influence of each scale is translated by a rescaling of the same function a^N.
Just as in the paper of Avellaneda [Ave87]; it is assumed that the microscopic influence a^N of each scale is diluted in the number N of scales: this is translated by following controls

$$a^N(x) = 1 + O(\frac{1}{\sqrt{N}}) \quad \text{in} \quad W^{1,\infty}(T_1^d) \tag{5.102}$$

$$\int_{T_1^d} a^N(x)dx\;, \quad \int_{T_1^d} \frac{1}{a^N(x)}dx \leq 1 + \frac{\beta}{N} \tag{5.103}$$

The scales $\epsilon_1, \epsilon_2, \dots$ are assumed rationally independent in the sense that for $n_i \in \mathbb{Z}$, $\sum_{i=1}^k \epsilon_i n_i = 0$ if and only if $n_1 = \cdots = n_k = 0$. This technical assumption is issued to obtain the functional mixing result: for $f_i \in L^2(T_1^d)$

$$\int_{T_1^d} \prod_{i=1}^k f_i(\frac{x}{\epsilon_i})dx = \prod_{i=1}^k \int_{T_1^d} f_i(\frac{x}{\epsilon_i})dx \tag{5.104}$$

Moreover as in [AB96] it is assumed that the scales separate quickly:

$$\sum_{k=2}^{\infty} k\Big(\frac{\epsilon_k}{\epsilon_{k-1}}\Big)^2 < \infty \tag{5.105}$$

For a real symmetric definite positive matrix, $K(x)$, the constant homogenized matrix K_{hom} associated to it is defined by the variational formulation: for $\eta \in \mathbb{R}^d$

$$^t\eta.K\eta = \inf_{u \in H^1(T_1^d)} \int_{T_1^d} {}^t\big(\eta + \nabla u(x)\big)K\big(\eta + \nabla u(x)\big)\, dx \tag{5.106}$$

In the asymptotic limit where the ratio between scales $\frac{\epsilon_{k+1}}{\epsilon_k}$ is equal to 0, the homogenized operator associated to 5.100 is characterized by a homogenized matrix A_N computed inductively by reiterated homogenization:

$$A_0 = I_d, \quad A_1 = (aI_d)_{hom}, \quad A_2 = (aA_1)_{hom}, \dots, A_k = (aA_{k-1})_{hom}, \dots \tag{5.107}$$

Here $\frac{\epsilon_{k+1}}{\epsilon_k}$ is not equal to 0 and the separation between scales in the reiterated homogenization procedure is not complete. However, under the assumptions of dilution 5.102, 5.103 and quick separation between scales 5.105, it is proven that the complete homogenization procedure associated to A_N control the homogenized matrix $(K_N)_{hom}$ associated to the multi-scale media K:

$$A_N - O(\frac{1}{N}) \leq (K_N)_{hom} \leq A_N + O(\frac{1}{N}) \tag{5.108}$$

The general technique used to obtain this is to replace the solution of the cell problem by its first order approximation in the method of asymptotic expansion and use it as a test function in the two variational formulations (5.106 for the upper bound and a formulation similar to 5.22 for the lower bound). But the error one makes by this way is of order of the ratio between scales $\frac{\epsilon_{k+1}}{\epsilon_k}$ multiplied by a constant which tends to grow with the number k of scales. Thats way the quick separation of scales 5.105 is needed for this technique so that quick decrease of $\frac{\epsilon_{k+1}}{\epsilon_k}$ "compensate" the growing error term with k.
In order to calculate A_N, an additional control is added to a^N: the differentiability of the homogenization procedure with respect to the dilution factor (also called concentration in DEM-T) $\frac{1}{N}$. That is to say, for any positive definite matrix C obeying the inequalities $\nu^{-1}I_d < C < \nu I_d$ with $\nu > 0$

$$\big(a^N C\big)_{hom} = C + \frac{M(C)}{N} + o\big(\frac{1}{N}\big) \tag{5.109}$$

where the remainder is uniformly small with respect to C and $M(C)$ is some symmetric matrix which depends continuously on C (this matrix function is assumed to be of class C^1)
Then just as in the paper of Avellaneda [Ave87], when the number of scales N grows towards infinity and the concentration $\frac{1}{N}$ towards 0, the asymptotic behavior of the effective medium $(K_N)_{hom}$ is given by the solution of the following equation, which is a rigorous form of the heuristic equations found in DEM theory.

$$\frac{dA(t)}{dt} = M(A), \quad A(0) = I_d \tag{5.110}$$

$$\lim_{N\to\infty} (K_N)_{hom} = A(1) \tag{5.111}$$

5.3 Rate of Convergence Towards the limit process

When homogenization takes place, two natural problems appear, the first one is to identify and characterize the limit object; the second one and generally harder one is to control the speed of Convergence towards equilibrium. In general much less work has been done in the latter field.

5.3.1 From short time estimates to long time behavior

Consider the self adjoint operator $H \in L^2(\mathbb{R}^d)$ given formally by (see [Dav93] for the introduction of this section).

$$Hf = -\sum_{i,j} \partial_i \{A_{ij}(x)\partial_j f\} \tag{5.112}$$

where $A(.)$ lies in the space $\mathcal{F}$ of measurable functions on $\mathbb{R}^d$ with values in the set of real symmetric matrices, and satisfying

$$\lambda^{-1} \leq A(x) \leq \lambda \tag{5.113}$$

for almost all $x \in R^d$, where $0 < \lambda < \infty$. As it has been shown in the chapter 3 many bounds on the heat kernel p can be written in the form

$$p(t,x,y) \leq c_{1,\delta} t^{-\frac{d}{2}} \exp[-\frac{d(A_{max},x,y)^2}{(4+\delta)t}] \tag{5.114}$$

$$p(t,x,y) \geq c_{2,\delta} t^{-\frac{d}{2}} \exp[-\frac{d(A_{min},x,y)^2}{(4-\delta)t}] \tag{5.115}$$

valid for all $\delta > 0$ and $t > 0$, where A_{min} and A_{max} lie in $\mathcal{F}$. In the above, $d(B,x,y)$ denote the Riemannian distance between x and y for the Riemannian metric B^{-1} given $B \in \mathcal{F}$. Explicitly

$$d(b,x,y) = \sup\{\psi(x) - \psi(y) \,:\, \psi \in \mathcal{D}_B\} \tag{5.116}$$

where $\mathcal{D}_B$ is the set of $\psi \in C^\infty(\mathbb{R}^d)$ such that

$$\sum_{i,j} B_{ij}(x)\partial_i\psi\partial_j\psi \leq 1 \tag{5.117}$$

for almost $x \in R^d$.

Aronson type estimates In [Aro67] Aronson proved the existence of the estimates of the form 5.114 and 5.115 with

$$A_{min,i,j}(x) = \alpha\delta_{ij}, \quad A_{max,i,j} = \beta\delta_{ij} \tag{5.118}$$

for some positive constants α and β independent of t and x.
In [Dav87], E.B. Davies proved the upper bound with $A_{max} = A$.
In is also interesting to note that in [DP89], E.B. Davies and M.M.H. Pang prove that

$$0 \leq p(t,x,y) \leq c_4 t^{-\frac{d}{2}} (1 + \frac{|x-y|^2}{t})^{\frac{1}{2}} \exp[-\frac{d(A,x,y)^2}{4t}] \tag{5.119}$$

and in one dimension the following explicit formula is available for $x \leq y$

$$d(a,x,y) = \int_x^y A(z)^{-\frac{1}{2}} dz \tag{5.120}$$

Short time behavior When A is continuous it is known that

$$\lim_{t\downarrow 0} t \ln p(t,x,y) = -\frac{d(A,x,y)^2}{4} \tag{5.121}$$

which gives the short time behavior; this limit reflects a large deviation principle which says that for x, y fixed and $t \downarrow 0$ the paths of the process concentrate on the geodesics minimizing the distance between x and y.

5.3.1.i Long time behavior in periodic medium

Now imagine that A is periodic of period T_1^d for instance. Fix the points x, y; for t small (if A is continuous) the behavior of the heat kernel is governed by the large deviation principle 5.121, the process remains close to the geodesics. For intermediate t; the process feel the particular form of A the heat kernel is controlled by Aronson type estimates 5.118. For t large the process only sees an effective medium and it has been conjectured by E.B. Davies in [Dav93] its heat kernel satisfies bounds of the form 5.114 and 5.115 where for all $x \in R^d$

$$\lim_{t\to\infty} A_{min}(x) = \lim_{t\to\infty} A_{max}(x) = A_{eff} \tag{5.122}$$

Where A_{eff} is the homogenized matrix associated to the effective medium (see the sub subsection 5.2.1.v).
This conjecture concerns the following important question: *At what speed, the heat kernel pass from the Aronson estimates behavior to the effective medium behavior ?*

J.R. Norris and D.W. Stroock result In dimension one; the lower bound is a corollary of J.R. Norris and D.W. Stroock's result [NS89]. More precisely Norris Stroock proved the lower bound in dimension d with

$$A_{min}^{-1} = g_t * A^{-1} \tag{5.123}$$

where $*$ denotes convolution,

$$g_t(x) = t^{-\frac{d}{2}} g(\frac{x}{\sqrt(t)}) \tag{5.124}$$

and g is a positive function on $\mathbb{R}^d$ which satisfies

$$\int g = 1, \quad \int g^{-1}|\nabla g|^2 < \infty \tag{5.125}$$

Then the Norris-Stroock lower bound satisfies

$$\lim_{t\to\infty} A_{min}(x) = K \tag{5.126}$$

where the constant matrix K is given by

$$K^{-1} = \int_{T_1^d} A(x)^{-1} \tag{5.127}$$

E.B. Davies result In [Dav93], E.B. Davies obtain the upper bound in the general non-periodic one-dimensional case, which completes the proof of the conjecture in one dimension. .
More precisely, it is proven in [Dav93] (dimension one) that

$$p(t, x, y) \le c_5(\lambda) t^{-\frac{1}{2}} \exp[-\frac{|x-y|^2}{4A_{eff}t}] \tag{5.128}$$

provided

$$t^{-\frac{1}{2}} \le \exp[-\frac{|x-y|^2}{4A_{eff}t}] \tag{5.129}$$

Norris result In [Nor92], J.R. Norris proves that

$$\lim_{\substack{|x-y|^2/t\to\infty \\ |x-y|^2/t(\ln t)^+\to 0}} \frac{\Big\{\ln t^{\frac{d}{2}} + \ln p(t,x,y)\Big\}}{\frac{d(A_{eff},x,y)^2}{4t}} = -1 \tag{5.130}$$

Thus precise asymptotic are obtained for $|x-y|^2/t \to \infty$:

- if at the same time $|x-y|/t \to \infty$ then the paths of the process concentrate on the geodesics and the behavior of the heat kernel is controlled by the Riemannian distance $d(A, x, y)$
- if at the same time $|x-y|^2/t(\ln t)^+ \to 0$, the process has the time to feel the periodic structure and the behavior is controlled by the homogenized metric $d(A_{eff}, x, y)$

5.3.2 Generalized Aronson estimates

In [Nor97] (see this article for this subsection), J.R. Norris consider the operator L on $L^2(\mathbb{R}^d)$ given by

$$\int_{\mathbb{R}^d} f L g dm = -\int_{\mathbb{R}^d} \nabla f (A+\Gamma)\nabla g\, dm + \int_{\mathbb{R}^d} f b \nabla g dx \tag{5.131}$$

where m is a Borel measure on $\mathbb{R}^d$ uniformly equivalent to Lebesgue measure with density μ,

$$A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad \Gamma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \tag{5.132}$$

are measurable, respectively symmetric and anti-symmetric and where $b \in R^d$ is a constant vector. There exists a constant $\lambda \in (0,\infty)$ so that for all $x, l \in R^d$

$$\frac{l^2}{\lambda} \le {}^t l.A(x).l \le \lambda l^2, \quad \frac{1}{\lambda} \le \mu(x) \le \lambda \tag{5.133}$$

$$|\Gamma(x)| \le \lambda \tag{5.134}$$

In the case $b \neq 0$, it is also assumed that for some bounded measurable vector field ξ on $\mathbb{R}^d$ all test-functions f verify

$$\int_{\mathbb{R}^d} f dm - \int_{\mathbb{R}^d} f dx = \int_{\mathbb{R}^d} \xi.\nabla f dx \tag{5.135}$$

with

$$|b||\xi(x)| \le \lambda \tag{5.136}$$

Formally

$$Lg = \operatorname{div}_m\big((A+\Gamma)\nabla g\big) + \frac{b}{\mu}.\nabla g \tag{5.137}$$

where div_m denotes the divergence associated with m

Theorem 5.3.1. *There exist a constant $C < \infty$, depending only on λ and the dimension, such that for all $x, y \in \mathbb{R}^d$ and $t > 0$*

$$\frac{1}{Ct^{\frac{d}{2}}} \exp\big(-C\frac{|x-y|^2}{t}\big) \le p(t,x,y+bt) \le \frac{C}{t^{\frac{d}{2}}} \exp\big(-\frac{|x-y|^2}{Ct}\big) \tag{5.138}$$

5.3.2.i Long time estimates in periodic bounded medium

Now assume in addition, that A, Γ, μ are periodic of period T_1^d and

$$m(T_1^d) = 1, \quad |b| \leq \lambda \tag{5.139}$$

then one can characterize an effective conductivity σ by the following variational formulation (σ_{sym} is its symmetric part):
for all $l, \xi \in R^d$

$$\begin{aligned} &{}^t(\xi - \sigma l)\sigma_{sym}^{-1}(\xi - \sigma l) \\ &= \inf_{f,H} \int_{T_1^d} |\frac{\xi}{\mu} - H\nabla_m - (A+\Gamma)(l - \nabla f) - (b.l)\xi + b\frac{f}{\mu}|_{A^{-1}}^2 dm \end{aligned} \tag{5.140}$$

Where the minimum is taken on $f \in C^\infty(T_1^d)$ such that $\int_{T_1^d} d\,dm = 0$, H varies in the set of smooth skew-symmetric $d \times d$ periodic matrices in $C^\infty(T_1^d)^{d(d-1)/2}$. $H\nabla_m$ is the divergence of H with respect to m that is to say; the vector field characterized by

$$\int_{T_1^d} (H\nabla, \omega) dm = -\int_{T_1^d} (H, d\omega) dm \tag{5.141}$$

for all one-forms ω.

$$\big(H\nabla\big)_i = \sum_{j=1}^d \partial_j H_{ij} \tag{5.142}$$

Theorem 5.3.2. *There exist are constants $\alpha \in (0,1)$ and $C < \infty$, depending only on λ and the dimension, such that for all $x, y \in \mathbb{R}^d$ and $t > 0$*

$$\begin{aligned} &\frac{1}{t^{\frac{d}{2}}} \exp\Big(- C\frac{e^{C\frac{|x-y|^2}{t}}}{t^\alpha} - C\big(1 + \frac{|x-y|^2}{t}\big)^{\frac{1}{2}} - \frac{{}^t(x-y)\sigma_{sym}^{-1}(x-y)}{4t}\Big) \\ &\leq p(t,x,y+bt) \\ &\leq C\frac{\big(1+\frac{|x-y|^2}{t}\big)^{\frac{d}{2}}}{t^{\frac{d}{2}}} \exp\Big(C\frac{e^{C\frac{|x-y|^2}{t}}}{t^\alpha} - \frac{{}^t(x-y)\sigma_{sym}^{-1}(x-y)}{4t}\Big) \end{aligned} \tag{5.143}$$

Thus precise homogenized asymptotic are obtained in the regime $|x-y|^2/t \to \infty$ and $|x-y|^2/t(\ln t)^+ \to 0$.

5.3.2.ii Long time estimates in periodic bounded differentiable medium

Now assume in addition, that A, Γ, μ are weakly differentiable and for all i, j; $x \in \mathbb{R}^d$

$$|\nabla A_{ij}| \leq \lambda, \quad |\nabla \mu| \leq \lambda \tag{5.144}$$

$$|\nabla \Gamma_{ij}| \leq \lambda \tag{5.145}$$

then

Theorem 5.3.3. *There exist are constants $\alpha \in (0,1)$ and $C < \infty$, depending only on λ and the dimension, such that for all $x, y \in \mathbb{R}^d$ and $t > 0$*

$$\begin{aligned} &\frac{1}{t^{\frac{d}{2}}} e^{-C(VE+E^{\frac{1}{2}})} \exp\Big(- \frac{{}^t(x-y)\sigma_{sym}^{-1}(x-y)}{4t}\Big) \\ &\leq p(t,x,y+bt) \\ &\leq C\frac{E^{\frac{d}{2}} e^{CVE}}{t^{\frac{d}{2}}} \exp\Big(- \frac{{}^t(x-y)\sigma_{sym}^{-1}(x-y)}{4t}\Big) \end{aligned} \tag{5.146}$$

where $V = \frac{|x-y|}{t}$ and $E = 1 + \frac{|x-y|^2}{t}$

Thus precise homogenized asymptotic are obtained in the regime $|x-y|^2/t \to \infty$ and $|x-y|/t \to 0$.

5.3.3 Further results

5.3.3.i Speed of convergence to equilibrium in medium with two distinct periodic scales

Rabi Bhattacharya (in [Bha99], see also [BDG99] by Bhattacharya - Denker and Goswami), analyze the large-time behavior of a class of time-homogeneous diffusion processes in $\mathbb{R}^d$ whose medium is characterized by a small scale T_1^d and a large scale RT_1^d.

$$dy_t = \sigma(y_t)d\omega_t + (b(y_t) + \beta(\frac{y_t}{R}))dt \tag{5.147}$$

where b and β are divergence-free drifts of period T_1^d.
This leads to phase changes in the behavior of the process as time increase through different time zones. In [Bha99] two distinct Gaussian phases (homogenized) occurs:

- The initial Gaussian phase is exhibited over $1 << t << R^{\frac{2}{3}}$; where b has been replaced by an homogenized drift and homogenization is realized on the smaller scale and the fluctuation of β is not felt
- Depending on geometric conditions on the velocity field β the final Gaussian phase occurs for times $t >> R^2(\ln R)^2$, $t >> R^2 \ln R$ or $t >> R^4(\ln R)^2$; where b and β are replaced by homogenized drifts and homogenization is realized on both scales.
- Particular examples of b and β show the existence of non Gaussian intermediate phases when $R \to \infty$ and the time stays in the intermediate phase given by R.

This interesting article was motivated in part by the "scale effect" in the dispersion of solute matter such as a chemical pollutant injected at a point in an underground water system in which the increase of dispersivity is explained by existence of multi-scale heterogeneities in the medium. It shows the dynamical image of the scales separation phenomenon. Indeed the divergence free drifts b, and β manifest there influence at well separated scales in time and space and above a certain scale their uniting creates a Gaussian diffusion but with greater diffusivity than in the molecular diffusion coefficient of the solute.
It is interesting to observe that the crucial key leading to results of this article is the control of the speed of convergence towards equilibrium (one of the means is to control of the spectral gap).
An other interesting observation of this article lies in the particular shear flow example $b = e_1 c_1 \sin(2\pi x_2)$ and $\beta = e_1 c_2 \cos(2\pi x_2)$. Indeed between the two Gaussian regime, when $R^{\frac{4}{3}} << t << R^2$ or $t \sim cR^2$, this diffusion shows a non Gaussian behavior.

5.3.3.ii Competition between large deviation and homogenization

In [FS99] Mark I. Freidlin and Richard B. Sowers consider in $\mathbb{R}^d$ the following kind of stochastic differential equation

$$dy_t = \sqrt{\epsilon}\sigma(\frac{y_t}{\delta_\epsilon})d\omega_t + b(\frac{y_t}{\delta_\epsilon})dt \tag{5.148}$$

Where σ and b are periodic of period T_1^d.
It is shown that there are three regimes depending on the relative rates at which the small viscosity parameter ϵ and the homogenization parameter δ tend to 0.

- $\lim_\epsilon \delta_\epsilon/\epsilon = 0$. Homogenization dominates, the large deviation of y_t are the same as those of a constant-coefficient system.

- $\lim_\epsilon \delta_\epsilon/\epsilon = c \in (0,\infty)$. The generator tends to an operator characterized by large deviation and homogenization
- $\lim_\epsilon \delta_\epsilon/\epsilon = \infty$. The large deviation principle for y_t is given by first finding the large deviation principle with δ fixed and then letting the period δ tend to zero.

See also [FX98], [KP91], [Bal95], [Mak93] for more results of this kind.

Part III

COMMENTS, MORE RESULTS AND INSIGHT

6. SUB-DIFFUSIVE MODEL

6.1 Insight

6.1.1 General set up

The purpose of this chapter is to give to the reader an insight of the variety of phenomena manifested by the solution of the stochastic differential equation (corresponding to 1.1)

$$\begin{cases} dy_t = d\omega_t - \nabla V(y_t)dt \\ y_0 = 0 \end{cases} \tag{6.1}$$

First observe that this SDE is a model of transport associated to the following partial differential equation

$$\frac{\partial}{\partial t} f = \frac{1}{2}\Delta f - \nabla V \nabla f \tag{6.2}$$

Of course one could consider a greater variety of PDE such as

$$\frac{\partial}{\partial t} f = \sum_{i,j=1}^{d} a_{i,j}(x)\partial_i \partial_j f + b.\nabla f \tag{6.3}$$

but the ideas and tools given here would remain the same (however as it will be shown in the next chapter the behavior might change). Moreover the model 6.2 allows to dissociate the influence of the variation of the local diffusivity $a_{i,j}$ (which is fixed to be 1/2 here) from the influence of variation of the local drift b (which, here, is the gradient of a potential V).
Thus this chapter focus on the richness of phenomena manifested by the diffusion associated to the invariant measure e^{-2V}. For a physicist V would represent the potential energy landscape on which a system y_t is evolving under the influence of a thermal noise $d\omega_t$ and the force $-\nabla V$ which reflects the propensity of the system to minimize the potential energy, V will be assumed to be time independent (the next step of exploration would be to make it time dependent).
First observe that if V is periodic or ergodic (and bounded) $\epsilon y_{t/\epsilon^2}$ converges (as $\epsilon \to 0$) to a Gaussian process with diffusivity matrix $D(V)$ for which one has a beautiful variational formulation.
Next observe that by the Aronson's estimates that if V is bounded then y_t exhibits a Gaussian behavior. This is well known, so now let's look a little bit beyond this picture and assume that V is unbounded.
If one knows the particular shape of V one might be able to say something on the particular behavior of y_t (particular to the shape) but this is not the case considered here. Now decompose the fluctuations of V into a infinite sum $V = \sum_{k=0}^{\infty} S_{1/R_k} U_k$ where R_k stands for the fluctuations of typical size R_k (R_k growing with k) and U_k (whose typical fluctuations are of size 1) stands for the shape of those fluctuations and S_R is the scaling operator $S_R : f(x) \to S_R f = f(Rx)$. Of course if one knows nothing about the U_k one can not say anything, in a real physical system it is reasonable to assume that the U_k are spatially ergodic, it would reflect the spatial homogeneity of the system for each typical scale, this would be the ideal picture to analyze but it will not be described in here because it constitutes the next step of exploration. Thus the first step is to analyze a simpler model where the ergodicity of the U_k is replaced by a periodicity condition (each U_k is periodic of period T_1^d) in

a physical system this would mean that the medium on which is evolving the system can be seen as the manifestation of a large number of scales with crystalline structures.
Moreover it is assumed that the norms $\|\nabla U_k\|_\infty$ are uniformly bounded (by K_1, actually in dimension one, an uniform bound on the Holder continuity is sufficient), this bounds means that the drift generated by the scale n is bounded by K_1/R_n and converges towards 0 with the scale (the bigger the scale, the longer one has to wait to feel its drift). Actually this condition is not absolutely necessary to have a well defined diffusion, however it will be shown that without this condition the diffusion can be not governed by the holistic influence of all the scales but by the influence of a single one that might change drastically with the scale (this case is interesting but not studied here.)
Observe that this medium is characterized by a smaller scale R_0 (which is physical) but it has no upper bound on the larger scales, of course one could say the size of the known universe is finite and this model is just a mathematician's fantasy and one would be right however one always observe physical systems over an interval of time which is finite, and it will be shown that to each spacial scale R_k corresponds a temporal scale t_k, and if the system has only a finite number of scales $R_0, \ldots, R_n$ then between the times (t_0, t_n) and the spacial scales (R_0, R_n) one will not see the difference (one has to wait for the time t_n or observe the system over distances R_n to see that the system has only a finite number of scales). Thus for those real physical systems it is important to keep in mind that the strange behavior of the diffusion y_t exhibited in this chapter will manifest itself between corresponding time and space scales.
R_k is assumed to grow to infinity as $k \to \infty$ at least geometrically: the ratio $r_n = R_n/R_{n-1}$ between two successive scales is assumed be lower bounded by $\rho_{\min} \geq 2$ to represent the different orders of magnitude (of course an other approach would be to decompose V on a continuum of scales but actually it will be shown that this approach is in a sense similar in the sense of overlapping ratios), the greater r_n is, the more separated the scales R_{n-1} and R_n are.
Thus $R_n \geq \rho_{\min}^n$, now a useful assumption is made: Each ratio r_n is an integer, with this assumption each R_n is an integer and each aggregation of scales V_0^n is periodic of period $T_{R_n}^d$. Of course this assumption is artificial in the sense that it is made to simplify the computations and the proofs (if one wants absolutely to translate this assumption into a physical scheme one would say that the crystalline structure of the scale n is a sub-crystalline structure of the scale $n+1$), nevertheless the ideas given here remain valid without this assumption even in the ergodic case (U_n are ergodic), what does change is the difficulty to translate them into mathematical proofs (actually in dimension one it is easy to get rid of this assumption, but it will not be done here because it will only makes the presentation less clear without bringing any new idea).

6.1.2 Heuristic analysis of the mean squared displacement

The purpose of this subsection is to give a short heuristic analysis of the model given in the subsection 1.1.1, more precisely the objective here is to introduce the though process, the philosophy and the basic concepts which will allow to obtain rigorous results on the solution of the SDE 6.1.
Observe that

$$y_t = x + \omega_t - \int_0^t \nabla V_0^n(y_s)\, ds - \int_0^t \nabla V_{n+1}^\infty(y_s)\, ds \tag{6.4}$$

Observe that the term $\int_0^t \nabla V_{n+1}^\infty(y_s)\, ds$ in 6.4 is bounded by $2K_1 t/R_{n+1}$, thus for t fixed as $n \uparrow \infty$, the global influence of the scales $n+1, \ldots, \infty$ is less and less felt by the diffusion, this is interesting ! but one has to be careful: observe that although bounded by $2K_1/R_{n+1}$, ∇V_{n+1} may vary over very small scales (since only the first derivatives of the U_n are uniformly bounded), thus one can not say that $\int_0^t \nabla V_{n+1}^\infty(y_s)\, ds$ behaves as the uniform drift $tV_{n+1}^\infty(x)$ whose value is fixed by the starting point x of the diffusion; $\int_0^t \nabla V_{n+1}^\infty(y_s)\, ds$ is a small drift but not uniform !

Thus it is natural to seek for a critical scale n_{flu} (flu stands for fluctuating scale) above which the influence of the term $\int_0^t V_{n+1}^\infty(y_s)\, ds$ can be neglected in front of the term $\int_0^t V_0^n(y_s)\, ds$. These brings

to the important question: how to determine n_{flu}? The answer is: it depends on which property of the diffusion one is looking at.
Actually since the name of this thesis begins with anomalous one will not be surprised that this chapter will focus on the properties which underline the anomaly of a diffusion:

- The mean squared displacement $\mathbb{E}[y_t^2]$
- The mean time to exit from a ball of radius r: $\mathbb{E}[\tau(B(0,r))]$
- The tail of the transition probability densities $\mathbb{P}[y_t.l \geq h]$ ($l \in \mathbb{S}^d$ where $\mathbb{S}^d$ is the unit sphere of $\mathbb{R}^d$)

Consider for instance the mean squared displacement $\mathbb{E}[y_t^2]$, write for $l \in \mathbb{S}^d$, χ_l^n the solution of the cell problem associated to V_0^n (which is periodic of period R_n: now one understands why it is useful to assume the ratios to be integers, this ensures the existence of a periodic solution to cell problem associated with an aggregation of finite number of scales). Then one can deduce from 6.4 that for $l \in \mathbb{S}^d$ (from now, assume to simplify the equations that the starting point x of the diffusion is 0 and choose $\chi_l^n(0) = 0$)

$$y_t.l = \chi_l^n(y_t) + \int_0^t (l - \nabla\chi_l^n(y_s))d\omega_s - \int_0^t \nabla V_{n+1}^\infty(y_s)(l - \nabla\chi_l^n(y_s))\, ds \tag{6.5}$$

and after a basic computation

$$\begin{aligned} \mathbb{E}[|y_t.l|^2] &\leq 3\mathbb{E}\big[\int_0^t |l - \nabla\chi_l^n(y_s)|^2\big]\big(1 + \|\nabla V_{n+1}^\infty\|_\infty^2 t\big) + 3\|\chi_l^n\|_\infty^2 \\ &\geq \mathbb{E}\big[\int_0^t |l - \nabla\chi_l^n(y_s)|^2\big]\big(\frac{1}{3} - \|\nabla V_{n+1}^\infty\|_\infty^2 t\big) - \|\chi_l^n\|_\infty^2 \end{aligned} \tag{6.6}$$

Now let's stop a while to have a close look at 6.6, this equation suggests that the influence of the larger scales can be neglected if $\|\nabla V_{n+1}^\infty\|_\infty^2 t < 1/6$ which is implied by

$$6K_1 t < R_{n+1}^2 \tag{6.7}$$

Thus it is natural to fix the value of n_{flu} to be the first integer such that $6K_1 t < R_{n_{flu}+1}^2$, thus for the mean squared displacement n_{flu} is fixed by the time t and it is natural to call the scales $n_{flu}+1, \ldots, \infty$ the drift scales since their influence appear to be limited by $\|\nabla V_{n+1}^\infty\|_\infty$.
Now observe the term $\|\chi_l^n\|_\infty^2$, $\|\chi_l^n\|_\infty$ reflects the typical distance put by a diffusion generated by $L_{V_0^n}$ so that the drift $\int_0^t \nabla V_0^n(y_s)\, ds$ produced by the smaller scales $0, \ldots, n$, behaves like a martingale. Observe that for

$$\mathbb{E}\big[\int_0^t |l - \nabla\chi_l^n(y_s)|^2\big] > 12\|\chi_l^n\|_\infty^2 \tag{6.8}$$

the error term $\|\chi_l^n\|_\infty^2$ can be neglected in front of the leading term $\mathbb{E}\big[\int_0^t |l - \nabla\chi_l^n(y_s)|^2\big]$, and the inequality 6.8 is valid for t big enough:

$$t > t_{1,n} \tag{6.9}$$

Now observe the term $\mathbb{E}\big[\int_0^t |l - \nabla\chi_l^n(y_s)|^2\big]$, if the generator of y_s were $L_{V_0^n}$, homogenization theory tells that this term would be equal to

$${}^t l D(V_0^n) l\, t + \mathbb{E}[\phi_l^n(y_t)] \tag{6.10}$$

where ϕ_l^n is the periodic solution of the ergodicity problem 5.10 associated to V_0^n, but here the generator of the diffusion is $L_{V_0^n + V_{n+1}^\infty}$, nevertheless the larger scales constitute a very small drift and

it is reasonable to think that their perturbation of the generator will perturb a little bit but will not destroy the homogenization picture 6.10 (this is an heuristic hocus pocus for the moment).
Thus it is reasonable to think that for

$$^{t}lD(V_0^n)l\,t > C\|\phi_l^n\|_\infty \tag{6.11}$$

$\mathbb{E}\big[\int_0^t |l - \nabla\chi_l^n(y_s)|^2\big]$ will be of order of ${}^{t}lD(V_0^n)l\,t$. Let's see now what has been obtained: for

$$t_{2,n} < t < R_{n+1}^2 \tag{6.12}$$

one has

$$\mathbb{E}[|y_t.l|^2] \propto {}^{t}lD(V_0^n)l\,t \tag{6.13}$$

Thus, a priori, to obtain the behavior of the mean squared displacement, one has to estimate the behavior of the effective diffusivity associated to an homogenization over the smaller scales $0, \dots, n$ and also to estimate $t_{2,n}$ which reflects the typical time put by the diffusion to homogenize on those smaller scales.
One might wonder how this will give the anomalous behavior of the diffusion: assume that the ratio between scales is constant $\rho_{\max} = \rho_{\min} = \rho$, and the effective diffusivity associated to each scale is constant and isotropic $D(U_n) = \lambda I_d$, homogenization theory says that if the fluctuations U_n of the medium are not constant functions then $0 < \lambda < 1$. Observe now the effective diffusivity associated to an aggregation of two scales V_0^1 for instance, then the asymptotic expansion method allows to see that as $\rho \uparrow \infty$, $D(V_0^1) \to \lambda^2$, thus if ρ is very large, for each scale that one adds, the effective diffusivity decreases by a factor $\lambda < 1$ and $D(V_0^n) \propto \lambda^{n+1}$. Now assume that the typical time to homogenize on those scales $t_{2,n}$ behaves like R_n^2; then the condition 6.12 fix the value of the cut-off scale n_{flu} to be

$$n_{flu} \propto \frac{\ln t}{2\ln\rho} \tag{6.14}$$

And it follows from 6.13 that

$$\mathbb{E}[|y_t.l|^2] \propto t^{1-\nu} \tag{6.15}$$

with

$$\nu = -\frac{\ln\lambda}{2\ln\rho} > 0 \tag{6.16}$$

Which says that the y_t is sub-diffusive. Of course this short analysis of the mean squared displacement was heuristic in the sense that the hard point is to make it rigorous, however it has allowed to introduce some basic concepts and ideas which will be discussed in the next subsection.

6.1.3 The strategy

Consider $U \in C^\infty(T_R^d)$, thus U reflects a periodic medium of period T_R^d $(R > 0)$, the patterns formed by the particular shape of U in its period are called soft obstacles. Consider now a Brownian motion evolving in this medium and submitted to the drift $-\nabla U$, the generator of this diffusion is L_U. From a heuristic point of view which can be justified through the Dirichlet form if U takes only values equal to 0 $+\infty$ (then the forms shaped by U are called hard obstacles) where the boundary of the region with $+\infty$ is smooth, then y_t is a Brownian motion moving in the 0 value region and with normal reflection against the boundary of the $+\infty$ value region.
Then one can associate U with characteristic mixing length (correlation length) $\xi_m(U)$ and a characteristic mixing time $\tau_m(U)$. They represent the "size" of the spatial correlation and the temporal

correlation of a diffusion evolving among the obstacles shaped by U.
In other words, $\xi_m(U)$ represents the typical length after which the diffusion has homogenized on the inhomogeneities of U and "sees" only an effective medium characterized by $D(U)$, the effective diffusivity of U (one might think that $\xi_m(U)$ is of order of the period length R, in general one will be right but it will be shown that this is not always the case); and $\tau_m(U)$ reflects the typical time needed by the diffusion to mean the inhomogeneities of U and "feel" only an effective medium. (note that $\xi_m(U)$ and $\tau_m(U)$ depends on molecular diffusivity of the medium which is 1 here since the diffusive transport is represented by a Brownian motion)
Now since U is smooth ∇U is also characterized by a visibility time $\tau_v(U)$ and a visibility length $\xi_v(U)$. In other words, $\xi_v(U)$ represents the length (and τ_v the delay) below which the diffusion starting from any point, is not too much influenced by the drift ∇U or it feels only a reflection against an infinite $d-1$ dimensional wall if U represents a hard obstacle with smooth boundary (the wall is then the tangent hyper-plane to the obstacle at x). If U is smooth it is natural to expect that $\tau_v(U) \propto 1/\|\nabla U\|_\infty$ and in the case of hard obstacle it is natural to expect that $\xi_v(U)$ is of order of the inverse of the typical curvature of the boundaries of the obstacles. Let's remember the previous example on the mean squared displacement, when U is smooth $\tau_v(U)$ does not necessarily represent the time below which $\nabla U(y_t)$ is very close to $\nabla U(x)$ where x is the starting point of the diffusion (in a sense here there is a conceptual difference between hard and soft obstacles).

How this is translated in the framework of a process evolving among an infinite number of scales: For instance for the mean squared displacement: It is clear that the only parameter here is t, for an aggregation of scales V_0^n if $\tau_m(V_0^n) < t$ then V_0^n is felt by the diffusion only through its effective diffusivity $D(V_0^n)$ and for the larger scales if $t < \tau_v(V_p^\infty)$ then the drift associated to V_p^∞ can be neglected. Now assume that

$$\tau_m(V_0^n) < t < \tau_v(V_{n+1}^\infty) \tag{6.17}$$

Then it follows that for this value of t the mean squared displacement behaves as if the diffusion had homogenized on the scales $0, \dots, n$ without feeling the scales $n+1, \dots, \infty$. Which leads to

$$\mathbb{E}[y_t^2] \propto t\,\mathrm{Trace}\left(D(V_0^n)\right) \tag{6.18}$$

and if $D(V_0^n)$ decreases geometrically to 0 with n as in the previous example, the diffusion shows an anomalous behavior. One might think that the parameters $\xi_m, \xi_v, \tau_m, \tau_v$ associated to each aggregation of obstacles are no more than conceptual tools, this is not the case: they appear everywhere in the computations for each property than one tries to characterize; now one might think that if they appear everywhere it is because of the method used in this work, this is an other point but those parameters are hidden in the results which characterize the behavior of a IHPD which are independent from the strategy used to prove them.

6.1.4 Working plan for the application of the strategy

Now it has become clear that to characterize the behavior of an IHPD one has to find estimates (the sharper, the better) of the mixing and visibility times and scales associated to each aggregation of scales. Moreover to prove the sub-diffusive behavior of the IHPD one has also to obtain an estimate of the speed of convergence towards 0 of the effective diffusivity associated with the aggregation of the first $n+1$ smaller scales.
One might think that this will be sufficient, unfortunately this is not as simple as that, indeed if the ratio between scales is bounded $\rho_{\max} < \infty$, although one can obtain very sharp estimates for $\xi_m, \tau_m, \xi_v, \tau_v$ and $D(V_0^n)$ one find oneself in front of the following inequalities

$$\tau_m(V_0^n) > \tau_v(V_{n+1}^\infty) \tag{6.19}$$

and

$$\xi_m(V_0^n) > \xi_v(V_{n+1}^\infty) \tag{6.20}$$

Which means that one can never find a time t such that

$$\tau_m(V_0^n) < t < \tau_v(V_{n+1}^\infty) \tag{6.21}$$

because the scales $n+1,\ldots,\infty$ become visible before homogenization on the smaller scales $0,\ldots,n$ has ended and those larger scales perturb this homogenization process. One might think, that this pathology is just an artefact created by the fact that the parameters $\xi_m, \tau_m, \xi_v, \tau_v$ are not sharp, but this is not the case; indeed simple examples shows that they are sharp. In fact this pathology is the reflection of an underlying overlap and interaction between scales, the smaller the ratio between scales and the stronger and deeper this interaction.

How to get rid of this pathology? One can not because it is inherent one has to do with it !

How to control it? to find the answer observe the following simple example: Consider $U \in C^\infty(T_1^d)$ and the diffusion y_t with generator L_U evolving in the periodic medium associated to U. For $t < \tau_v(U)$, y_t will behave like a Brownian motion plus a small drift, for $t > \tau_m(U)$, y_t will behave like a Gaussian process with effective diffusivity $D(U)$. It follows that $\tau_v(U) < \tau_m(U)$, then what happens between $\tau_v(U)$ and $\tau_m(U)$? Nobody knows because it depends on U, between those two times the particular shape of U manifests itself in the behavior of the diffusion; homogenization theory does not control the influence of U between those two times but hide it in the solution of the cell problem χ_l associated to U. Can the influence of U be controlled between those two times? Yes, by the Aronson estimates (which are a control of the transition probability densities) and the control on the Green functions (which leads to control on the exit times), in those controls U appears as a perturbation of the Laplace operator.

Now let's return to the IHPD, and let $t > 0$, write

$$n_{ef}(t) = \sup\{p \in \mathbb{N} \,:\, \tau_m(V_0^p) < t\} \tag{6.22}$$

and

$$n_{flu}(t) = \inf\{p \in \mathbb{N} \,:\, \tau_v(V_{p+1}^\infty) > t\} \tag{6.23}$$

$$n_{per} = n_{flu} - n_{ef} \qquad n_{dri} = n_{flu} + 1 \tag{6.24}$$

Then the scales $0,\ldots,n_{ef}$ are effective scales (ef for effective) in the sense that at the time t those scales are seen by the diffusion as an effective medium with effective diffusivity $D(V_0^{n_{ef}})$.

The scales $n_{dri},\ldots,\infty$ are drift scales (dri for the drift) in the sense that at the time t their influence is limited by the norm $\|\nabla V_{n_{dri}}^\infty\|_\infty$.

What about the remaining n_{per} scales $n_{ef}+1,\ldots,n_{ef}+n_{per} = n_{dri}-1$? One can not consider those scales as effective scales since the mixing time associated to each of them is bigger than t, neither can one consider them as drift scales since their visibility time is smaller than t. In fact those scales are perturbation scales (per for perturbation) in the sense that the particular shape of each of those scales is manifesting itself in the behavior of the diffusion at the time t. In fact "interacting scales" would have been a better name in the sense that it would have reflected the underlying phenomenon however it has been chosen to call them "perturbation" scales because they will enter in the computations and the proofs as a perturbation of the effective scales (if one has no information about the particular shape of those scales, a priori the only thing that one can do with them is to consider them as perturbation scales, however with a precise knowledge of their shapes and internal symmetries one would be able to make them enter into computations as particular scales and keep their specifities).

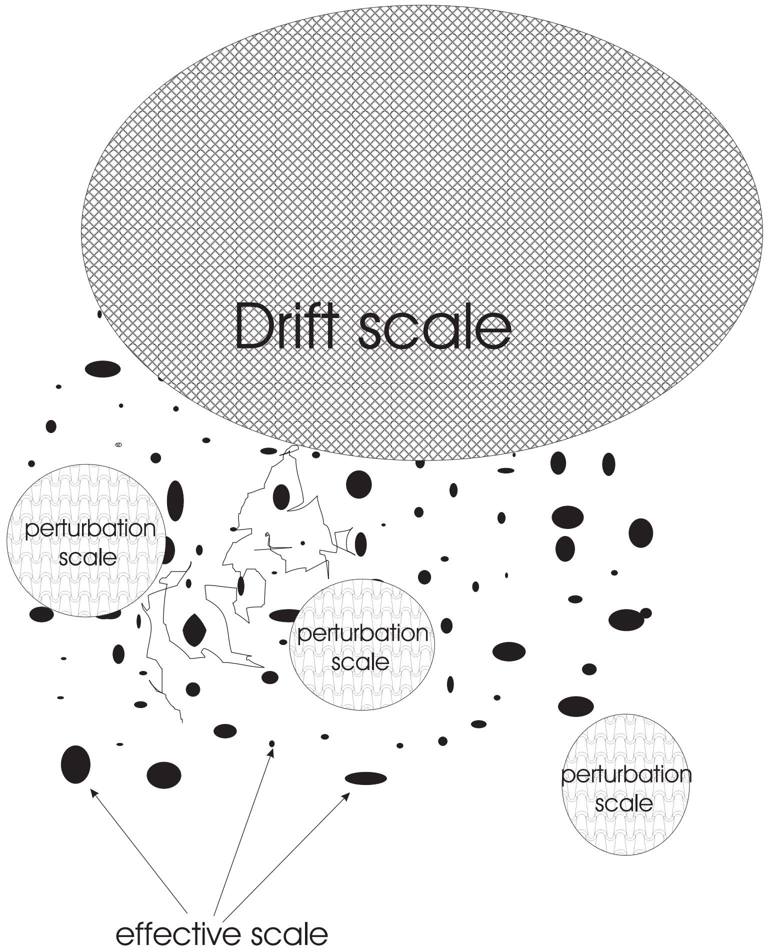

Fig. 6.1: *Effective, perturbation and drift scales.*

The scales $0,\dots,n_{flu}$ (note that $n_{flu} \geq n_{ef}$) are fluctuating scales in the sense that at the time t the fluctuation of the medium at those scales is felt by the diffusion and one can consider none of them as a drift scale.
If the medium has only a finite number of scales and thus a maximal scale $n_{\max}$, this makes no difference in the behavior of the diffusion if $n_{dri} \leq n_{\max}$ and to the scale $n_{\max}$ corresponds a time and length scales $\tau_{\max}$, $\xi_{\max}$. Thus in the time interval $(0,\tau_{\max})$ and length interval $(0,\xi_{\max})$ the results given in this chapter for a process evolving on a medium characterized by an infinite number of scales remain valid. After $\tau_{\max}$ standard homogenization on the scales $0,\dots,n_{\max}$ will take place giving a standard Gaussian process with effective diffusivity $D(V_0^{n_{\max}})$.

Now one can see the additional work that has to be undertaken: control the influence of the perturbation scales. Moreover observe that Aronson estimates and comparison of Green functions reflects a perturbation of the Laplace operator, here the object which is perturbed is the operator $\frac{1}{2}\Delta - \nabla V_0^{n_{ef}}\nabla$ and one can not use the strategy used to obtain the Aronson estimates or to compare Green functions because they will give back an estimate of the influence of $V_{n_{ef}+1}^{\infty}$ which will totally destroy the slow down of the diffusion due to the effective scales. (one can use Aronson estimates and Harnack inequality to perturb the Laplace operator but not the operator associated to the effective scales ! a new strategy has to be found).
How these perturbation scales will manifest themselves in the behavior of the mean squared displacement? Assume that $D(V_0^n) \propto \lambda^n I_d$ with $0<\lambda<1$, observe that if their influence were null one would have

$$\mathbb{E}[y_t^2] \propto t\lambda^{n_{ef}(t)} \tag{6.25}$$

The scales $n_{ef}+1,\dots,n_{ef}+n_{per}$ will perturb this relation into

$$C_1 t\lambda^{n_{ef}(t)}\frac{1}{\mu^{n_{per}(t)}} \leq \mathbb{E}[y_t^2] \leq C_2 t\lambda^{n_{ef}(t)}\mu^{n_{per}(t)} \tag{6.26}$$

with $\mu>1$, now one can guess that if the perturbation is too strong it might destroy the sub-diffusive behavior, assume that this is not the case then one would obtain when the ratio between scales is bounded that

$$\lambda^{n_{ef}(t)}\mu^{n_{per}(t)} \propto t^{-\nu_2} \tag{6.27}$$

$$\lambda^{n_{ef}(t)}\mu^{n_{per}(t)} \propto t^{-\nu_1} \tag{6.28}$$

with $\nu_1>\nu_2>0$ and

$$C_1 t^{1-\nu_1} \leq \mathbb{E}[y_t^2] \leq C_2 t^{1-\nu_2} \tag{6.29}$$

Which gives the sub-diffusive behavior of the mean squared displacement, the lower bound an the upper bound are not because $\mathbb{E}[y_t^2]$ can really oscillate between those two values if the medium is not self similar.
Now one might wonder what happens in the case of fast separation between scales that is to say $R_n \sim \rho^{n^\alpha}$ with $\alpha>1$? Indeed, with this fast separation between scales the diffusion has more and more time to homogenize on the smaller scales before feeling the large ones, and in fact one obtains that

$$\tau_m(V_0^n) < \tau_v(V_{n+1}^{\infty}) \tag{6.30}$$

Thus there exists times t such that there exists $n\in\mathbb{N}$ with

$$\tau_m(V_0^n) < t < \tau_v(V_{n+1}^{\infty}) \tag{6.31}$$

and there is no perturbation scales, thus one will be able to prove sub-diffusivity in a simple way for those times however a closer look shows that there also exist intervals (t_1^n, t_2^n) such that for $t \in (t_1^n, t_2^n)$ no $n \in \mathbb{N}$ does verify the inequalities 6.31 and those intervals correspond to the manifestation of the particular shape of the fluctuation of the media at each scale n. For those time intervals the number of perturbation scales will be equal to 1, one can not get rid of them, they are inherent.

In resume here is the list of the main tasks that will be undertaken in this work:

- Estimate $D(V_0^n)$
- Estimate the mixing times and lengths $\tau_m(V_0^n), \xi_m(V_0^n)$ (actually the estimation of the visibility times and lengths $\tau_v(V_{n+1}^\infty), \xi_v(V_{n+1}^\infty)$ is a trivial task)
- Obtain a sharp control of the influence of the perturbation scales (this is the hardest part)
- Explore the anomalous behavior of the mean squared displacement
- Explore the anomalous behavior of the hitting times (for each property both cases: $\rho_{\max} < \infty$ and $\rho_{\max} = \infty$ will be considered)
- Explore the anomalous behavior of the tail of the probability densities $\mathbb{P}[y_t \leq h]$ (here it will be necessary to improve results on the speed of convergence towards the asymptotic process in a periodic medium by taking into account the perturbation and improving the speed, one might wonder how n_{flu} will be fixed here since there are two parameters t and h, it will be shown that it is fixed by the ratio t/h and this fact has important consequences)
- Explore the pathologies which might appear.

6.2 Anomalous behavior of an IHPD through a simple example

In this section a simple example of anomalous behavior of an one-dimensional IHPD will be given as an introduction to more general results. Since everything can be computed in this simple example it will allow to see that a condition $\rho_{\min} > \rho_0(K_0, K_1, d, \lambda_{\min}, \lambda_{\max})$ is indispensable to guarantee the geometric decrease towards 0 of the effective diffusivity of V_0^n and the sub-diffusive behavior of the IHPD.
Thus consider a self similar IHPD in dimension one: for all n, $U_n = U$ and $r_n = \rho$.
Then (the following corollary is the corollary 8.3.2 of chapter 8).

Corollary 6.2.1. *Let y_t be a self similar infinitely homogenized potential diffusion. Then*

$$\mathbb{E}_0[\tau(0,r)] = r^{2+\nu(r)} \tag{6.32}$$

with

$$\nu(r) = \frac{\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)}{\ln \rho} + \epsilon(r) \tag{6.33}$$

with $\epsilon(r) \to 0$ as $r \to \infty$.

Where $\mathcal{P}_\rho$ is the topological pressure associated to the shift s_ρ (see section C.1) and since it is convex one has $\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) \geq 0$. From this corollary it is easy to see that y_t shows a clear anomalous behavior (clear anomalous meaning $\mathbb{E}[\tau(0,r)] \sim r^{2+\gamma}$ with $\gamma > 0$) if and only if $\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) > 0$. Now the theorem C.1.2 says that this happens if and only if U does not belong to the closed subspace of $\mathcal{C}(T_1^d)$ generated the elements $F(x) - F(\rho^k x)$ with $F \in C(T_1^d)$ and

$k \in \mathbb{N}$. Moreover it is easy to see from the corollary 8.2.2 that there exists $\rho_0(K_1, D(U))$ such that if U is not the constant function, for $\rho > \rho_0(K_1, D(U))$ one has $\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) > 0$ and as $\rho \to \infty$

$$\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) \to \ln \int_{T_1^d} e^{2U} dx + \ln \int_{T_1^d} e^{-2U} dx > 0 \tag{6.34}$$

In resume if U is not a constant function, there exists a constant $\rho_0(K_1, D(U))$ such that for $\rho > \rho_0$, and r big enough $\mathbb{E}_0[\tau(0,r)] \sim r^{2+\nu}$ with $\nu > 0$. The interval $(\rho_0, +\infty)$ is called "separating ratios". What happens in the region $(1, \rho_0]$? the corollary 6.2.1 and the proposition 9.5.1 say that $\ln \mathbb{E}_0[\tau(0,r)] = 2\ln r(1+\epsilon(r))$ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \Big\| \sum_{k=0}^{n-1} \big(S_{\rho^k} U - \int_{T_1^d} U(x)dx\big) \Big\|_\infty = 0 \tag{6.35}$$

Can this phenomenon happen? The section 9.5 answers yes with the simple example

$$U(x) = \sin(x) - \sin(81x) \tag{6.36}$$

for this particular shape of fluctuation of the medium V, for $\rho = 3, 27$ or 81. Thus for this simple example, $\mathbb{E}[\tau(0,r)]$ is anomalous (sub-diffusive $\sim r^{2+\nu}$ with $\nu > 0$) for $\rho \in \{2\} \cup \{4, \dots, 26\} \cup \{28, \dots 80\} \cup \{82, \dots, +\infty\}$ and normal ($\sim r^2$) for $\rho = 3, 27, 81$.
This is interesting. In the interval $(1, \rho_0]$ an IHPD may show a normal or an anomalous behavior according to the value of the ratio between scales ρ and the regions of normal behavior can be separated by regions of anomalous behavior !
What creates this phenomenon is a strong overlap or interaction between scales that's why the region $(1, \rho_0)$ will be called "overlapping ratios". More precisely what does it mean interaction or overlap between scales? If one look at the example 6.36 with $\rho = 3$ one would see that $U(x)$ which is supposed the represent the fluctuation of the medium V at each scale 3^n by $U(x/3^n)$ is itself characterized by fluctuation over at least two scales 1 and 1/81 and for $\rho \in (1, 81]$ a fluctuation of size ξ is decomposed over at least three successive U_n. Thus overlap between scales means that for all length scale ξ big enough in the decomposition

$$V(x) = \sum_{n=0}^{\infty} U_n(\frac{x}{R_n}) \tag{6.37}$$

the fluctuation of V at the size ξ is not represented by a single $S_{1/R_n} U_n$ but by several ones. Now one could say, may be there is something wrong with the decomposition of V indeed in the example 6.36 with $\rho = 3$,

$$V(x) = \sum_{n=0}^{\infty} \big(\sin(\frac{x}{3^n}) - \sin(\frac{81x}{3^n})\big) \tag{6.38}$$

is a rather strange way to write $V(x) = \sin(x) + \sin(3x) + \sin(9x) + \sin(27x)$ From where does come this pathology?
It seems to come from the point of view which has been chosen in this model in the sense that in this model the shapes U_n and the ratios r_n are given arbitrary to build a medium V through the formula 6.37. An other point of view would have been: one knows V, find a decomposition of the form 6.38 in which the fluctuation of V at each scale ξ is decomposed over at most two U_n and at each scale R_n over a single one. But this is not the point of view considered here, because before considering this point of view one must be able to characterize an IHPD where the fluctuations U_n would be ergodic and not periodic.
Which point of view is better? it depends on the application. It is clear that if one knows V the point of view of decomposing V in the best way is more adapted. However if one do not know V and

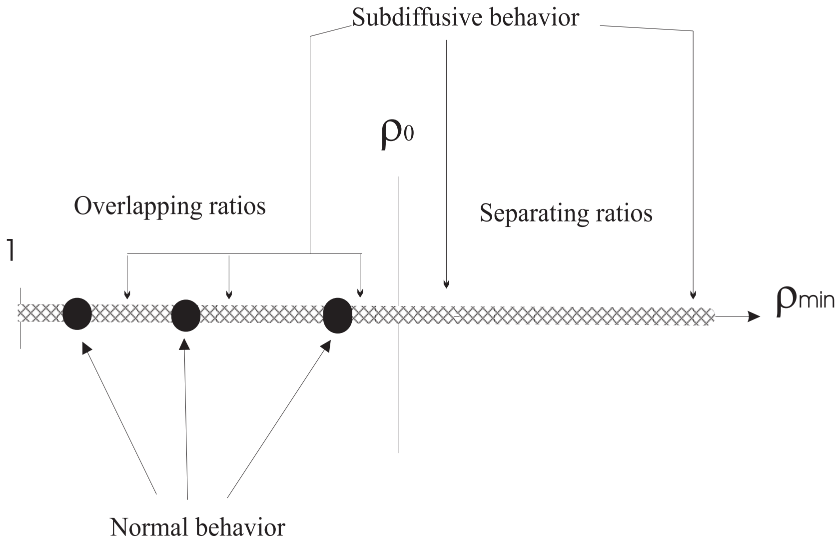

Fig. 6.2: *Ratios and behavior.*

only the shapes of the fluctuations and one wants to explore the properties which might appear with different choices of ratios r_n it is clear that the point of view considered here is more adapted but it is important to keep in mind that since the summation 6.37 is over an infinite number of elements, in the overlapping ratios interval V might be very "sensitive to" or "unstable under" a slight fluctuation of the ratios which would be reflected in the instability of the anomaly under a fluctuation of those ratios. Of course this opens new areas to explore: in the example 6.36 the regions of normality are only points separated by intervals, imagine now that the ratios can be real numbers and not only integers, what are the regions of normality? Can they be intervals? Can they be dense in the region of anomaly? What is their Hausdorff dimension? Moreover in the particular example 6.36 the regions where the IHPD is not strongly sub-diffusive are quite trivial in the sense that in those regions V is bounded, but one can imagine that for other examples V could be unbounded the IHPD could have a weak sub-diffusive behavior in those regions in the sense that $\mathbb{E}[\tau(0,r)] \sim r^2 g(r)$ with $g(r) \to \infty$ as $r \to \infty$ but $(\ln g(r))/\ln r \to 0$. This exploration is postponed to a sequel work.

In this work results will be given for a general IHPD underlying its sub-diffusive behavior for $\rho_{\min} > \rho_0$, one might say that this condition on $\rho_{\min}$ is artificial in the sense that with a better strategy one would be able to prove the sub-diffusivity for all the ratios $r_n \geq 2$, but this is not the case, if the only knowledge that one has on the fluctuations is $K_0, K_1, \lambda_{\min}$ and $\lambda_{\max}$ one can not get rid of this condition.
Indeed assume that $U \in C^\infty(T_1^1)$ (non constant) is given, consider the self-similar IHPD such that for all n, $U_n = U$ and $r_n = \rho$. Imagine that one wants to know when this IHPD is clearly sub-diffusive, then the answer is given by the formula 6.33: this IHPD is clearly sub-diffusive when $\mathcal{P}(2U) + \mathcal{P}(-2U) > 0$ but to give this answer one must have a precise information on the shape of U, imagine now that one does not have this information and the only thing one knows is $D(U)$ and that $\text{Osc}(U) \leq K_0$ and $\|\nabla U\|_\infty \leq K_1$ then the only way to be sure that this IHPD is sub-diffusive is that ρ belongs to the separating scale ratios region (ρ_0, ∞) (the counter example has been given above if ρ belongs to the overlapping ratios) and the only way to evaluate ρ_0 is trough K_0, K_1 and $D(U)$ (note that the presence of K_1 is natural in the sense that it ensures that U can not have fluctuations over both scales 1 and $1/\rho$). Thus ρ_0 the boundary between the overlapping scales and the separating scales is computed according to the knowledge that one can have on U. If one knows only $K_0, K_1, \lambda_{\max}, \lambda_{\min}$ and the dimension d then it will computed according to these parameters; if one knows the precise shape of U then ρ_0 is the first ratio above which the IHPD is sub-diffusive. In other words, through all this chapter, results will be given showing that a general

IHPD is sub-diffusive for $\rho_{\min} > \rho_0(d, K_0, K_1, \lambda_{\max}, \lambda_{\min})$ and it is important to remember that the presence of this ρ_0 is indispensable (to avoid the overlapping ratios), if one wants to obtain the sub-diffusive behavior of a general IHPD with ratios fixed below ρ_0 then one is obliged to introduce new parameters describing the particular shapes of the fluctuations U_n. In the overlapping ratios domain, the influence of the perturbation scales is larger than the influence of the effective scales and increasing the ratios between scales in this domain can make the IHPD pass from sub-diffusive behavior to a normal behavior.
In fact the proper question to ask is :"is constant ρ_0 given through this chapter the optimal one ?" and the answer is no because the clarity of the presentation and the proofs has been privileged however the proofs are constructive and one can follow them to compute the optimal constant (but one would obtain very heavy expressions), this question is interesting because once one has the optimal constant, one sees which class of medium V one can decompose in order to obtain a sub-diffusive behavior.

In resume it has been obtained that

Theorem 6.2.1. *For a self similar IHPD in dimension one, if U is not a constant function, there exists a constant $\rho_0(K_1, D(U))$ such that for $\rho > \rho_0$,*

$$\mathbb{E}_0[\tau(0, r)] = r^{2+\nu+\epsilon(r)} \tag{6.39}$$

with $\nu > 0$ given by the topological pressure

$$\nu = \frac{\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)}{\ln \rho} \tag{6.40}$$

and $\epsilon(r) \to 0$ as $r \to \infty$. Moreover there are examples of U such that there exists ratios ρ_1, ρ_2 ($\rho_1 + 10 < \rho_2$) in the interval $(1, \rho_0]$ such that if $\rho = \rho_1$ or ρ_2 then $C_1 r^2 \leq \mathbb{E}[\tau(0, r)] \leq C_2 r^2$ and if $\rho \in (\rho_1, \rho_2) \cap \mathbb{N}$, $\mathbb{E}[\tau(0, r)]$ follows the anomalous behavior given in the equation 6.39 with $\nu > 0$ as above.

6.3 Effective diffusivities

The key point leading to the sub-diffusive behavior of an IHPD is the geometric decrease towards 0 of the effective diffusivities of the aggregation of scales $D(V_0^n)$. From a heuristic point of view it can be understood in the following sense: the farther the IHPD travels, the more numerous the felt scales are, the more it is slown down. In a heuristic mathematical formulation, this sentence becomes: if $t \sim \rho^{2n_{ef}}$ and $D(V_0^n) \sim \lambda^n$ with $\lambda < 1$ then (with $\nu > 0$)

$$\mathbb{E}[y_t^2] \sim t D(V_0^{n_{ef}(t)}) \sim t^{1-\nu} \tag{6.41}$$

6.3.1 Dimension One

The one-dimensional case is particular because the effective diffusivities can be explicitely computed. Indeed the following formula is available:

$$D(V_0^n) = \frac{R_n^2}{\int_0^{R_n} e^{2V_0^n(x)} dx \int_0^{R_n} e^{-2V_0^n(x)} dx} \tag{6.42}$$

6.3.1.i General IHPD

The following theorem is the theorem 8.2.2 of chapter 8.

Theorem 6.3.1. *For $\rho_{\min} > 2K_1 e^{2K_0}$*

$$\prod_{k=0}^{n-1} \frac{1}{\int_{T_1^1} e^{2U_k(x)} dx \int_{T_1^1} e^{-2U_k(x)} dx} \frac{1}{(1 + \frac{2K_1 e^{2K_0}}{r_k})^2} \leq D(V^{n-1}) \tag{6.43}$$

and

$$D(V^{n-1}) \leq \prod_{k=0}^{n-1} \frac{1}{\int_{T_1^1} e^{2U_k(x)}dx \int_{T_1^1} e^{-2U_k(x)}dx} \frac{1}{(1-\frac{2K_1e^{2K_0}}{r_k})^2} \tag{6.44}$$

Remark 6.3.1. If $\lambda_{\max} < 1$, this theorem imply that for $\rho_{\min} > \rho_0(K_0, K_1, \lambda_{\max})$

$$\lambda_1^n \leq D(V_0^{n-1}) \leq \lambda_2^n \tag{6.45}$$

with

$$\lambda_1 = \frac{\lambda_{\min}}{1-\frac{2K_1e^{2K_0}}{\rho_{\min}}} \qquad \lambda_2 = \frac{\lambda_{\max}}{1-\frac{2K_1e^{2K_0}}{\rho_{\min}}} < 1 \tag{6.46}$$

Observe that if $\lambda_{\min} = \lambda_{\max}$ then as $\rho_{\min} \uparrow \infty$

$$\frac{1}{n} \ln D(V_0^{n-1}) \to \ln \lambda \tag{6.47}$$

uniformly in n

The following corollary is the corollary 8.2.1, it allows to see that the influence of the overlap between scales disappear in the fast separation between scales regime.

Corollary 6.3.1. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$\lim_{n\to\infty} \frac{1}{n} \ln D(V^{n-1}) = \frac{1}{\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx} \tag{6.48}$$

6.3.1.ii Self similar IHPD

Assume that the IHPD is self-similar with ratio between scales $\rho \in \mathbb{N}/\{0,1\}$ and periodic potential $U \in C^\infty(T_1^1)$.
The following theorem is the theorem 8.2.1

Theorem 6.3.2.

$$\lim_{n\to\infty} -\frac{1}{n} \ln\left(D(V^{n-1})\right) = \mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) \tag{6.49}$$

where $\mathcal{P}_\rho$ is the topological pressure associated to the shift s_ρ (see section C.1).
The theorem C.1.2 says that $\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) > 0$ if and only U does not belong to the closed subspace of $\mathcal{C}(T_1^d)$ generated the elements $F(x) - F(\rho^k x)$ with $F \in C(T_1^d)$ and $k \in \mathbb{N}$; the proposition 9.5.1 says that this is equivalent to

$$\lim \inf_{n\to\infty} \frac{1}{n} \| \sum_{k=0}^{n-1} S_{\rho^k} U\|_\infty > 0 \tag{6.50}$$

Moreover it is easy to see that as $\rho \to \infty$

$$\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U) \to \ln \int_{T_1^d} e^{2U}dx + \ln \int_{T_1^d} e^{-2U}dx \tag{6.51}$$

which is strictly positive if U is not constant.
Thus when the IHPD is self similar, then the behavior of $D(V_0^{n-1})$ can be exactly computed thanks to a simple application of the thermodynamic formalism and the theory of level 3-large deviations (see section C.1).

6.3.2 All dimensions

This sub subsection gives some of the results proven in the chapter 9.
In dimension greater than 1 no explicit formula is generally available (in dimension 2, one has an explicit formula under the assumption $D(U_n) = D(-U_n)$). Moreover DEM theories can not be used because first the influence of each scale is not completely diluted in their numbers, next the ratio between scales is not infinite but is finite and fixed independently from the number of scales and for the same reason standard multi-scaled (or reiterated) homogenization is powerless.
Nevertheless variational formulation giving an upper bound and lower bound for $D(V_0^{n-1})$ are available, and the general technique used to obtain multi-scale homogenization results for those multi-scale media is to replace the solution of the cell problem by its first order approximation in the method of asymptotic expansion and use it as a test function in a variational formula. But the error made by this way is of order of the ratio between scales multiplied by a constant that tends to grow with the number of scales.
That's why this method is also powerless to describe materials for which the ratio between scales is fixed in dependently from the number of scales and this is the situation of an IHPD with bounded ratios $\rho_{\max} < \infty$. One might think that if this method is powerless it is because the one has plugged in the variational formulation only the first order approximation in the asymptotic expansion of the cell problem χ_l and may be by using the second, third or $n-th$ order approximation one would be able to obtain a control of $D(V_0^{n-1})$. This is not the case, and the computation of sub subsection 5.2.1.iii is here to show that things will get only worse (expressions more heavy and headaches more frequent).
An alternative strategy must be sought for, the following theorem (which is the theorem 9.2.2 of chapter 9) is one of the results of this alternative strategy:

Theorem 6.3.3. *If $\rho_{min} \geq C_{1,d,K_0,K_1}$ then for all $n \geq 1$*

$$\lambda_{\max}(D(V_0^{n-1})) \leq (1 + \frac{C_{2,d,K_0,K_1}}{\rho_{\min}^{\frac{1}{2}}})^n \prod_{k=0}^{n-1} \lambda_{\max}(D(U_k)) \tag{6.52}$$

and

$$\lambda_{\min}\big(D(V^n)\big) \geq (1 + \frac{C_{2,d,K_0,K_1}}{\rho_{\min}^{\frac{1}{2}}})^{-n} \prod_{k=0}^{n-1} \lambda_{\min}\big(D(U_k)\big) \tag{6.53}$$

$$C_{1,d,K_0,K_1} = C_d e^{(6d+16)K_0}(1+K_1)^3 \tag{6.54}$$

and

$$C_{2,d,K_0,K_1} = C_d e^{(3d+8)K_0}(1+K_1)^{\frac{1}{2}} \tag{6.55}$$

Of course for the clarity of the presentation the constants $C_{1,d,K_0,K_1}, C_{2,d,K_0,K_1}$ given above are not the optimal ones. Actually this theorem is a corollary of more general results which allow to control the whole matrix $D(V_0^{n-1})$ (see propositions 9.3.5 and 9.4.1)
Before giving the alternative strategy observe some simple consequences of this theorem :

Corollary 6.3.2. *if one has for all n, $\lambda_{\max}\big(D(U_n)\big) \leq \lambda_{\max} < 1$, then if for all n*

$$r_n > \rho_{\lambda_{\max},d,K_0,K_1} \tag{6.56}$$

then

$$C_1\lambda_1^n \leq D(V^n) \leq C_2\lambda_2^n \tag{6.57}$$

with $0 < \lambda_1 \leq \lambda_2 < 1$ and

$$\rho_{\lambda_{\max},d,K_0,K_1} = \Big[\frac{C(d,K_0,K_1)\lambda_{\max}}{1-\lambda_{\max}}\Big]^2 \tag{6.58}$$

This corollary (which corresponds to corollary 9.2.2) gives the boundary $\rho_0 = \rho_{\lambda_{\max},d,\alpha,K_0,K_1}$ of the separating scales region. Observe that in this region $D(V_0^{n-1})$ decreases towards 0 with a geometric speed. Below this boundary, in the overlapping ratios region the particular shape of the fluctuations U_n starts to manifest themselves in the holistic behavior of $D(V_0^{n-1})$ which can lead to the lower boundedness of $D(V_0^{n-1})$.
Without giving additional information on the particular shape of the fluctuations, one can not guess the precise behavior of $D(V_0^{n-1})$ in the overlapping region, however by the Voigt' Reiss inequality one can show that the geometric speed of convergence of $D(V_0^{n-1})$ towards 0 imply

$$\lim\inf_{n\to\infty} \frac{1}{n} \|V_0^n - \frac{1}{R_n^d} \int_{T^d_{R_n}} V_0^n(x)\,dx\|_\infty > 0 \tag{6.59}$$

which gives a criterion to check whether the ratios r_n belongs to a region of weak anomaly (or normality).

Now observe an other simple consequence of the theorem 6.3.3 is the behavior of the multi-scale effective diffusivities of a self similar IHPD:

Let $R \in \mathbb{N}/\{0,1\}$ and $U \in C^\infty(T_1^d)$. Write

$$V_0^{n-1} = \sum_{k=0}^{n-1} (S_R)^k U \tag{6.60}$$

then one has the following theorem (corresponding to 9.2.1):

Theorem 6.3.4. *If $R \geq C_{1,d,U}$ then for all $n \geq 1$*

$$\lambda_{\max}\big(D(V_0^{n-1})\big) \leq \Big(\lambda_{\max}\big(D(U)\big)\Big)^n (1 + \frac{C_{2,d,U}}{R^{\frac{1}{2}}})^n \tag{6.61}$$

$$\lambda_{\min}\big(D(V_0^{n-1})\big) \geq \Big[\frac{\lambda_{\min}\big(D(U)\big)}{1 + \frac{C_{2,d,U}}{R^{\frac{1}{2}}}}\Big]^n \tag{6.62}$$

with

$$C_{1,d,U} = C_d e^{(6d+16)\,\mathrm{Osc}(U)} (1 + \|\nabla U\|_\infty)^3 \tag{6.63}$$

and

$$C_{2,d,U} = C_d e^{(3d+8)\,\mathrm{Osc}(U)} (1 + \|\nabla U\|_\infty)^{\frac{1}{2}} \tag{6.64}$$

As a first reaction to this theorem, it is interesting to deduce the following corollary (corresponding to 9.2.1)

Corollary 6.3.3. *If in all the directions $l \in \mathbb{S}^d$ of the space $l.\nabla U$ is not the null function then $D(U) < 1$ and for*

$$R > \rho_{d,U} = \Big[\frac{C_{d,\mathrm{Osc}(U),\|\nabla U\|_\infty} \lambda_{\max}\big(D(U)\big)}{1 - \lambda_{\max}\big(D(U)\big)}\Big]^2 \tag{6.65}$$

$D(V^n)$ tends geometrically towards 0 with an explicit control of the speed of convergence given by the theorem 9.2.1.

This is the key leading to the sub-diffusive behavior in a smooth periodic pre fractal. It is interesting also to observe that when U is isotropic that is to say the minimal and maximal eigenvalues of $D(U)$ are equal then the multi-scale effective diffusivity $D(V^n)$ behaves like $\lambda\big(D(U)\big)^n (1 + \frac{\text{error}}{R^{\frac{1}{2}}})^n$ but one must be careful this does not mean that $D(V^n)$ is isotropic.

6.3.2.i Alternative strategy

The proof of the main result allowing to homogenize on an arbitrary large number of scales with bounded ratios is mainly based on three ideas and observations.

1. When homogenization takes place on two scales separated by a ratio R, a translation of the first one with respect to the second one does not change much the effective diffusivity (see lemma 9.3.2, the perturbation can easily be controlled).
2. The distance between the solution of the cell problem and itself translated by e_k/R is small with respect to the effective diffusivity of the medium (see lemma 9.3.3).
3. The effective diffusivity of n different scales is obtained by recurrence by adding the smaller scale to the $n-1$ bigger ones (here the point of view is technically different from the one of DEM theory where at each step a bigger scale is added to a matrix of smaller ones).

Consider $V,T \in C^\infty(T_1^d)$ and $R \in \mathbb{N}/\{0,1\}$. Thanks to the first observation, one can see that (for $l \in \mathbb{S}^d$) ${}^t lD(S_RV+T)l$ is close to $\int_{T_1^d} {}^t lD(S_R\Theta_y V+T)l dy$ which corresponds to the effective diffusivity of the sum of the two scales, meaned with respect to a relative translation between them (see the equation C.5 for the definition of the shift operator S_R and the equation C.20 for the translation operator Θ_y). Thus what one needs to evaluate is for the upper bound

$$\int_{x,y\in T_1^d\times T_1^d} l.(l-\nabla\chi_l(x,y)) m_{S_R\Theta_y V+T}(dx)dy \tag{6.66}$$

where $x \to \chi_.(x,y)$ is the solution of the cell problem associated to $S_R\Theta_y V+T$. Now this integration in y allows to see that $\int_{T_1^d} {}^t lD(S_R\Theta_y V+T)l dy$ is close to $D(V,T,R=\infty)$ (corresponding to complete separation between the scales) in the sense that the error is of the order of

$$\begin{aligned}(\int_{x\in T_1^d} (\nabla\chi_l(x,0) - \nabla\chi_l(x+\frac{e_k}{R},0))^2 m_{S_RV+T}(dx))^{\frac{1}{2}} \\ \times ({}^t lD(T)l)^{\frac{1}{2}} e^{C_d \operatorname{Osc}(V)}\end{aligned} \tag{6.67}$$

and now, thanks to the second observation the first term is of the order of ${}^t lD(S_RV+T)l(e^{4\frac{\|\nabla T\|_\infty}{R}}-1)$ and since in the recurrence the T contains $n-1$ scales and V only one, the term $e^{C_d \operatorname{Osc}(V)}$ does not explode and the term ${}^t lD(T)l$ is close to ${}^t lD(V,T,R=\infty)l$.

The idea to add smaller and smaller scales (contrary to DEM theories) might appear a tautology but this is not the case. In iterative homogenization, the smaller scales are homogenized first, next the bigger ones. Here it is shown that reversing this iteration allows to obtain sharp estimates in this homogenization procedure.

6.3.2.ii Connection between cohomology and homogenization, dimension two

In higher dimensions, the constant $\rho_{d,U}$ associated to the corollary 9.2.1 appears as an upper bound to the regions of normal behavior, when U is characterized only by $\lambda_{\max}\big(D(U)\big)$, $\|\nabla U\|_\infty$ and $\operatorname{Osc}(U)$. Moreover by the Voigt Reiss's inequality

$$\liminf_{n\to\infty} -\frac{1}{n}\ln\Big(\lambda_{\min}\big(D(V^n)\big)\Big) \le \mathcal{P}_R(2U)+\mathcal{P}_R(-2U) \tag{6.68}$$

Thus if U belongs to the closed subspace of $\mathcal{C}(T_1^d)$ generated the elements $T(x)-T(R^kx)$, then

$$\lim_{n\to\infty}\frac{1}{n}\ln\Big(\lambda_{\min}\big(D(V^n)\big)\Big) = 0 \tag{6.69}$$

And the diffusion does not show a clear anomaly, this suggests that regions of normality separated by regions of anomaly exists (they can be built on simple examples).

Now an interesting question arises: if $R \leq \rho_{d,\alpha,U}$ and is bounded above by a region of normality (are normal region only points? or can they be open an non void ?) then what is the mechanism behind this the geometric decrease of $D(V^n)$ towards 0, what kinds of large deviations are hidden behind this sort of transition of phase? This question will be investigated here in dimension, two. Indeed there is a strong connection between homogenization and cohomology which allows to obtain the following result (which corresponds to the theorem 9.3.1):

Theorem 6.3.5. *For $d = 2$ one has*

$$\begin{aligned}\lambda_{\max}\big(D(U)\big)\lambda_{\min}\big(D(-U)\big) &= \lambda_{\min}\big(D(U)\big)\lambda_{\max}\big(D(-U)\big) \\ &= \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}\end{aligned} \tag{6.70}$$

from which one deduces that if $D(U) = D(-U)$ then

$$\lambda_{\max}\big(D(U)\big) = \lambda_{\min}\big(D(U)\big) = \frac{1}{\sqrt{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}} \tag{6.71}$$

Moreover

Theorem 6.3.6. *In the self-similar case, if $d = 2$ and for all n, $D(V_0^n) = D(-V_0^n)$ then*

$$\lim_{n\to\infty} -\frac{1}{n} \ln\big(\lambda(D(V_0^{n-1}))\big) = \frac{\mathcal{P}_R(2U) + \mathcal{P}_R(-2U)}{2} \tag{6.72}$$

where $\mathcal{P}_R$ is the topological pressure associated to the shift s_R.
As an example of medium satisfying the condition of the previous theorem one can give the following corollary

Corollary 6.3.4. *In the self-similar case , if $d = 2$ and for all n, $U_n(-x) = -U_n(x)$ then*

$$\lim_{n\to\infty} -\frac{1}{n} \ln\big(\lambda(D(V_0^{n-1}))\big) = \frac{\mathcal{P}_R(2U) + \mathcal{P}_R(-2U)}{2} \tag{6.73}$$

6.3.2.iii Perspectives

These statements show clearly that when the scales are not self-similar and non symmetric (can be chosen at random) the geometric speed of convergence of $D(V^n)$ towards 0 can be controlled without the necessity to use large deviations techniques, however it is interesting to wonder how this is translated in the theory of shifts dynamical systems. For instance note that $V^0 = U_0$ and

$$V^{n+1} = S_{r_{n+1}}(V^n + U_{n+1}) \tag{6.74}$$

and the latter inductive definition will be interesting to explore in the shift spaces, what notion will replace the pressure? What kind of large deviations might be hidden behind the behavior of the eigenvalues of the matrix $D(V^n)$ in any dimension? Those questions will be postponed to a future work.
All the aspects of the connections between the geometry of homogenization, scaling and cohomology have not been explored here (in dimension $d \geq 3$), this investigation is postponed to a future work.

6.4 Sub-diffusive behavior

The sub-diffusive behavior of an IHPD is a consequence of the decrease towards 0 of the effective diffusivities.

6.4.1 Dimension one

(see chapter 8) In dimension one, the mixing length ξ associated to a periodic potential of period T_R^1 is of the size of the period R (the mixing length corresponds to norm of the solution of the associated cell problem 5.10 and the associated ergodicity problem 5.4 which can be exactly computed). Thus for each aggregation of scales V_0^n, $\xi(V_0^n) = R_n$.

6.4.1.i Exit times

Since $\xi(V_0^n) = R_n$ the effective scale corresponding to the exit time from the ball $B(0,r)$ is fixed by

$$n_{ef}(r) = \sup\{n \geq 0 \,:\, R_n \leq r\} \tag{6.75}$$

This is clear, but now one can wonder how to determine the drift scale since the drift is not apparent in the expression of $\tau(0,r)$. This is done by observing how the presence of the scales $n_{ef}+1, \dots, \infty$ perturb the value of the value of the exit time $\mathbb{E}_0[\tau(0,r)]$. This is done by the following corollary:

Perturbation of the exit times (see sub subsection 13.5.2.ii) Let Ω be an open set of $\mathbb{R}$. for $U \in C^\infty(\bar{\Omega})$ write $\mathbb{E}^U$ the expectation associated to the diffusion generated by the operator and τ its exit time from Ω.

$$L_U = \frac{1}{2}\Delta - \nabla U \nabla \tag{6.76}$$

The following corollary corresponds to the corollary 13.5.2.

Corollary 6.4.1. *For $U, P \in C^\infty(\bar{\Omega})$, $x \in \Omega$*

$$e^{-4\,\mathrm{Osc}(P)} \leq \frac{\mathbb{E}_x^{U+P}[\tau]}{\mathbb{E}_x^{U}[\tau]} \leq e^{4\,\mathrm{Osc}(P)} \tag{6.77}$$

This corollary says that the exit times $\mathbb{E}_x^U[\tau]$ is stable under a perturbation of the operator L_U by the drift $-\nabla P$, what is important is that this perturbation is completely controlled by $\mathrm{Osc}(P)$ whatever U might be. In fact this control comes from a sharp control on the perturbation of the Green functions $G_{e^{-2U}}(x,y)$ and $G_{e^{-2(U+P)}}(x,y)$ associated to the operators $-\nabla(e^{-2U}\nabla)$ and $-\nabla(e^{-2(U+P)}\nabla)$ with Dirichlet conditions on the boundary of Ω. (the following corollary corresponds to the corollary 6.4.2)

Corollary 6.4.2. *For $d = 1$ and $U, P \in C^\infty(\bar{\Omega})$ one has*

$$e^{-6\|P\|_\infty} \leq \frac{G_{e^{-2(U+P)}}(x,y)}{G_{e^{-2(U)}}(x,y)} \leq e^{6\|P\|_\infty} \tag{6.78}$$

and this corollary is implied by a new analytical inequality (the following theorem is deduced from the theorem 13.5.1)

Theorem 6.4.1. *For $d = 1$, Ω an open set of $\mathbb{R}$, $U \in C^2(\bar{\Omega})$ and ϕ and ψ two $C^2(\bar{\Omega})$ functions with Dirichlet conditions on $\partial\Omega$ and both sub harmonic with respect to the operator $-\nabla(e^{-2U}\nabla)$ one has*

$$\int_\Omega \big|\nabla\phi(x)\nabla\psi(x)\big| e^{-2U(x)}\,dx \leq 3\int_\Omega \nabla\phi(x)\nabla\psi(x) e^{-2U(x)}\,dx \tag{6.79}$$

Determination of the perturbation scales Now let's go back to the determination of the drift scale, in $B(0,r)$, it follows by the corollary 6.4.1 that $E[\tau(0,r)]$ behaves like $r^2/D(V_0^{n_{ef}})$ multiplied by a perturbation term which is of order of the exponential of $\mathrm{Osc}_{B(0,r)}(V_{n_{ef}+1}^{\infty})$ (where $\mathrm{Osc}_{B(0,r)}$stands for the supremum minus the minium taken in the ball $B(0,r)$); thus the drift scale is determined by the minimization of this term ($V_{n_{ef}+1}^{\infty}$ is the sum of an infinite number of fluctuations U_n, the first n_{per} will be bounded by the infinite norm $K_0 n_{per}$ and the remaining $n_{dri},\dots,\infty$ by an uniform bound on their gradient $2K_1$). And it is easy to see that the number of perturbation scales can here be limited to be equal to one (this is a direct consequence of the fact that the mixing length are of the order of the periodicities in dimension one).

These considerations lead to the sub-diffusive behavior of the exit times but one can see that the sub-diffusive behavior is weaker when the ratios between scales are not bounded, this is due to the fact that the diffusion must travel longer and longer distances to feel the slow down created by larger and larger scales.

Sub-diffusivity with bounded ratios The following theorem corresponds to the corollary 8.3.3 of chapter 8.

Theorem 6.4.2. *Let y_t be an infinitely homogenized potential diffusion such that, $\rho_{\min} > 4K_1 e^{2K_0}$, $\rho_{\max} < \infty$ and $\lambda_{\max} < 1$. Then*

$$C_1 r^{2+\nu(r)} \le \mathbb{E}_0[\tau(0,r)] \le C_2 r^{2+\nu(r)} \tag{6.80}$$

where C_1, C_2 depends only on K_0, K_1 and $\rho_{\min}$ and

$$0 < -\frac{\ln \lambda_{\max}}{\ln \rho_{\max}} - \frac{8K_1 e^{2K_0}}{\rho_{\min} \ln \rho_{\max}} \le \nu(r) \le -\frac{\ln \lambda_{\min}}{\ln \rho_{\min}} + \frac{4K_1 e^{2K_0}}{\rho_{\min} \ln \rho_{\min}} \tag{6.81}$$

This theorem is the generalization of the corollary 6.2.1 when the IHPD is not self similar. Here the value given for $\nu(r)$ can really depend on r (it is generally not constant), this explains why one can give only bounds of the form 6.81. Observe that in those bounds the term $K_1 e^{2K_0}/(\rho_{\min} \ln \rho_{\max})$ acts as an error term and is small in front of $-\ln \lambda_{\max}/\ln \rho_{\max}$ and $-\ln \lambda_{\min}/\ln \rho_{\min}$. Observe also that when $\rho_{\min} = \rho_{\max} = \lambda$, $\lambda_{\min} = \lambda_{\max} = \lambda$ (which does not mean that the IHPD is self similar because for that one needs the additional condition for all n $U_n = U$) one has

$$\nu(r) \sim -\frac{\ln \lambda}{\ln \rho} \tag{6.82}$$

In fact the corollary 6.4.2 is deduced from a more general theorem 8.3.1 which allow to control the exit times of an IHPD in very general situations. For instance one could choose the soft obstacles and ratios U_n, r_n with a strong dependence on the scale n and at the end one would obtain a strong variation of ν with r.

Sub-diffusivity with unbounded ratios The following theorem is the corollary 8.3.4 of chapter 8.3.4

Theorem 6.4.3. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$C_1 r^2 e^{g(r)} \le \mathbb{E}_0[\tau(0,r)] \le C_2 r^2 e^{g(r)} \tag{6.83}$$

where C_1, C_2 *depends only on* K_0, K_1, ρ, α *and*

$$g(r) = (\ln r)^{\frac{1}{\alpha}} \frac{\ln\left(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\right)}{(\ln \rho)^{\frac{1}{\alpha}}} \tag{6.84}$$

Observe that this theorem says how by a fine tuning of the parameter α (which reflects the speed of the separation between scales) the IHPD behavior changes from weakly sub-diffusive to strongly sub-diffusive.

6.4.1.ii Mean squared displacement

Bounded ratios The following theorem corresponds to the theorem 8.5.3.

Theorem 6.4.4. *Assume* $\lambda_{\max} < 1$, $\rho_{\min} > 10e^{-\frac{30}{\ln \lambda_{\max}}(K_1+4K_0^2)}$, $t > R_9$ *and* $\rho_{\max} < \infty$ *then*

$$\mathbb{E}[y_t^2] = t^{1-\nu(t)} \tag{6.85}$$

$$\nu(t) \leq -\frac{\ln \lambda_{\min}}{2\ln \rho_{\min}} + \frac{\frac{2K_1 e^{2K_0} \ln \rho_{\min}}{\rho_{\min}} + 16K_0 \ln(\frac{16}{15\lambda_{\min}})}{(\ln \rho_{\min})^2} + \epsilon(t) \tag{6.86}$$

$$\nu(t) \geq -\frac{\ln \lambda_{\max}}{2\ln \rho_{\max}} - \frac{\frac{2K_1 e^{2K_0} \ln \rho_{\min}}{\rho_{\min}} + 8K_0 \ln(\frac{9}{8\lambda_{\min}})}{\ln \rho_{\min} \ln \rho_{\max}} - \epsilon(t) \tag{6.87}$$

where $\epsilon(t) \to 0$ *as* $t \to \infty$ *and*

$$-\frac{\ln \lambda_{\max}}{2\ln \rho_{\max}} - \frac{\frac{2K_1 e^{2K_0} \ln \rho_{\min}}{\rho_{\min}} + 8K_0 \ln(\frac{9}{8\lambda_{\min}})}{\ln \rho_{\min} \ln \rho_{\max}} > 0 \tag{6.88}$$

Observe that this theorem gives the sub-diffusive behavior of the IHPD, Note also that the terms $\frac{\frac{2K_1 e^{2K_0} \ln \rho_{\min}}{\rho_{\min}} + 16K_0 \ln(\frac{16}{15\lambda_{\min}})}{(\ln \rho_{\min})^2}$ and $\frac{\frac{2K_1 e^{2K_0} \ln \rho_{\min}}{\rho_{\min}} + 8K_0 \ln(\frac{9}{8\lambda_{\min}})}{\ln \rho_{\min} \ln \rho_{\max}}$ are error terms (small) in front of $-\frac{\ln \lambda_{\min}}{2\ln \rho_{\min}}$ and $-\frac{\ln \lambda_{\max}}{2\ln \rho_{\max}}$. The upper bound and the lower bound for $\nu(t)$ in 6.86 are not equal because it does really fluctuate between those bounds (this is due to the fact that the IHPD is not self similar). Observe also that if $\lambda_{\max} = \lambda_{\min} = \lambda$ and $\rho_{\max} = \rho_{\min} = \rho$ then

$$1 - \nu(t) \sim 1 + \frac{\ln \lambda}{2\ln \rho} \tag{6.89}$$

In fact the theorem 6.4.4 is deduced from more general theorems 8.5.1 and 8.5.2 which allow to control the mean squared displacement of an IHPD in very general cases. For instance one could choose the ratios and ratios U_n, r_n with a strong dependence on the scale n and at the end one would obtain from this theorem a strong variation of ν with t.

Unbounded ratios The following theorem corresponds to the theorem 8.5.4

Theorem 6.4.5. *Assume that for all* k, $U_k = U$ *and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ *then*

$$C_1 t e^{-g(t)} \leq \mathbb{E}_0[y_t^2] \leq C_2 t e^{-g(t)} \tag{6.90}$$

where C_1, C_2 depends only on K_0, K_1, ρ, α and

$$g(t) = (\ln t)^{\frac{1}{\alpha}} \frac{\ln\left(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\right)}{(2\ln\rho)^{\frac{1}{\alpha}}}(1+\epsilon(t)) \tag{6.91}$$

with $\epsilon(t) \to 0$ as $t \to \infty$.

Observe that $\mathbb{E}[y_t^2]/t \to 0$ as $t \to \infty$ but for all $1 > \beta > 0$, $\mathbb{E}[y_t^2]/t^{1-\beta} \to \infty$. Moreover this theorem shows how the behavior of the diffusion passes from a slightly anomalous one to a strongly anomalous one.

$$\mathbb{E}[y_t^2] \sim \frac{t}{\left(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\right)^{\frac{1}{(2\ln\rho)^{\frac{1}{\alpha}}}(\ln t)^{\frac{1}{\alpha}}}} \tag{6.92}$$

Perturbation scales The heuristic proof of the mean squared displacement has been given above, however the essential point was missing: how is it possible to take into account the perturbation scales in the mean square displacement? The answer to this question will allow to look closer at the significance of the cell problem.

Consider $U \in C^\infty(T_1^d)$, in homogenization theory the solution of the cell problem χ_l seems to play an essential role in the sense that it contains all the information on the homogenization over a periodic medium U but if one look closer, one would see that the essential point to obtain homogenization is not the existence of a periodic solution χ_l to the cell problem, but the existence of linear harmonic (with respect L_U) functions $F_l = l.x - \chi_l$, of course when U is periodic those point of views are equivalent, but this is not the case when U is only ergodic or contains an infinite number of scales. Moreover observe that when U is periodic, the corresponding effective diffusivity is given by $D(U) = m_U\left({}^t\nabla F_. \nabla F_.\right)$, and the solution of ergodic problem is given by $\phi_l = |F_l|^2 - {}^t l D(U) l \psi_l$ which allows to introduce ψ_l the solution of the Poisson equation $L_U \psi_l = 1$ in $\mathbb{R}^d$, observe the importance of F_l in those equations.

When U is only ergodic, one can still find harmonic functions F_l growing linearly as $l.x$ (this is clear in dimension one, in greater dimensions this will be shown in a sequel work) but the difference $l.x - F_l(x)$ is not bounded that's why the solution of the cell problem does not exit, nevertheless F_l exists on contains all the information that lead to homogenization results. What about the solution to the ergodic problem $L_U \phi_l = |\nabla F_l|^2 - {}^t l D(U) l$? Again when U is periodic, one can find ϕ_l periodic solution of that equation and this is used to say that $\mathbb{E}[\int_0^t |\nabla F_l(y_s)|^2\, ds]$ behaves like ${}^t l D(U) l\, t$ plus a bounded term $\mathbb{E}[\phi_l(y_t)] - \phi(y_0)$, but when U is ergodic but not periodic such a solution periodic solution does not exist. Nevertheless one can find solutions to Poisson equation $L_U \psi_l = 1$ growing like $(l.x)^2$ (plus something growing less quickly in the transverse directions for $d \geq 2$, this is the subject of a sequel work). Now write for $c > 0$, $\phi_l^c = (F_l)^2 - c\,{}^t l D(U) l \psi_l$, for $c = 1$, ϕ_l^1 is generally not bounded however one can see that for $c > 1$, ϕ_l^c is upper bounded and for $c < 1$ ϕ_l^c is lower bounded, this is interesting ! because it means that for all $\epsilon > 0$, $\mathbb{E}[\int_0^t |\nabla F_l(y_s)|^2\, ds]$ is upper bounded by $(1+\epsilon)\,{}^t l D(U) l\, t + C_\epsilon$ and lower bounded by $(1-\epsilon)\,{}^t l D(U) l\, t - C_\epsilon$. Now one understands how to do homogenization without cell problem and without the necessity to sit on the particle (and see the medium moving).

Note that the level lines of those linear harmonic functions F_l can be seen as corresponding to the equipotentials of a capacitor with parallel plates perpendicular to the direction l with distance and tension $2L$ (L is very large) between them and permittivity e^{-2U}, saying that homogenization operates is equivalent to say that those equipotentials behave as those of a linear harmonic function (F_l has a clear physical signification, χ_l is just here to evaluate how close is F_l from $l.x$).

What about the case when $U = V$ contains an infinite number of scales? In dimension one F_l exists and can be exactly computed $F_l(x) = \int_0^x e^{2V(y)}\, dy$, it is harmonic with respect to L_V but it does not behave linearly with x, this is due to the fact that homogenization operates at all the scales. Let's remember the purpose of this sub subsection, it is to control the mean square displacement, and it has been shown above that the drift scale is fixed by

$$n_{dri} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \leq t\} + 2 \tag{6.93}$$

if $R_n = \rho^n$ is behaves like $\ln t/(2\ln\rho)$. The visibility time of the scales $V_{n_{dri}}^{\infty}$ behaves like

$$\tau_v(V_{n_{dri}}^{\infty}) \sim R_{n_{dri}}^2 \tag{6.94}$$

and the mixing time corresponding to the scales $V_0^{n_{dri}-1}$ behaves like

$$\tau_m(V_0^{n_{dri}-1}) \sim R_{n_{dri}-1}^2/D(V_0^{n_{dri}-1}) \tag{6.95}$$

and when the ratios are bounded one has $\tau_m(V_0^{n_{dri}-1}) > \tau_v(V_{n_{dri}}^{\infty})$ which makes error terms larger than effective terms in the computations. To find how to determine the perturbation scales and bypass this difficulty one must look at the origin of $\tau_m(V_0^{n_{dri}-1})$, if one tries to compute $\mathbb{E}_0[y_t^2]$ for $t \sim R_{n_{dri}}^2$, and if one assumes that homogenization has fully operated on $V_0^{n_{dri}-1}$ then one would obtain that $\mathbb{E}_0[y_t^2] \sim D(V_0^{n_{dri}-1})t$+error, where "error" is of the order of $\|\chi_l^{V_0^{n_{dri}-1}}\|_\infty^2 + \|\phi_l\|_\infty \sim R_{n_{dri}-1}^2$ which represents the error created by the "distance" between y_t and its martingale behavior fixed by $F_l^{V_0^{n_{dri}-1}}(y_t)$ at the scale $R_{n_{dri}-1}$ and the error one makes by assuming that $\int_0^t |\nabla F_l^{V_0^{n_{dri}-1}}(y_t)|^2\,ds$ behave like $D(V_0^{n_{dri}-1})t$. In resume if one tries to compare $\mathbb{E}[y_t^2]$ to effective behavior of the scale $R_{n_{dri}-1}$: $D(V_0^{n_{dri}-1})t$ one makes an error which is bigger than the latter term. The solution is to compare $\mathbb{E}[y_t^2]$ to $\mu D(V_0^{n_{dri}-1})t$ hoping that for μ big enough $\mathbb{E}[y_t^2] \leq \mu D(V_0^{n_{dri}-1})t + C_\mu$ where C_μ is an error smaller than the main term $D(V_0^{n_{dri}-1})t$ and for μ small enough $\mathbb{E}[y_t^2] \geq \mu D(V_0^{n_{dri}-1})t - C_\mu$ with the error term C_μ smaller than the main term. This is done by decomposing $V_0^{n_{dri}-1}$ as a sum over effective scales $V_0^{n_{ef}}$ and perturbation scales $V_{n_{ef}+1}^{n_{ef}+n_{per}}$ and observing their respective analytical influence on the linear harmonic function $F_l^{V_0^{n_{dri}-1}}$ associated to the medium $V_0^{n_{dri}-1}$.

Deformation of the linear harmonic functions Consider $W \in C^\infty(T_1^1)$ and $T \in C^\infty(T_R^1)$ (of period $\mathbb{R} \in \mathbb{N}$). Consider F^U the linear harmonic function associated to the medium $U = W + T$ which is periodic of period R. This medium is characterized by two scales 1 and R. The purpose of this paragraph is to determine the influence of each scale on F^U. Observe that the cell problem $\chi^U = x - F^U(x)$ associated to L_U is periodic of period R and since $\|\chi^U\|_\infty = R$ one makes an error of order R by assuming that the diffusion y_t generated by L_U behaves like its associated martingale behavior $\int_0^t \nabla F^U(y_s)\,ds$. Moreover by assuming that $\mathbb{E}[\int_0^t |\nabla F^U(y_s)|^2\,ds$ behaves like $D(U)t$ one makes an error of the size of R^2 which corresponds to the norm of the ergodicity problem. Thus when one assumes that $\mathbb{E}[y_t^2]$ behaves like its homogenized behavior $D(U)t$, one makes an error of order R^2 which becomes negligible for $D(U)t >> R^2$.
But now one is interested on what happens for $1/D(W) < t < R^2/D(U)$, in this interval it is natural to think that homogenization on the smaller scale W is complete but the scale T acts as a perturbation scale, how is this translated in the behavior of F^U the linear harmonic function containing all the information about the homogenization on U?
As one can decompose F^U into $l.x$ a linear term corresponding to a flat medium minus χ^U the solution of the cell problem of period R corresponding to the fluctuations over the scale R, one can also decompose F^U into

$$F^U(x) = F^T(x) + \chi^{W,T}(x) \tag{6.96}$$

where $F^T(x)$ is the linear harmonic function associated to the medium T (of period R) and $\chi^{T,W}$ which is close to the solution of the cell problem associated to W but perturbed by T. $\chi^{T,W}$ is not of period 1 but R however

$$\|\chi^{W,T}\|_\infty \leq 2e^{2\,\mathrm{Osc}(T)}[1 + 4\|\nabla T\|_\infty] \tag{6.97}$$

In other words if one compares F^U to $l.x$ which is equivalent to say that one has homogenized on both scales of U, one makes an error of the order of R, however if one compares F^U to F^T which is equivalent to say that one has homogenized only on the smaller scale W one makes an error of order of $e^{2\,\mathrm{Osc}(T)} << R$ and this is the key to obtain the multi-scale behavior of the mean squared displacement. Then one has to control the influence of T on the mean squared displacement and this is done thanks to the following inequality

$$e^{-4\,\mathrm{Osc}(T)}x^2 \leq |F^T(x) - x|^2 \leq e^{4\,\mathrm{Osc}(T)}x^2 \tag{6.98}$$

6.4.1.iii Heat kernel tail

There are mainly two ways to prove the anomalous behavior of the transition probability densities tail, the first one is trough the control on the exit times (this is the strategy used by Barlow-Bass for the Sierpinski carpet), the second one is trough an improvement of the speed of convergence towards the asymptotic process in homogenization theory (see sub subsection 5.3.1.i). The latter strategy will be used and developed in the section 6.5.

J. R. Norris in an interesting paper [Nor97] has shown that the homogenized behavior of the heat kernel $p(t, x, y)$ corresponding to a periodic operator of period T_1^d starts at least for $t \ln t >> |x - y|^2$ (one must have also $|x - y|^2 << t$ to be far from the heat kernel diagonal regime); in this chapter it will be shown that it starts for $t >> |x - y|$ (and that boundary is sharp) in any dimension and this is the key leading to the multi-scale control of the tail of the heat kernel. Since the there is much to say about it, the multi-scale results will be given first but let's remember that if one wants to understand the deep origin of those results one must look at the section 6.5.

In short, one is in front of an IHPD, and one wants to show that the heat kernel tail

$$\mathbb{P}_0(y_t \geq h) \tag{6.99}$$

manifests a sub-diffusive behavior for t and h in a region to be determined. How to do this? How to fix the drift scale and the effective scales since here there are two parameters t and h?

One knows that to give an anomalous upper bound to the heat kernel tail 6.99 it is sufficient to evaluate the Laplace transform of the IHPD $\mathbb{E}[e^{\lambda y_t}]$ for $\lambda > 0$ and then optimize on λ in the following inequality

$$\mathbb{P}_0[y_t \geq h] \leq \mathbb{E}[e^{\lambda(y_t - h)}] \tag{6.100}$$

Then how to evaluate the Laplace transform $\mathbb{E}[e^{\lambda y_t}]$? Observe that one can decompose y_t into a fluctuating scales and drift scales by

$$y_t = \chi^{V_0^{n_{flu}}}(y_t) + \int_0^t \nabla F^{V_0^{n_{flu}}}(y_s)d\omega_s - \int_0^t \nabla V_{n_{dri}}^{\infty}.\nabla F^{V_0^{n_{flu}}}(y_s)ds \tag{6.101}$$

In this equation one sees that the fluctuating scales act trough their martingale behavior (generating the error term $\chi^{V_0^{n_{flu}}}(y_t)$) and the drift scales trough their drift behavior, now just plug this decomposition in the exponential in 6.100 to obtain (by playing with the deformation of the linear harmonic functions) from which it follows that

$$\mathbb{P}_0[y_t \geq h] \leq Ce^{\lambda(e^{CK_0 n_{per}}R_{n_{ef}} - h)}e^{\|\nabla V_{n_{dri}}^{\infty}\|_{\infty}^2 t/4}\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F^{V_0^{n_{flu}}}(y_s)|^2 ds}] \tag{6.102}$$

(the drift scales and effective scales have not been specified yet). Now things start to become serious, indeed one must evaluate

$$\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F^{V_0^{n_{flu}}}(y_s)|^2 ds}] \tag{6.103}$$

how to do this?
Observe that if homogenization were complete on the fluctuating scales one would have

$$\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F^{V_0^{n_{flu}}}(y_s)|^2 ds}] \sim e^{8\lambda^2 t D(V_0^{n_{flu}})} \tag{6.104}$$

but homogenization is never complete over fluctuating scales however if one separates those scales into effective scales n_{ef} and perturbation scales n_{per} and call $\mathcal{F}_t$ the filtration generated by the IHPD one can show that for $z < t$ one can homogenize on the effective scales by paying the price of an error created by the perturbation scales

$$\mathbb{E}[\int_z^t |\nabla F^{V_0^{n_{flu}}}(y_s)|^2 ds|\mathcal{F}_z] \leq e^{2K_0 n_{per}} D(V_0^{n_{flu}})(t-z) + CR_{n_{eff}}^2 \tag{6.105}$$

Can one deduce from the conditional expectations 6.105 a sharp control on the Laplace transform 6.103? The answer is yes and this is a result which is far from being trivial whose explanation is postponed to the section 6.5, one of the consequences of this result is that the homogenized behavior of the heat kernel $p(t,x,y)$ in a perturbed periodic medium starts for $t >> |x-y|$. In short this result says that for λ small enough

$$\lambda < CR_{n_{ef}} \tag{6.106}$$

the Laplace transform can be upper bounded by the behavior of the conditional expectations 6.105 and

$$\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F^{V_0^{n_{flu}}}(y_s)|^2 ds}] \leq C_{\lambda,R_{ef}} e^{8\lambda^2 t e^{2K_0 n_{per}} D(V_0^{n_{flu}})} \tag{6.107}$$

finally one obtains after an optimization on $\lambda \propto h/(tD(V_0^{n_{flu}}))$.

$$\mathbb{P}_0[y_t \geq h] \leq \text{small terms} \times e^{\|\nabla V_{n_{dri}}^{\infty}\|_\infty^2 t/4} e^{-Ce^{2K_0 n_{per}}\frac{h^2}{D(V_0^{n_{flu}})t}} \tag{6.108}$$

Now one sees how to determine the drift scales: it is fixed so that

$$\|\nabla V_{n_{dri}}^{\infty}\|_\infty^2 t/2 < Ce^{2K_0 n_{per}} \frac{h^2}{D(V_0^{n_{flu}})t} \tag{6.109}$$

Thus it is fixed by the value of t/h, in other words, the greater t/h, the greater the number of fluctuating and effective scales. This fact is important because it explains the anomalous shape of the tail of the heat kernel. To understand this fact observe the following example with this value of n_{dri}: assume $R_n \sim \rho^n$ and $D(V_0^n) \sim \lambda^n$ with $\lambda < 1$, then one has

$$n_{dri} \sim \frac{\ln \frac{t}{h}}{\ln \frac{\rho}{\lambda^{\frac{1}{2}}}} \tag{6.110}$$

and

$$\begin{aligned} \mathbb{P}_0[y_t \geq h] &\leq \text{small terms} \times e^{-\frac{h^2}{t\lambda^{n_{ef}}}} \\ &\leq e^{-\frac{h^2}{t}(\frac{t}{h})^\nu} \end{aligned} \tag{6.111}$$

with

$$\nu \sim -\frac{\ln \lambda}{\ln \rho} \tag{6.112}$$

What about the condition λ small enough 6.106? one must not forget it ! in fact this condition says one must have $t > C_{K_0,K_1,R_2}h$. This is interesting ! it means that even in a multi-scale medium, multi-scale homogenizations starts for $t >> h$ (the presence of an infinite number of scales does not perturb this fact).
Actually there is an other condition which has been put under the carpet in the above computation, one can find it in the equation 6.102. One must have $h > 2e^{CK_0 n_{per}} R_{n_{ef}}$, in other words h must be greater than the mixing length associated to the effective scales, it is easy to understand this necessity because if try to evaluate $\mathbb{P}[y_t \geq h]$ for h smaller than this mixing length then one has to take into account the particular shape of those scales (thus one can not say that homogenization has operated on those scales). But now observe that the size of the mixing length of the effective scales grow with n_{ef} which is in the first approximation proportional to $t/(hD(V_0^{n_{ef}})^{\frac{1}{2}})$ this lead to the condition

$$\frac{h^2}{t} > C_{K_1,\rho}(\frac{t}{h})^{\frac{\ln\lambda}{2\ln\rho}} \tag{6.113}$$

This is interesting ! indeed if the medium were periodic a condition $h^2/t >> 1$ would mean that one is far from the heat kernel diagonal regime, here in a medium with an infinite number of scales one finds again this condition that one must be far from the heat kernel diagonal regime but h^2/t can be allowed to be very small in front of one (observe that $\frac{\ln\lambda}{2\ln\rho} < 0$) this additional flexibility is created by the slow down of the diffusion.

Now one can understand the origin and the signification of the results which will be given below on the sub-diffusive behavior of the heat kernel's tail.

Bounded ratios The following theorem corresponds to the theorem 8.6.2

Theorem 6.4.6. *Assume* $\rho_{\max} < \infty$, $\lambda_{\max} < 1$,
$\rho_{\min} > C_{16}$

$$\frac{h^2}{t} \geq C_{11}(\frac{t}{h})^{\frac{\ln\lambda_{\max}}{2\ln\rho_{\max}}+\frac{C_{12}}{(\ln\rho_{\min})^2}} \tag{6.114}$$

and

$$\frac{t}{h} \geq C_{13} \tag{6.115}$$

then for $l \in \mathbb{S}^d$

$$\mathbb{P}[l.y_t \geq h] \leq C_{14}e^{-C_{15}\frac{h^2}{t}(\frac{t}{h})^{\nu}} \tag{6.116}$$

with

$$\nu = -\frac{\ln\lambda_{\max}}{\ln\rho_{\max}} - \frac{C_6}{\ln\rho_{\min}\ln\rho_{\max}} > 0 \tag{6.117}$$

Where C_{16}, C_{15} *depend on* $K_0, K_1, \rho_{\min}, \rho_{\max}, \lambda_{\max}$*;* C_{11} *depends on* K_0, K_1, $\rho_{\max}, \rho_{\min}$*;* C_{13} *on* K_0, K_1, R_2 *and* C_6, C_{12} *on* K_0, K_1

It is not surprising to have the condition 6.115 since even with one scale the homogenized behavior of the transition probability densities starts for $t > h$. Observe also that the condition 6.114 corresponds to the condition that the behavior of the diffusion is far from the heat kernel diagonal regime, however here since $\frac{\ln\lambda_{\max}}{2\ln\rho_{\max}} + \frac{C_{12}}{(\ln\rho_{\min})^2} < 0$ one can have $h^2/t << 1$ before reaching this

regime.
Observe that the equation 6.116 is equivalent to

$$\mathbb{P}[l.y_t \geq h] \leq C_{14} e^{-C_{15}(\frac{h^{d_w}}{t})^{\frac{1}{d_w-1}}} \tag{6.118}$$

with $d_w = 1 + \frac{1}{1-\nu}$ which is the form found for a diffusion in the Sierpinski carpet. It is interesting to notice that this particular form is due to the fact that the fluctuating scale is fixed by the ratio t/h.
Observe also that for a self similar diffusion

$$\nu \sim -\frac{\ln \lambda}{\ln \rho} \tag{6.119}$$

In fact the theorem 6.4.6 is deduced from a more general theorem 8.6.1 that allows to control the IHPD in very general cases. For instance one could choose the ratios and ratios U_n, r_n with a strong dependence on the scale n and at the end one would obtain from this theorem a strong variation of ν with t/h.

Unbounded ratios The following theorem corresponds to the theorem 8.6.3

Theorem 6.4.7. *Assume that for all k, $U_k = U$ (U non constant) and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then for

$$C_1 < \frac{t}{h} < C_2 h \tag{6.120}$$

one has

$$\mathbb{P}[l.y_t \geq h] \leq C_3 e^{-C_4 \frac{h^2}{t} g(\frac{t}{h})} \tag{6.121}$$

with

$$g(x) = (\frac{1}{\lambda})^{(\frac{x}{\ln \rho})^{\frac{1}{\alpha}}(1+\epsilon(x))} \tag{6.122}$$

and $\epsilon(x) \to 0$ as $x \to \infty$
Where the constants C_1, C_2 depend on ρ, α, K_0, K_1 and C_4 on ρ, K_0, K_1, λ.

Remark 6.4.1. Observe that $\frac{t}{h^2} \ln \mathbb{P}[l.y_t \geq h] \to -\infty$ as $t/h \to \infty$. Moreover this theorem shows how the behavior of the diffusion passes from weakly anomalous to strongly anomalous.

6.4.2 All dimensions

(see chapter 10 for this subsection) In fact, if one looks at the proofs leading to the sub-diffusive behavior of an IHPD in dimension one; one would see that there are mainly two layers in those proofs. The first one is probabilistic and it is straightforward to extend it to dimensions greater than one, the second one is analytical and concerns the deformations of linear harmonic functions which is in a sense close to perturbation of elliptic operators with non regular coefficients. This is the generalization of this analytical layer to higher dimensions which constitutes the additional work to be undertaken.

First of all, one must estimate the mixing length associated to the effective scales, this can be done through the following analysis. Let $U \in C^\infty(T_R^d)$ (period R), it has been shown that if the dimension is equal to one then the mixing length associated to an homogenization on U is equal to the period R. What about the case $d \geq 2$? Does a universal result exists saying that the mixing length must be of the order of the period $\xi_m(U) \leq C_d R$ even for $d \geq 2$. As not intuitive it might appear the answer is no ! Pathologies with very long range correlations $\xi_m(U) >> R$ does exist (for any $C > 0$ one can find $U \in C^\infty(T_1^d)$ such that $\xi_m(U) > CR$), they will be given in the subsection 6.6.1.
Does it mean that one has to restrict the fluctuations U_n to avoid those pathologies to be able to say something? The answer is no because $\xi_m(U)$ can be uniformly controlled by $\mathrm{Osc}(U)$ and here it has been assumed that there exists a uniform bound on the oscillations of all the fluctuations of the multi-scale medium: for all n, $\mathrm{Osc}(U_n) \leq K_0$.
This is the subject of the chapter B in which it is shown that for U of period R one has

$$\xi_m(U) \leq C_d e^{(3d+2)\,\mathrm{Osc}(U)} R \tag{6.123}$$

This control is an application of the theory of elliptic operators with discontinuous coefficients.

6.4.2.i Anomaly starting from the invariant measure

The purpose of this sub subsection is to underline the sub-diffusive behavior of the exit times of an IHPD in all dimensions. First of all how the effective scales are chosen? The answer is now easy since the control 6.123 imply that for the multi-scale medium corresponding to a general IHPD one has

$$\xi_m(V_0^{n_{ef}}) \leq C_d R_{n_{ef}} e^{(3d+2)K_0 n_{ef}} \tag{6.124}$$

Thus the effective scales corresponding to the estimation of the expectation of the exit time from the ball of center 0 and radius r ($\mathbb{E}[\tau(0,r)]$) are simply fixed by the necessity that their mixing length must be smaller than r:

$$n_{ef} = \sup\{p \in \mathbb{N} \,:\, \xi_m(V_0^p) < \frac{r}{2}\} \tag{6.125}$$

If the perturbation created by the remaining scales $n_{ef}+1, \ldots, \infty$ were equal to 0 one would have

$$\mathbb{E}_0[\tau(0,r)] \sim \frac{r^2}{\lambda_{\max}(D(V_0^{n_{ef}}))} \tag{6.126}$$

which would lead directly to the sub-diffusive behavior of the exit times by remembering that the effective diffusivities $D(V_0^{n_{ef}})$ decrease geometrically towards 0 with the number of effective scales n_{ef} and the latter increase as the logarithm of the radius of the ball: if for instance $R_n \sim \rho^n$ and $\lambda_{\max}(D(V_0^{n_{ef}})) \sim \lambda^{n_{ef}}$ ($\lambda < 1$) one would have $n_{ef} \sim \ln r/(\ln \rho + (3d+2)K_0)$ which would lead in the first approximation in $1/\ln \rho$ that

$$\mathbb{E}_0[\tau(0,r)] \sim r^{2-\frac{\ln \lambda}{\ln \rho}} \tag{6.127}$$

This would give the anomalous behavior but the influence of the scales $n_{ef}+1, \ldots, \infty$ is in general not null, one has to control it. And this is the hard core of the work. Because to do this one must in a sense compare the Green function (in fact only the exit times but the techniques are similar) associated to the operator L_V of the IHPD in the ball $B(0,r)$ with Dirichlet conditions on the boundary to the Green function associated to the generator $L_{V_0^{n_{ef}}}$ corresponding to the effective scales alone. With some work this is possible if one assumes that homogenization on the scales $0, \ldots, n_{ef}$ were complete in the sense that in the operator L_V and $L_{V_0^{n_{ef}}}$ only the effective diffusivity

$D(V_0^{n_{ef}})$ corresponding to the aggregation of those scales appears. But this is not the case ! what one has to perturb is something which is not the standard Laplace operator and one can not use usual techniques based on the Harnack inequality or the parabolic Harnack inequality (the constants associated to those inequalities blow up with the number of effective scales because those scales are not necessarily homogeneous and isotropic), an new strategy must be found to obtain strong stability results for operators of the form $-\nabla e^{-2U}\nabla$ where $U \in C(B(\bar{0}, r))$.

Mean sub-diffusive behavior If one considers Green functions of symmetric elliptic operators as quadratic forms, then it is easy to compare them. In terms of exit times this imply that one can compare expectation of the exit times associated to the generators L_V and $L_{V_0^{n_{ef}}}$ if one takes their spacial mean with respect to the invariant measure $e^{-2V} / \int_{B(0,r)} e^{-2V(x)}dx$.
In other words one can show that for any $U, P \in C^\infty(\bar{\Omega})$, and Ω a smooth bounded open subset of $\mathbb{R}^d$, if one writes $\mathbb{E}^U$, $\mathbb{E}^{U+P}$ the expectations associated to the diffusions generated by L_U and L_{U+P} and $\tau(\Omega)$ the exit time from Ω; m_U^Ω the following probability measure on Ω:

$$m_U^\Omega(dx) = \frac{e^{-2U(x)}\,dx}{\int_\Omega e^{-2U(x)}\,dx} \tag{6.128}$$

then (proposition 10.0.2)

$$\begin{aligned} \int_\Omega \mathbb{E}_x^U\big[\tau(\Omega)\big] m_{U+P}^\Omega(dx) &\leq e^{2\,\mathrm{Osc}(P)} \int_\Omega \mathbb{E}_x^{U+P}\big[\tau(\Omega)\big] m_{U+P}^\Omega(dx) \\ &\geq e^{-2\,\mathrm{Osc}(P)} \int_\Omega \mathbb{E}_x^{U+P}\big[\tau(\Omega)\big] m_{U+P}^\Omega(dx) \end{aligned} \tag{6.129}$$

Now it is straightforward to control the perturbation induced by the scales $n_{ef}+1, \ldots, \infty$ on the exit time $\tau(0, r)$ for a general IHPD, one might wonder how the drift scales are distinguished from the perturbation scales. Observe that the application of the inequality 6.129 to the IHPD is done with $P = V_{n_{ef}+1}^\infty$, thus n_{dri} is chosen to minimize the error term $\mathrm{Osc}(P)$ on $B(0, r)$, that is to say

$$n_{dri} = \inf\{n \in \mathbb{N} \,:\, R_n \geq r\} \tag{6.130}$$

and the scales $n_{ef}+1, \ldots, n_{dri}-1$ are perturbation scales so that

$$\mathrm{Osc}(P) \leq n_{per} K_0 + 2K_1 \tag{6.131}$$

(the fluctuations U_n for $n \geq n_{dri}$ are bounded by the norm of their gradient $\|U_n\|_\infty r/R_n$ that's why they are called drift scales). Observe that contrary to the one-dimensional case, the number of perturbation scales is not limited to one and tends to grow with r, this is due to the fact that the mixing length of each scale can be very large in front of its respective period.

Bounded ratios between scales In resume, those considerations lead to the following theorem (theorem 10.1.1) if the IHPD has bounded ratios $\rho_{\max} < \infty$.

Theorem 6.4.8. *One has for $r > C_{16}$,*

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) = r^{2+\nu(r)} \tag{6.132}$$

with for $\rho_{\min} > C_{13}$

$$\nu(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}\big(1 + \frac{C_7}{\ln \rho_{\min}}\big) + \frac{1}{\ln r} C_6 \tag{6.133}$$

and

$$\nu(r) \geq \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}\big(1 - \frac{C_{12}}{\ln \rho_{\min}}\big) - \frac{1}{\ln r} C_{11} > C_{15} > 0 \tag{6.134}$$

Where the constants C_{11}, C_{12}, C_7, C_6 *depends on* d, K_0, K_1*;* C_{13} *on* $d, K_0, K_1,$ $\lambda_{\max}$ *and* C_{15}, C_{16} *on* $d, K_0, K_1, \lambda_{\max}, \rho_{\max}$

This theorem gives the sub-diffusive behavior of exit times of the IHPD, the upper bound in the control of ν is not equal to its lower bound because the non self similarity can really creates fluctuations between those two bounds.
Observe that if $\lambda_{\max} = \lambda_{\min} = \lambda$ and $\rho_{\max} = \rho_{\min} = \rho$

$$\nu(r) \sim \frac{\ln \frac{1}{\lambda}}{\ln \rho} \tag{6.135}$$

Now if one wonders what happens for $\rho_{\min}$ below the constant C_{13}, one has to remember what has been said about overlapping ratios, one can have intervals of ratios corresponding to anomalous behavior surrounded by points of normal behavior and without specifying the particular shape of the fluctuations U_n one can not say in which region one is. However to be in a region of strong anomaly it is easy to see that one must have at least (use the Voigt Reiss inequality on the multi-scale effective diffusivities),

$$\limsup_{n\to\infty} \frac{1}{n} \|V_0^n - \frac{1}{R_n^d}\int_{T^d_{R_n}} V_0^n(x)dx\|_\infty > 0 \tag{6.136}$$

In fact the theorem 6.4.8 is deduced from a more general proposition 10.1.1 which allow to control the IHPD in very general cases. For instance one could choose the ratios and ratios U_n, r_n with a strong dependence on the scale n and at the end one would obtain from this theorem a strong variation of ν with r.

Fast separation between scales Now one can also wonder what happens with fast separating scales, then the following theorem (which corresponds to theorem 10.1.2) gives the answer:

Theorem 6.4.9. *Assume that* $R_n = R_{n-1}[\frac{\rho^{n^\alpha}}{R_{n-1}}]$ *(*$\rho, \alpha > 1$*) and* $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ *then*

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) = \frac{r^2}{\lambda^{\beta(r)}} \tag{6.137}$$

with for $r > C_{16}(d, K_0, K_1)$

$$\beta(r) = \big(\frac{\ln r}{\ln \rho}\big)^{\frac{1}{\alpha}}(1 + \epsilon(r)) \tag{6.138}$$

with $\epsilon(r) \to 0$ *as* $r \to \infty$

Observe that this theorem shows how the diffusion becomes more and more anomalous as the separation between scales is less and less large: $\alpha \downarrow 1$

6.4.2.ii Anomaly starting from any point

The proof of the anomaly of the exit times starting from any point is similar in the computation of the effective and perturbation scales to that of the anomaly starting from the invariant measure however there is a crucial difference: the control of the influence of the scales $n_{ef}+1, \ldots, \infty$ is harder.

In fact to obtain such a control for all kinds of fluctuations $U_n \in C^\infty$ it is necessary and sufficient to prove the following conjecture:

Strong stability conjecture I Let $U \in C^\infty(T_1^d)$ and $P \in C^\infty(\overline{B(0,1)})$. Write $\mathbb{E}^{S_R U}$, $\mathbb{E}^{S_R U+P}$ the expectations associated to the diffusions generated by $L_{S_R U}$ and $L_{S_R U+P}$ and $\tau(B(0,1))$ the exit time from the d dimensional unit ball $B(0,1)$.

Conjecture 6.4.1. *The exists $C_d > 0$ a constant depending only on the dimension such that for*

$$R > C_d e^{C_d(\mathrm{Osc}(U)+\mathrm{Osc}(P))} \qquad \mathrm{Osc}(P) < \frac{1}{C_d}\mathrm{Osc}(U) \tag{6.139}$$

and

$$\|\nabla P\|_\infty < \frac{R}{C_d} \tag{6.140}$$

one has

$$E_0^{S_R U+P}\big[\tau(B(0,1))\big] \leq C_d e^{C_d \,\mathrm{Osc}(P)} \sup_{x\in B(0,1)} E_x^{S_R U}\big[\tau(B(0,1))\big] \tag{6.141}$$

and

$$E_0^{S_R U+P}\big[\tau(B(0,1))\big] \geq C_d e^{-C_d \,\mathrm{Osc}(P)} \inf_{x\in B(0,\frac{1}{2})} E_x^{S_R U}\big[\tau(B(0,1))\big] \tag{6.142}$$

In other words to be able to say something about anomaly starting from any point for all kinds of IHPD one must be able to prove the conjecture 6.4.1 which says that if the size of the period $1/R$ is small enough and the perturbation P small and smooth enough then the exit times associated to the operator $L_{S_R U}$ are stable under a perturbation of this operator by the fluctuations of P.

Actually in the chapter 10 the proofs are made under the following conjecture 6.4.2 which is stronger than 6.4.1 (it does not changes much to re write them under the conjecture 6.4.1) because there are good reasons to believe that the conjecture 6.4.2 is true.

Strong stability conjecture II Let $U, P \in C^\infty(B(\bar{0},1))$. Write $\mathbb{E}^U$, $\mathbb{E}^{U+P}$ the expectations associated to the diffusions generated by L_U and L_{U+P} and $\tau(B(0,1))$ the exit time from the d dimensional unit ball $B(0,1)$.

Conjecture 6.4.2. *The exists $C_d > 0$ a constant depending only on the dimension such that*

$$E_0^{U+P}\big[\tau(B(0,1))\big] \leq C_d e^{C_d \,\mathrm{Osc}(P)} \sup_{x\in B(0,1)} E_x^{U}\big[\tau(B(0,1))\big] \tag{6.143}$$

and

$$E_0^{U+P}\big[\tau(B(0,1))\big] \geq C_d e^{-C_d \,\mathrm{Osc}(P)} \inf_{x\in B(0,\frac{1}{2})} E_x^{U}\big[\tau(B(0,1))\big] \tag{6.144}$$

Here the results corresponding to the chapter 10 will be given, nevertheless with a slight change although in the chapter 10 they are given conditionally to the conjecture 6.4.2, here they will be given conditionally to the fact than the exit times created by the smaller scales of the IHPD are stable under the influence of its own larger fluctuations. Which leads to introduce the following condition:

Stability condition of an IHPD An IHPD is said to satisfy the stability condition 6.4.1 if and only if:

Condition 6.4.1. *Under the notation of 6.4.2, the exists $\mu > 0$ such that for all $n \in N$, all $z \in \mathbb{R}^d$, and all $r > 0$,*

$$E_z^V\big[\tau(B(z,r))\big] \leq \mu e^{\mu \,\mathrm{Osc}_{B(z,r)}(V_{n+1}^\infty)} \sup_{x\in B(z,r} E_x^{V_0^n}\big[\tau(B(z,r))\big] \tag{6.145}$$

and

$$E_z^V\big[\tau(B(z,r))\big] \geq \frac{1}{\mu} e^{-\mu \,\mathrm{Osc}_{B(z,r)}(V_{n+1}^\infty)} \inf_{x\in B(z,\frac{r}{2})} E_x^{V_0^n}\big[\tau(B(z,r))\big] \tag{6.146}$$

In fact with the conjecture 6.4.2 says that all the IHPD do satisfy the stability condition 6.4.1.

Anomalous exit times With the definition of effective scales n_{ef} given above:

$$n_{ef}(r) = \sup\{n \geq 0 \ : \ e^{(n+1)(9d+15)K_0} R_n^2 \leq C_d^1 r^2\} < \infty \tag{6.147}$$

which corresponds to the maximal scale n such that its associated mixing length is less or equal to the radius of the ball $B(0,r)$ divided by two: $\xi_m(V_0^n) \leq r/2$ one can define a mean ratio between scales $\rho_{ef}(r)$ and a mean maximal eigenvalue $\lambda_{\max}^{ef}(r)$ for each fluctuation U_n, if the multi-scale medium associated to the IHPD is not self similar, those parameters do vary with r, nevertheless at the scale r they represent the self similar multi-scale medium which would have the same effect on the IHPD as its own non self similar medium U_n, r_n.

$$\lambda_{\max}^{ef}(r) = (\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)\big)^{\frac{1}{n_{ef}(r)+1}} \tag{6.148}$$

$\lambda_{\max}^{ef}(r)$ is called the geometric mean maximal eigenvalue. It reflects the following image: At a scale of order r the maximal eigenvalue of the effective medium characterized by the scales $0,\ldots,n_{ef}+1$ behaves as if those scales were totally separated and the diffusivity of each scale were characterized by the same maximal eigenvalue $\lambda_{\max}^{ef}(r)$ (all associated to the same eigenvector: whose direction does not change with the scale).

$$\ln \rho_{ef}(r) = \frac{\ln r}{n_{ef}(r)} \tag{6.149}$$

$\rho_{ef}(r)$ reflects the following image: The behavior of the IHPD at the scale r is the same as a diffusion with $n_{ef(r)}$ effective scales, the maximal eigenvalue associated to each scale being $\lambda_{\max}^{m}(r)$ and the ratio between each scale being $\rho_{ef}(r)$. Then the following theorem (which corresponds to the theorem 10.2.1) says that the IHPD is at the first approximation in $1/\ln \rho_{\min}$ totally controlled my this geometric mean eigenvalue $\lambda_{\max}^{ef}(r)$ and ratio $\rho_{ef}(r)$.

Theorem 6.4.10. *If the IHPD satisfies the stability condition 6.4.1 and $\lambda_{\max} < 1$, then for $\rho_{\min} > C_{1,d,K_0,K_1,\lambda_{\max},\mu}$, $r > C_{2,d,K_0,K_1,\rho_{\max},\mu}$ one has*

$$\begin{aligned} \mathbb{E}_x\big[\tau(B(x,r))\big] &\leq C_{32,d,K_0,K_1,\mu} r^{2+\sigma(r)(1+\gamma)} \\ &\geq C_{33,d,K_0,K_1,\mu} r^{2+\sigma(r)(1-\gamma)} \end{aligned} \tag{6.150}$$

$$\sigma(r) = \frac{\ln \frac{1}{\lambda_{\max}^{ef}(r)}}{\ln \rho_{ef}(r)}, \qquad \gamma = C_{2,d}\frac{K_0}{\ln \rho_{\min}} < 0.5 \tag{6.151}$$

$$\frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}(1 + \frac{C_{34,d,K_0,K_1,\mu}}{\ln \rho_{\min}})^{-1} \leq \sigma(r) \tag{6.152}$$

and

$$\sigma(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}(1 + \frac{C_{35,d,K_0,K_1,\mu}}{\ln \rho_{\min}}) \tag{6.153}$$

This theorem says that the behavior of the exit times is fixed by the geometric mean effective diffusivity $\lambda_{\max}^{ef}(r)$ and ratio $\rho_{ef}(r)$ at the scale r; the parameter γ plays the role of an error term generated by the perturbation scales. Observe that this theorem is very general in the sense that one can have $\rho_{\max} = \infty$ and underline all kinds of exotic behaviors by choosing ratios r_n a function of n oscillating between $\rho_{\min}$ and very high values with n. Observe also that if $\rho_{\max} < \infty$ then this theorem gives the anomalous behavior of the exit times since

$$0 < rac\ln \frac{1}{\lambda_{\max}} \ln \rho_{\max}(1 + \frac{C_{34,d,K_0,K_1,\mu}}{\ln \rho_{\min}})^{-1} \leq \sigma(r) \tag{6.154}$$

and it follows that

$$\mathbb{E}_x[\tau(x,r)] = r^{2+\nu(r)+\epsilon(r)} \tag{6.155}$$

with $\epsilon(r) \to 0$ as $r \to \infty$ and

$$0 < \nu_1 \leq \nu(r) \leq \nu_2 \tag{6.156}$$

Observe also that if $\rho_{\max} = \rho_{\min} = \rho$ and $\lambda_{\max} = \lambda_{\min} = \lambda$ then in the first approximation in $1/\ln \rho$

$$\mathbb{E}_x[\tau(B(x,r))] \sim r^{2+\frac{\ln \frac{1}{\lambda}}{\ln \rho}} \tag{6.157}$$

Fast separation between scales

Theorem 6.4.11. *If the IHPD satisfies the stability condition 6.4.1,* $R_n = R_{n-1}[\frac{\rho^{n^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ *and* $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ *then*

$$\mathbb{E}_0\big[\tau(B(0,r)\big] = \frac{r^2}{\lambda^{\beta(r)}} \tag{6.158}$$

with for $r > C_{16}(d, K_0, K_1)$

$$\beta(r) = \big(\frac{\ln r}{\ln \rho}\big)^{\frac{1}{\alpha}}(1 + \epsilon(r)) \tag{6.159}$$

with $\epsilon(r) \to 0$ *as* $r \to \infty$

Observe that for this theorem shows how the diffusion becomes more and more anomalous as $\alpha \downarrow 1$. In fact the stability condition 6.4.1 is not necessary because with fast separating ratios homogenization operates (and it is easy to control the error in the asymptotic expansion for the expression of $\mathbb{E}_x[\tau(0,r)]$, then the result is given by an adaptation of the Aronson estimates, this is quite straightforward).

6.4.2.iii Sub-diffusive behavior of the tail of the heat kernel (starting from any point)

From the anomaly of the exit times one can deduce the anomaly of the density of probability of transitions by adapting a strategy used by M.T. Barlow and R. Bass for the Sierpinski Carpet. Since the anomaly of the exit times starting from any point is needed the following results are based on the stability condition 6.4.1 of the IHPD.
The following theorem corresponds to the corollary 10.3.2

Theorem 6.4.12. *If the IHPD satisfies the stability condition 6.4.1,* $\rho_{\max} < \infty$ *and* $\lambda_{\max} < 1$. *Then for* $\rho_{\min} > C(d, K_0, K_1)$ *and*

$$C_{40} r \leq t \leq C_{41} r^{2+\sigma(r)(1-3\gamma)}$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_7 \frac{r^2}{t}\big(\frac{t}{r}\big)^{\nu'}$$

with

$$0 < c < \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}(1 - \frac{C_{50,d,K_0}}{\ln \rho_{\min}}) \leq \nu'(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}(1 - \frac{C_{50,d,K_0}}{\ln \rho_{\min}}) \tag{6.160}$$

$C_{50,d,K_0} < 0.5 \ln \rho_{\min}$ *and the constants* C_{40}, C_{41}, C_{42} *depend on* $d, K_0, K_1, \rho_{\max}, \rho_{\min}$. *All the constants depending on* K_0 *also depend on* μ.

$\sigma(r)$ and γ are those given in the theorem 6.4.10 observe that the condition

$$t \leq C_{41} r^{2+\sigma(r)(1-3\gamma)}$$

reflects the fact the heat kernel must be far from its diagonal regime but what is interesting is that the parameter $r^{2+\sigma(r)(1-3\gamma)}$ which appear here corresponds in the first approximation in $1/\ln \rho_{\min}$ to the expectation of the exit time from the ball $B(0,r)$ for the IHPD (for a Brownian motion it would be r^2).
The condition

$$C_{40} r \leq t$$

corresponds to the fact that even with a periodic medium the homogenized behavior of the heat kernel starts for $t >> r$. Observe that if $\rho_{\max} = \rho_{\min}$ and $\lambda_{\max} = \lambda_{\min}$ then at the first order in $1/\ln \rho_{\min}$, ν' behaves like

$$\nu' \sim \frac{\ln \frac{1}{\ln \lambda}}{\ln \rho} \tag{6.161}$$

Anomaly of the transition probability densities with fast separating scales

Theorem 6.4.13. *If the IHPD satisfies the stability condition 6.4.1, $R_n = R_{n-1}[\frac{\rho^{n^\alpha}}{R_{n-1}}]$ ($\rho, \alpha > 1$) and $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ then for*

$$C_{60} r \leq t \leq C_{61} r^2$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_{63} \frac{r^2}{t} g(\frac{t}{r})$$

with

$$g(x) = (\frac{1}{\lambda})^{(\frac{x}{\ln \rho})^{\frac{1}{\alpha}}(1+\epsilon(x))} \tag{6.162}$$

where $\epsilon(x) \to 0$ as $x \to \infty$ and the constant C_{60} to C_{63} depends on $\rho, \alpha, K_0, K_1, d$. All the constants depending on K_0 also depend on μ.

Observe that $\frac{t}{h^2} \ln \mathbb{P}[l.y_t \geq h] \to -\infty$ as $t/h \to \infty$. Moreover this theorem shows how the behavior of the diffusion passes from a slightly anomalous one to a strongly anomalous one.

6.5 Davies conjecture, exponential martingales and homogenization

The section concerns the chapter 12. The purpose of this section is to show that the homogenized behavior of the heat kernel $p(t,x,y)$ associated to a periodic generator starts for $t >> |x-y|$. This gives an answer to the Davies conjecture for the upper bound and obtain a sharp lower bound estimate of the tail of the heat kernel in all dimensions allowing to complete the picture describing the behavior of $p(t,x,y)$ (see the section 5.3). For previous results concerning that subject see the work of E. B. Davies [Dav87],[Dav93]; E.B. Davies and M.M.H. Pang [DP89]; J.R. Norris and D.W. Stroock [NS89]; J.R. Norris [Nor92], [Nor97]; R. Bhattacharya, M. Denker and A. Goswami [BDG99]; A. Dembo [Dem96]. In fact the key theorem 6.5.1 for this result concerns more general objects than periodic generators, it can be applied for instance to ergodic media since its input is only the behavior of the conditional quadratic variation of the diffusion.
For instance in this chapter it has been used to give the anomalous behavior of an IHPD for which the medium has an infinite number of scales, below it will be used to draw the three different regimes of the heat kernel associated to generator $\frac{1}{2}\Delta - \nabla U \nabla$ with $U \in C^\infty(T_1^d)$ as an example of its utility.

The three different regimes of the heat kernel in a periodic medium

1. **Large deviation regime**: for $|x-y| >> t$ the paths of the diffusion concentrate on the geodesics and

$$\ln p(t,x,y) \sim -\frac{|x-y|^2}{2t} \tag{6.163}$$

2. **Homogenization regime**: for $1 << |x-y| << t$ and $|x-y|^2 >> t$, homogenization takes place and

$$p(t,x,y) \sim \frac{1}{t^{\frac{d}{2}}} \exp(-\frac{|x-y|^2}{2D(e_{y-x})t}) \tag{6.164}$$

3. **Heat kernel diagonal regime**: for $|x-y|^2 << t$, the behavior is fixed by the diagonal of the heat kernel and

$$p(t,x,y) \sim \frac{C_0(x)}{t^{\frac{d}{2}}} \tag{6.165}$$

Note that $|x-y| << 1$ and $|x-y| << t$ imply $|x-y|^2 << t$ thus all the regimes are here.
In fact for the homogenization regime ($1 << |x-y| << t$ and $|x-y|^2 >> t$) it will be shown that

$$p(t,x,y) \lesssim \frac{1}{t^{\frac{d}{2}}} \exp(-\frac{|x-y|^2}{2D(e_{y-x})t}) \tag{6.166}$$

and writing y_t the diffusion associated to $p(t,x,y)$, for $l \in \mathbb{R}^d$, $|l| = 1$, $\lambda > 0$ one has the following tail estimate

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{4\sqrt{2\pi}} \int_X^\infty e^{-z^2/2} dz \tag{6.167}$$

with

$$X \sim \frac{\lambda}{\sqrt{{}^t l D(U) l t}} \tag{6.168}$$

one can combine the strategy given in the proof with the Aronson estimates to obtain a sharp lower bound for the behavior of the heat kernel as it is done for the upper bound in the corollary 6.5.2. This is quite straightforward in dimension one and needs some care in higher dimensions, this will be the subject of a sequel work.

6.5.1 The key theorem, an exponential inequality for martingales

The following theorem corresponds to the theorem 12.1.1. Consider M_t a continuous square integrable $\mathcal{F}_t$ adapted martingale such that $M_0 = 0$ and for $\lambda, t > 0$, $\mathbb{E}[e^{\lambda M_t}] < \infty$.
Assume that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t_2 > t_1 \geq 0$ one has a.s.

$$\mathbb{E}[\int_{t_1}^{t_2} d < M, M >_s |\mathcal{F}_{t_1}] \leq \int_0^{t_2 - t_1} f(s) ds$$

With $f(s) = f_1$ for $s < t_0$ and $f(s) = f_2$ for $s \geq t_0$ with $t_0 > 0$ and $0 < f_2 < f_1$.

Theorem 6.5.1. *For the martingale given above one has*

1. *for all*

$$0 < |\lambda| < \frac{1}{\big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}} \tag{6.169}$$

one has

$$\mathbb{E}[\exp(\lambda M_t)] \leq e^{3(1-1/g(\lambda))} \exp(\frac{g(\lambda)}{2}\lambda^2 f_2 t) \tag{6.170}$$

with $g(\lambda) = \frac{1}{1-\lambda^2(f_1-f_2)t_0 e}$ *which verify* $1 \leq g \leq 2$

2. *for all*

$$0 < \nu < \frac{1}{2e(f_1 - f_2)t_0} \tag{6.171}$$

one has

$$\mathbb{E}[\exp(\nu < M, M >_t)] \leq \exp(\nu f_2 t)\frac{\exp\big(\nu t_0(f_1 - f_2)\big)}{((f_1 - f_2)\nu t_0)^2} \tag{6.172}$$

As one can see, this theorem use the knowledge on the conditional behavior of the quadratic variation of a martingale to upper bound its Laplace transform, and one knows that once one has a sharp control on the Laplace transform, one has a sharp control on the probability of going far. Observe also that $g \to 1$ when $\lambda \to 0$ so the control is sharp. The condition λ small enough 6.169 is absolutely necessary in the sense that it marks the boundary between the large deviation regime and the homogenization regime. It is interesting to note that in the proof this boundary corresponds to a point above which a series becomes divergent.

6.5.2 Application to bound from above the tail estimate of a martingale

The direct application of the key theorem is the following result which corresponds to the corollary 12.1.1

Corollary 6.5.1. *Let* M_t *be the martingale given in theorem 6.5.1.*
Write $C_1 = \big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}/f_2$. *Then for*

$$r = \frac{C_1 x}{t} < 1 \tag{6.173}$$

one has

$$\mathbb{P}(M_t \geq x) \leq e^{\frac{3}{2}r^2} \exp\big(- (1 - r^2)\frac{x^2}{2f_2 t}\big) \tag{6.174}$$

Note that $0 < g_1(r) \leq 1$ and that g_1 converges towards 1 with the speed r^2 as $r \to 0$, so this upper bound gives an estimate the speed of convergence towards the limit process of the behavior of the martingale. Note that the homogenization behavior starts when $r < 1$ and converges towards the asymptotic process as $x/t \to 0$ with the speed given above.

6.5.3 Application to homogenization in periodic media

As an example, the theorem 6.5.1 will be applied here to obtain estimates on the speed of convergence towards the asymptotic process of periodic potential form diffusions.

$$dy_t = d\omega_t - \nabla U(y_t)dt \tag{6.175}$$

where $U \in C^1(T_1^d)$ $(U(0) = 0)$

The following corollary corresponds to the corollary 12.1.2.

Corollary 6.5.2. *Consider $p(t,x,y)$ the transition density probabilities of the diffusion 6.175 with respect to the measure m_U. then for*

$$20k_1|x-y| < t, \qquad k_2 < \frac{|x-y|}{\sqrt{t}}, \qquad |x-y| > 4C_\chi \tag{6.176}$$

one has

$$p(t,x,y) \le \frac{E_1}{t^{\frac{d}{2}}} \exp\big(-(1-E)\frac{|y-x-2C_\chi|^2}{2D(e_{y-x})t}\big) \tag{6.177}$$

where k_1, k_2, C_χ, E_1 are constants depending only on d and $\mathrm{Osc}(U)$*. Moreover*

$$E = 8(\frac{k_1|x-y|}{t})^2 + 2\frac{\sqrt{t}}{|x-y|} \le \frac{1}{10} \tag{6.178}$$

Remark 6.5.1. Since the constants appearing in this corollary does not depend on $\|\nabla U\|_\infty$ but only on $\mathrm{Osc}(U)$ it is an easy task to extend this result to case where U is only bounded (left to the reader, see for instance the theorem 1.2 of [CQHZ98]).
Observe that this corollary gives a sharp upper bound corresponding to the homogenized behavior of the heat kernel.

In fact this is only an example and one can consider a wider class of periodic diffusions. Indeed, the corollary 6.5.2 is deduced from the theorem 6.5.2 can be used to give estimates on the rate of convergence to the limit process of a diffusion in a periodic media as soon as a cell problem is well defined and Aronson kind of estimates do exist(see the next subsection).
For instance one can consider the operator 5.131 considered by J.R. Norris, one has to combine the theorem 12.1.2 to the generalized Aronson type estimates obtained by J.R. Norris [Nor97] (see subsection 5.3.2) in order to obtain estimates on the rate of convergence towards the limit process of the diffusions associated to those operators (this application is left to the reader).

6.5.4 Application to the upper bound estimate of the transition probability densities of a diffusion

Here the key theorem will be applied to give a sharp upper bound for the heat kernel of a diffusion for which a sort of cell problem can be defined ("sort of" because the so called cell problem solution χ does only need to be upper bounded). The medium associated to the diffusion is not specified (it may be ergodic) the only thing that is needed is a sharp control on the conditional quadratic variation of the martingale associated to the diffusion.
Consider y_t is a diffusion on $\mathbb{R}^d$ such that for $t > 0$

$$y_t = x + \chi(t) + M_t \tag{6.179}$$

where $\chi(t)$ is a uniformly (in t) bounded random vector process ($\|\chi\|_\infty \le C_\chi$) and M_t is a continuous square integrable $\mathcal{F}_t$ adapted martingale such that $M_0 = 0$ and for $\lambda, t > 0$, $\mathbb{E}[e^{\lambda M_t}] < \infty$.
Assume also that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $l \in \mathbb{R}^d$ with $|l| = 1$ for all $t_2 > t_1 \ge 0$ one has a.s.

$$\mathbb{E}[\int_{t_1}^{t_2} d < M.l, M.l >_s |\mathcal{F}_{t_1}] \le \int_0^{t_2-t_1} f(s)ds$$

With $f(s) = f_1$ for $s < t_0$ and $f(s) = {}^t lDl < f_1$ for $s \ge t_0$ with $t_0 > 0$ and $0 < f_2 < f_1$.
where D is a positive definite symmetric matrix.

Assume also that the diffusion y_t has symmetric Markovian probability densities $p(t,x,y)$ with respect to the measure $m(dy)$ such that for all $x,y \in \mathbb{R}^d$ and $t>0$

$$p(t,x,y) \leq \frac{C_2}{t^{\frac{d}{2}}} \tag{6.180}$$

and for $\delta > 0$

$$\mathbb{P}_x(|y_t - x| \geq \delta) \leq C_3 e^{-C_4 \frac{\delta^2}{t}} \tag{6.181}$$

where C_2, C_3, C_4 are constants.
The following theorem corresponds to the theorem 12.1.2

Theorem 6.5.2. *Assume that y_t is the diffusion described above. Then with $k_1 = \big(2e(f_1 - \lambda_{\min}(D))t_0\big)^{\frac{1}{2}}/\lambda_{\min}(D)$ and $k_2 = 30 + 10d\lambda_{\max}(D)(1+C_4)$*

$$20k_1|x-y| < t, \qquad k_2 < \frac{|x-y|}{\sqrt{t}}, \qquad |x-y| > 4C_\chi \tag{6.182}$$

one has

$$p(t,x,y) \leq \frac{E_1}{t^{\frac{d}{2}}} \exp\big(-(1-E)\frac{|y-x-2C_\chi|^2}{2D(e_{y-x})t}\big) \tag{6.183}$$

with

$$E_1 = C_2\Big(\frac{e^{3/2}}{2\lambda_{\min}(D)C_4} + 2^d C_3\Big) \tag{6.184}$$

and

$$E = 8\Big(\frac{k_1|x-y|}{t}\Big)^2 + 2\frac{\sqrt{t}}{|x-y|} \leq \frac{1}{10} \tag{6.185}$$

Remark 6.5.2. Note that $E \to 0$ as $\frac{|x-y|}{t} + \frac{\sqrt{t}}{|x-y|} \to 0$, this gives an estimate on the speed of convergence towards asymptotic process. The exact homogenized behavior appears in the asymptotic regime $|x-y|/t \to 0$ and $|x-y|^2/t \to \infty$.
It is interesting to note that the homogenization regime begins as soon as the time t is of order of the distance $x-y$ (which must be at least of the order of C_χ).
The condition $k_2\sqrt{t} < |x-y|$ is a natural one in the sense that if it says that the behavior of the diffusion is not too close to the center of the Gaussian, however with $20k_1|x-y| < t$ the large deviation regime is replaced by a homogenized regime. Note also that if one is only interested in the behavior of the diffusion in the direction $y-x$ all that is needed is that $\chi.e_{y-x}$ is upper bounded (χ may have a greater generality than the solution of the cell problem).

6.5.5 Lower bound. Sharp estimate of the speed of convergence towards the asymptotic process

The key theorem can also be used to give a lower bound on the tail of the heat kernel in a periodic medium, this is the object of this subsection.
Consider the diffusion y_t on $\mathbb{R}^d$ associated to the generator ($U \in C^\infty(T_1^d)$)

$$L = \frac{1}{2}\Delta - \nabla U.\nabla \tag{6.186}$$

(as usual this is only an example, one can consider wider class of generators as soon as a cell problem is well defined) The following theorem corresponds to the theorem 12.2.1.

Theorem 6.5.3. *For $l \in \mathbb{S}^d$, $\lambda > C_6(d, \mathrm{Osc}(U))$ and*

$$C_7(d, \mathrm{Osc}(U))\lambda < t \tag{6.187}$$

one has

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{4\sqrt{2\pi}} \int_X^\infty e^{-z^2/2} dz \tag{6.188}$$

with

$$X = \frac{\lambda}{\sqrt{{}^t l D(U) l t}}(1 + E) \tag{6.189}$$

and

$$E = \frac{C_8(d, \mathrm{Osc}(U))}{\lambda} + C_5(d, \mathrm{Osc}(U))\sqrt{\frac{\lambda}{t}} \leq \frac{1}{10} \tag{6.190}$$

Remark 6.5.3. Observe that all the constants appearing above only depends on d and $\mathrm{Osc}(U)$, thus it is easy to extend this result to the case when U is only bounded and periodic (left to the reader). Note also that $E \to 0$ as $1/\lambda + \sqrt{\frac{\lambda}{t}} \to 0$ giving the speed of convergence towards the asymptotic process. Note also that one can consider a wider class of periodic diffusions such as the one 5.131 considered by J.R. Norris (this extension is left to the reader)

one can combine the strategy given in this theorem with the Aronson estimates to obtain a sharp lower bound for the behavior of the heat kernel as it has been done for the upper bound in the corollary 6.5.2. This is quite straightforward in dimension one and needs some care in higher dimensions (because one needs to show that for well chosen time changes one can approximate a multidimensional martingale by a Gaussian process, this will be the subject of a sequel work).

6.6 Pathologies

6.6.1 The long range correlation pathology

In dimension one, the mixing length associated to a smooth periodic potential $U \in C^\infty(T_1^d)$ is upper bounded by the period which is one here. One might think that this is also the case for $d \geq 2$, the purpose of the pathology presented in the subsection is to show that this intuition is false for $d \geq 2$. Indeed observe the figure 6.3. This is an illustration of a periodic obstacle U (consider it as a reflecting obstacle for a Brownian motion for the moment), of period T_1^d, for clarity, the x_2 axis has been stretched in comparison to the x_1. Observe that basically each period is decomposed into three regions A, B and C parallel to the $(0, x_1)$ axis and the central B one is separated from the two others by two reflecting walls. There are tunnels linking the central region to the two others (above and below along the $(0, x_2)$ axis) and observe that those "correlations tunnels" can be as long as one wants along the direction $(0, x_1)$. Because of this particular structure what happens in the central region is not correlated to what happens half a period above and below along the $(0, x_2)$ plus half a period along the $(0, x_1)$ axis; it is correlated to what happens half a period above and below along the $(0, x_2)$ plus a translation of ξ_m (which can big as big as one wants) along the $(0, x_1)$ axis. How is this translated in the mathematical terms? Just consider the solution of the cell problem χ_1 associated to U along the $(0, x_1)$ axis, observe that one can write this periodic solution as $\chi = x_1 - F_1(x)$ where $F_1(x)$ is the harmonic function associated to the generator L_U with linear growth along the $(0, x_1)$ axis (an periodic along the $(0, x_2)$ axis), because of those correlation tunnels F_1 has a fluctuation of order ξ_m along the $(0, x_2)$ axis and one obtains that $\|\chi_1\|_\infty \sim \xi_m$. And since χ reflects the difference between the diffusion and its martingale behavior it has been obtained that the mixing length of the diffusion built on the periodic medium U is of order of $\xi_m >> 1$ which can be chosen as large as

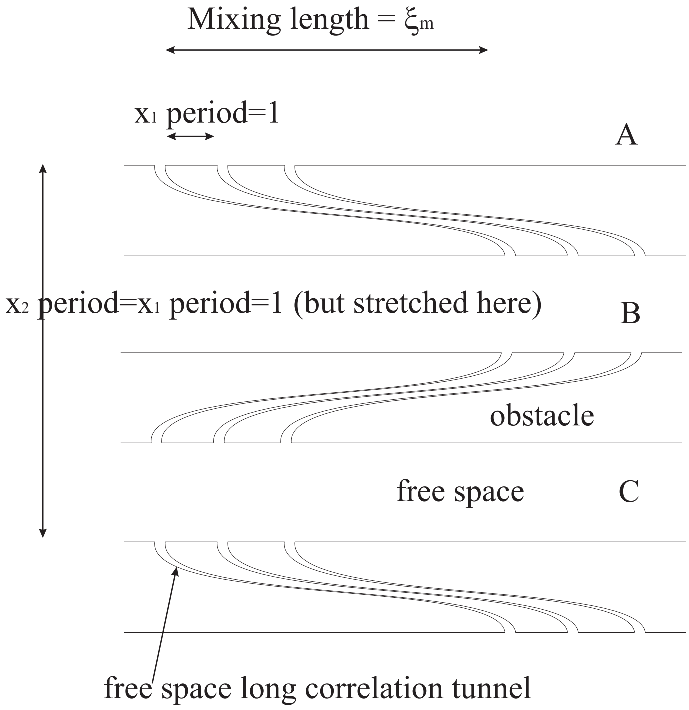

Fig. 6.3: *Long range correlation periodic obstacles.*

needed.
Here, this little presentation has been done with a hard reflecting obstacle, nevertheless one can approximate it by smooth periodic versions $W_n \in C^\infty(T_1^d)$ whose associated solution of the cell problem will converge to χ as $\mathrm{Osc}(W_n) \to \infty$.
Now one understands why a condition such as for all n, $\mathrm{Osc}(U_n) \leq K_0 < \infty$ is necessary, if one takes it away one must be aware that the mixing length might explode (one will have to introduce a geometric condition to avoid this pathology).

6.6.2 The Critical Point Pathology

This pathology has been found by Alano Ancona [Anc99]. The strategy used to control the perturbation scales of an IHPD in dimension one (for the study of the mean squared displacement) is based on the deformation of the linear harmonic functions. More precisely given $U \in C^\infty(T_1^1)$, write χ the solution of the associated cell problem and $F(x) = x - \chi(x)$ the linear harmonic function associated to the generator L_U. In dimension one it is elementary to show that for $x \in \mathbb{R}$

$$|F(x)| \geq e^{-2\,\mathrm{Osc}(U)}|x| \tag{6.191}$$

and this trick is used to control the influence of the perturbation scales on the mean squared displacement of the diffusion.
Thus it is natural to wonder whether this trick can be extended to higher dimensions, more precisely for $U \in C^\infty(T_1^d)$ $(d \geq 1)$ write $(x \in \mathbb{R}^d)$ $F_. = x - \chi_.$ (with $L_U \chi_l = -l \nabla U$) the linear harmonic vector associated to L_U, is it true that F must satisfy $\|F(x)\| \geq C\|x\|, x \in \mathbb{R}^d$ for some $C = C_U > 0$?
As it has been pointed out by A. Ancona, this question is equivalent to the following one:
Is $F_.$ a diffeomorphism of $\mathbb{R}^d$ (onto $\mathbb{R}^d$)? Which is equivalent to wonder whether the Jacobian of $F_.$ is non degenerate at every $x \in \mathbb{R}^d$.
In resume the answer to those questions is yes if it is not possible for $F_.$ to have a critical point for $d \geq 2$.
A. Ancona in [Anc99] shows that $F_.$ can not have a critical point for $d = 2$ (which suggests that the one-dimensional trick might work) but for $d \geq 3$ there exists a periodic function $U \in C^\infty(T_1^d)$ such

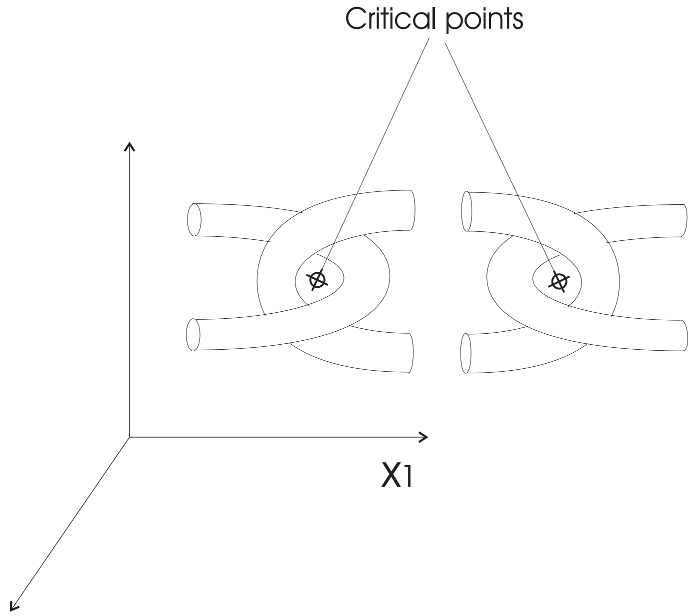

Fig. 6.4: *Obstacles with critical points.*

that the associated linear harmonic function in $\mathbb{R}^d$ has critical points.
The counter example given by A. Ancona is drawn in the figure 6.4 (a single period has been illustrated). The pathology is built by considering reflective tubular (the diffusion can enter into them by their cut ends) half circles (and considering smooth approximations of that pattern). This structure is interesting in itself because it shows that for $d \geq 3$ the geometry of the obstacles can correlate distant points so that if one observe the propagation of heat among those them one will see its gradient vanishing and changing upside down in some points of the space.
Now for $d \geq 3$ imagine that all the fluctuations U_n associated with the IHPD are all characterized by the fact that their corresponding linear L_{U_n} harmonic vector admits a critical point at the origin 0, what is their exact influence on the IHPD starting from the origin? This question has not been investigated in this work, nevertheless it might hide interesting behaviors.

6.7 Perturbation

6.7.0.i Mathematical Interpretation and consequences

See the chapter 13 for this section.
The proof of the sub-diffusive behavior starting from any point for ($d \geq 2$) is based on the stability condition 6.4.1. This condition says that the exit times of the diffusion whose generator is associated to the smaller scales, are stable under the influence of the larger ones. If one can find an IHPD such that this condition is violated one would have found an IHPD whose behavior can be quite weird at some fixed points of the medium. Actually there are good reasons to believe that such an IHPD can not be found, in other words, they necessarily all satisfy the condition 6.4.1. This fact is implied by the conjecture 6.4.1 which is true in dimension one. This conjecture is also implied by the stronger conjecture 6.4.2 which is also true in dimension one.

Comparison of elliptic operators To prove the stability condition 6.4.1 or the conjectures 6.4.1, 6.4.2 one is lead to compare two different elliptic operators. The chapter 3 gives the usual techniques to do so. But if one look closer at those techniques one would see that they all corresponds to a comparison with the Laplace operator, if one tries to use them to compare to operators which are not the Laplace operator one would obtain their equivalence nevertheless with constants which tends to explode. The reason is simple, for instance if one tries to compare the Green functions associated to

two elliptic operators $-\nabla(e^U\nabla)$, $-\nabla(e^{U+V}\nabla)$ with Dirichlet condition on $B(0,1)$ one can do so by comparing them to the Laplace operator $-\Delta$ and using for instance the result of Stampacchia (see subsection 3.6.1) but one will obtain a bound on the ratio between the Green functions G_{U+V}/G_U which will depend on U and will explode as $\mathrm{Osc}(U) \to \infty$.
The Harnack inequality and Aronson estimates are also comparisons with respect to the Laplace operator (the constants explode as $\mathrm{Osc}(U) \to \infty$). Actually the strategy given by Davies to obtain Gaussian bounds could be developed to obtain anomalous bounds nevertheless one would have to improve the constants given for the Parabolic Harnack inequality by P. Li and S. T. Yau, in [LY86]. Actually when the scales separates quickly it is easy to use the techniques of the chapter 3 to prove the stability of the IHPD (left to the reader, it is not done here because this work focus on bounded ratios between scales).

Deformation of elliptic operators Those conjectures are based on the comparison of two different operators which is quite not pleasant to handle. The purpose of the chapter chapter 13 is to show how two operators can be compared by an analytical inequality verified by a single one.
More precisely it is possible to show by the techniques developed in the chapter 13 that the conjecture 6.4.1 is implied by to the following conjecture:

Conjecture 6.7.1. *There exist a constant C_d depending only on the dimension of the space such that for $\lambda \in C^\infty(\overline{B(0,1)})$ such that $\lambda > 0$ on $\overline{B(0,1)}$ one has*

$$\begin{aligned}\sup_{x\in B(0,1)} \int_{B(0,1)} \lambda(y)|\nabla_y G_\lambda(x,y).\nabla_y \int_{B(0,1)} G_\lambda(y,z)\lambda(z)\,dz|\,dy \leq \\ \sup_{x\in B(0,1)} C_d \int_{B(0,1)} G(x,z)\lambda(z)\,dz\end{aligned} \tag{6.192}$$

where $G_\lambda(x,y)$ are is the Green function of the operator $-\nabla(\lambda\nabla)$ on $B(0,1)$ with Dirichlet conditions on the boundary.

The conjecture 6.7.1 is itself implied by the conjecture 6.7.2

Conjecture 6.7.2. *For $\Omega \subset \mathbb{R}^d$an open subset with smooth boundary, there exist a constant $C_{d,\Omega}$ depending only on the dimension of the space and the open set such that for $\lambda \in C^\infty(\bar\Omega)$ such that $\lambda > 0$ on $\bar\Omega$ and $\phi, \psi \in C^2(\bar\Omega)$ null on $\partial\Omega$ and both sub harmonic with respect to the operator $-\nabla(\lambda\nabla)$, one has*

$$\int_\Omega \lambda(x)|\nabla\phi(x).\nabla\psi(x)|\,dx \leq C_{d,\Omega}\int_\Omega \lambda(x)\nabla\phi(x).\nabla\psi(x)\,dx \tag{6.193}$$

This conjecture is true in dimension one with $C_{d,\Omega} = 3$ (this constant is an homotopy invariant, this is proven by the corollary 13.5.1). In fact one can show that (the following corollary corresponds to the corollary 13.5.4)

Corollary 6.7.1. *Let Ω be a smooth bounded open subset of $\mathbb{R}^d$. Assume that ϕ, ψ are both convex or both concave and null on $\partial\Omega$, then*

$$\int_\Omega |\nabla_x\phi(x).\nabla_x\psi(x)|\,dx \leq 3\int_\Omega \nabla_x\phi(x).\nabla_x\psi(x)\,dx \tag{6.194}$$

this result is deduced from the following theorem (which corresponds to 13.5.3)

Let $\lambda \in C^\infty(\bar\Omega)$ such that $\lambda > 0$ on $\bar\Omega$, then $\psi \in C^2_D(\Omega)$ is said to be strongly sub harmonic (resp. strongly super harmonic) with respect to the operator $-\nabla(\lambda\nabla)$ if for all $x \in \Omega$, all $e \in \mathbb{S}^d$ $-\frac{\partial}{\partial e}\big(\lambda(x)\frac{\partial}{\partial e}\phi\big) \geq 0$ (resp. ≤ 0)

Theorem 6.7.1. *For all $\psi, \phi \in C^2(\Omega)$ strongly sub harmonic or super harmonic with respect to the operator $-\nabla(\lambda\nabla)$ and null on $\partial\Omega$ one has*

$$\int_\Omega |\nabla_x\phi(x)\lambda(x)\nabla_x\psi(x)|\,dx \leq 3\int_\Omega \nabla_x\phi(x)\lambda(x)\nabla_x\psi(x)\,dx \tag{6.195}$$

In fact the 6.7.1 is deduced from a more general result given in the chapter 13 (see the theorem 13.5.2). In fact the chapter 13 shows that the conjecture 6.7.2 is equivalent to the stability of the Green functions associated to the operator $-\nabla(\lambda\nabla)$ under a small isotropic deformation of that operator, this is the object of the corollary 13.4.1.

6.7.0.ii Physical interpretation and consequences

The conjecture 6.7.2 has an interesting signification (and consequences) in the framework of electrostatic theory, one can see Ω as a dielectric cavity with conducting boundary and the conjecture 6.7.2 is equivalent to the stability of the electrostatic potential created by a density of negative charges under a small isotropic perturbation of the dielectric constant of the material.
In fact it has other mathematical and physical consequences (see the chapter 13, all will not be given here). One of them is directly connected to the notion of localization of the electrostatic energy. As it is shown in the section 13.1 there has been an interesting debate among physicists on that subject, and as it has been underlined by R. F. Feynman ([Fey79] page 142) "the idea of locating the energy in the field is incompatible with the assumption existence of punctual charges. One way out of the difficult would be to say that elementary charges, such as an electron, are not points but are really small distribution of charge. Alternatively, we could say that there is something wrong in our theory of electricity at very small distances, or with the idea of the local conservation of the energy. There are difficulties with either point of view. These difficulties have never been overcome; there exists to this day."
It is interesting to note that this difficulty is one of the causes of the infinite terms appearing in quantum electrodynamics (the other being the infinite number of degree freedom of the field but it is easy to get rid of that one by a change of the origin of the energies).
If the conjecture 6.7.2 is true then the energy displaced by an introduction of a small quantity of charges is stable (see the chapter 13) if it is false, then there would be a strange point in the idea of the local conservation of the energy: by spending a small amount of energy one could displace a huge amount of energy (without a priori bound) ! And the chapter 13 shows also how this notion of local conservation of energy is linked to the stability of the electrostatic potentials under a small perturbation of the permittivity of the material.

6.7.1 Strong conjectures

In fact heuristic considerations lead to conjecture that:

Conjecture 6.7.3. *For $\Omega \subset \mathbb{R}^d$an open subset with smooth boundary, there exist a constant $C_{d,\Omega,1}$ depending only on the dimension of the space and the open set such that for M a symmetric smooth coercive matrix on $\bar{\Omega}$ and $\phi, \psi \in C^2(\bar{\Omega})$ null on $\partial\Omega$ and both sub harmonic with respect to the operator $-\nabla(M\nabla)$, one has*

$$\int_\Omega |\nabla\phi(x)M\nabla\psi(x)|\,dx \leq C_{d,\Omega,1}\int_\Omega \nabla\phi(x)M\nabla\psi(x)\,dx \tag{6.196}$$

Conjecture 6.7.4. *For $\Omega \subset \mathbb{R}^d$an open subset with smooth boundary, there exist a constant $C_{d,\Omega,2}$ depending only on the dimension of the space and the open set such that for M a symmetric smooth* *coercive matrix on* $\bar{\Omega}$ *and* $\phi, \psi \in C^2(\bar{\Omega})$ *null on* $\partial\Omega$ *and both sub harmonic with respect to the operator* $-\nabla(M\nabla)$*, one has*

$$\int_\Omega \|\nabla\phi(x)\|\|M\nabla\psi(x)\|\,dx \leq C_{d,\Omega,2}\int_\Omega \nabla\phi(x)M\nabla\psi(x)\,dx \tag{6.197}$$

The conjecture 6.7.3 is equivalent to the strong stability of the Green function associated to the operator $-\nabla(M\nabla)$ (with Dirichlet condition on the boundary) under an isotropic perturbation of the permittivity M (see theorem 13.4.1 and the theorem 13.4.3).
If $C_{d,\Omega,1} = 3$ as it is true in dimension one, this would mean that the energy displaced in the electrostatic cavity by the importation of new charges (of the same sign as those present in the cavity) is always less or equal to work accomplished.
The conjecture 6.7.3 is equivalent to the strong stability of the Green function associated to the operator $-\nabla(M\nabla)$ (with Dirichlet condition on the boundary) under a non isotropic perturbation of the permittivity M (see corollary 13.4.2 and 13.4.4).

6.8 Link with the Infinitely Ramified Fractals

The purpose of this section is to investigate on the connections between an IHPD and so called Brownian motions constructed of fractals such as the Sierpinski Carpet.

6.8.1 The Sierpinski carpet

What is the link between an IHPD and the reflected Brownian motion on the Sierpinski pre-Carpet? Take for all $n \in \mathbb{N}$, U_n periodic of period T_1^d ($d \geq 2$) such that

$$U_n(x) = \begin{cases} +\infty & \text{for } x \in [1/3, 2/3]^d \\ 0 & \text{for } x \in [0,1]^d - [1/3, 2/3]^d \end{cases} \tag{6.198}$$

Choose $\rho_{\min} = \rho_{\max} = 3$, then observe that the function $U(x)$ on $\mathbb{R}^d$ given by the equation 1.2 is equal to 0 on the generalized pre-Sierpinski Carpet and equal $+\infty$ outside of it.
Of course, one has to choose U_n smooth for all n, but one can approximate 6.198 to obtain a soft version of the carpet where the reflection on the faces of the cubes is replaced by a strong drift.
Observe also that similarly, one can construct a soft version of the random Sierpinski carpets studied in [HKKZ98].
Moreover, since the U_n are allowed to vary in $\mathcal{C}^\infty(T_1^d)$ without any symmetries, it will be shown that interesting pathologies may appear.
Thus the Sierpinski pre-carpet is a particular case of the multi-scale medium associated to a self similar IHPD, nevertheless there is no uniform bound on the norm and gradient of the U_n ($K_0 = K_1 = \infty$) for the hard version of the pre-Sierpinski carpet. And the theorems given above do not give any rigorous result for this particular case because they are too general and reflect an other point of view in the following sense:

- The point of view of the construction of the Brownian motion on the generalized Sierpinski carpet is to control the diffusion through the knowledge of the particular shape and symmetries of the medium on which it is evolving. More precisely the first layer of the proofs are calibrated to exactly fit the corresponding symmetries and reflecting planes. This point of view is interesting in the following sense: it allows to go far deeper in the analysis of the diffusion and to obtain very precise results for some particular multi-scale media which can be used to help the imagination in more general situations. Moreover its corresponding breakthrough is accompanied with new mathematical tools and results which can be used in more general situations.

- The point of view of the construction of an IHPD is to characterize a diffusion evolving on a medium with an infinite number of scales with the greater generality, of course there is a price to pay for this generality, since the only knowledge on the medium that one has is $\rho_{\max}, \rho_{\min}, \lambda_{\max}, \lambda_{\min}, K_0, K_1$, the proofs must always be done by imagining the worst case corresponding to those parameters, and with the parameters $K_0 = K_1 = \infty$ corresponding to the Sierpinski carpet the worst case says that the diffusion does not move at all or it can also be normal (both cases are possible because the ratio between scales always belongs to the overlapping ratios interval), thus either one assumes $K_1 < \infty$, either one gives more information on the particular shape of the fluctuations by the introduction of new parameters in order to be able to say something because with $K_1 = \infty$, the overlapping ratios interval is not upper bounded ! (equal to $(1, \infty)$). To develop this point of view one is lead to develop the tools of homogenization theory.

6.8.2 From soft obstacles to hard obstacles

Of course the next step in the development of the IHPD is to consider multi-scale media such that $K_0 = K_1 = \infty$, indeed this work on the IHPD is mainly based on homogenization theory and this theory allows to consider wider objects than soft fluctuations U_n one can consider reflecting obstacles (also called hard obstacles by opposition to soft obstacles). The purpose of this subsection is to show how the tools developed in this work could be used and to underline the additional work to be undertaken.
If one look closer at the proofs given in this work, one would see that there are basically two layers, the first one is analytical and the second probabilistic. The probabilistic layer can be directly used without any changes however the analytical one is based on the behavior of solutions of elliptic operators with discontinuous coefficients, and the operators will be modified to the Laplace operator with Neumann condition on the boundaries of the obstacles, this is the location of the additional work.
It is clear that the notion of effective scales remains unchanged, indeed one can still compute mixing length associated to $\xi_m(V_0^{n_{ef}})$, however with $K_0 = \infty$ those mixing length could grow very quickly to infinity (see the long range correlation pathology) that's why one have to introduce a parameter that will ensure that the mixing length of the effective scales will remain of the size of their period; how to do this? If the boundaries of the obstacles are regular a cell problem $\chi_.$ associated to the homogenization on those effective scales is still well defined, one would have to ensure that its associated norm (which one to chose is not clear for the moment) will remain bounded.
What about the drift scales? Indeed with hard obstacles $K_1 = \infty$ and the influence of a reflecting obstacle can not be considered has a small drift. Nevertheless one can assume that the radius of curvature of the boundaries of the hard obstacles tend towards infinity with their size, and one can associate a visibility length $\xi_v(U_n)$ (equal to the radius of the curvature) to each obstacle corresponding to the following image: below $\xi_v(U_n)$ the diffusion feels the influence of the obstacles U_n only through the reflection against an hyperplane and one knows that such a reflection will not change the anomalous behavior of the diffusion.
What about the perturbation scales? This will be the hard part for which specific new parameters will have to be introduced in order to ensure the stability of the reflected diffusion under the influence of a few scales.
Of course one would have also to control the speed of convergence towards 0 of the multi-scale effective diffusivities (not necessarily geometric if the medium is not self similar).

6.8.3 Limit process

An interesting problem has not been investigated in this work (which is focused on the anomaly of the diffusion), it is the existence of a limit process. In other words, for a diffusion y_t evolving on a periodic medium $U \in C^\infty(T_1^d)$ one knows that $\epsilon^{\frac{1}{2}} y_{t/\epsilon}$ converges in law to a Gaussian diffusion z_t with

covariance matrix $D(U)$, z_t in a sense, reflects the long time behavior of y_t.
What about a Infinitely Homogenized Potential Diffusion associated to a medium V with an infinite number of scales? Is it possible to find a spacial rescaling function $\epsilon \to k(\epsilon)$ such that as $\epsilon \downarrow 0$, $k(\epsilon)y_{t/\epsilon}$ converges in law to some limit process? The answer is of course yes, take $k(\epsilon) = 0$ but this is not very interesting indeed; to obtain an interesting answer one must refine the question by "... a limit process which is non degenerated?". Now the answer is in general no and the explanation is very simple.
Indeed at the time t, the slowdown of the IHPD is reflected by multi-scale effective diffusivity $D(V_0^{n_{ef}})$; now observe that this matrix can be non isotropic and one can have $\lambda_{\max}\big(D(V_0^{n_{ef}})\big)/\lambda_{\min}\big(D(V_0^{n_{ef}})\big) \to +\infty$! in such a situation it is easy to see that for any choice of $k(\epsilon)$ can not converge to a limit process living in a d-dimensional space, either it will blow up in the direction of the maximal eigenvalue, either it will converge to a point in the direction of the minimal eigenvalue; and if the direction of the associated eigenvectors is not stable with t one can imagine that this will produce a very weird rotation of the degenerate axes of the diffusion with the time t.
In a sense to find such an answer is natural since there are no a priori reasons for an IHPD to have a unique long time behavior since there are an infinite number of scales which are not self similar.
Now imagine that one still needs to obtain a limit process, then there are two strategies to force the diffusion to do so:

- Either one accelerates differently the diffusion along direction corresponding to each eigenvector of $D(V_0^{n_{ef}})$: $k(\epsilon)y_{\epsilon_i t}e_i$
- Either one spatially rescale differently the diffusion along direction corresponding to each eigenvector of $D(V_0^{n_{ef}})$: $k_i(\epsilon)y_{\epsilon t}e_i$

Now observe that the first one will produce a diffusion whose dynamic at the time t is not Markovian !
The second one, corresponds to a deformation of the space with a biased lense whose magnification are different along the different axis of the eigenvectors.
What if the diffusion is one dimensional or $D(V_0^{n_{ef}})$ is always isotropic? Then the problem of the existence of a degenerate axis vanish, however is the medium is not self similar it is easy to see that although $k(\epsilon)y_{t/\epsilon}$ might be tight in some non degenerate space, it will only converge along specific subsequences of ϵ because the limit is not unique. This pathology is created by the oscillation of the invariant measure (the next subsection will come back to it) at the scale of the observation (because the medium is not self similar).
What about if $D(V_0^{n_{ef}})$ is isotropic and the medium self similar? In that case the question is important but postponed to a sequel work.

Now observe that the existence of a limit process requires an a priori choice which has been put under the carpet above. Indeed for the construction of the Brownian motion on the Sierpinski carpet, the diffusion is not constructed by rescaling the space but by adding smaller and smaller obstacles, although those two point of view seems equivalent when the medium is self similar, they are certainly not when the medium is not self similar. This choice is also reflected in the construction of the invariant measure associated to the limit process (or to the Dirichlet form), this is the subject of the next subsection.

6.8.4 Soft Pre Fractal Measure

See the section C.2 for this part.
The medium on which an IHPD is built in this work, is not a fractal in the usual sense of this term. An IHPD is uniquely controlled by the drift $-\nabla V$ characterized by an infinite number of scales, however it is more convenient to describe it through the invariant measure $e^{-2V(x)}\,dx$ of the associated generator (and its is a also the proper way because it has a precise physical and

mathematical signification). In this work the invariant measure will sometimes be called smooth pre-fractal measure or smooth periodic pre-fractal (this notion is introduced and discussed in the section C.2)will be introduced and analyzed (this name is given according the name "Sierpinski pre-Carpet" introduced by H. Osada).

Definition 6.8.1. A smooth pre-fractal measure is a collection $\{(r_n, U_n)_{n\in\mathbb{N}}\}$ where for each n, $r_n \in \mathbb{N}/\{0,1\}$ and $U_n \in C^\infty(T_1^d)$ such that $U_n(0) = 0$ and

$$K_1 = \sup_{n\in\mathbb{N}} \|\nabla U_n\|_\infty < \infty \tag{6.199}$$

It constitute the medium on which the diffusion takes place. From a physical point of view its density e^{-2V} can be seen as an energy landscape with an infinite number of potential pits (all with approximately same depth).
Observe that the condition $U_n(0) = 0$ is not necessary to have a well defined drift ∇V however it is necessary to have a well defined invariant measure e^{-2V} (something is hidden behind this fact, the section 6.8.5 will come back to it).
Of course the first thing that one would like to do in front of a pre-fractal measure is to characterize it by a sort of fractal dimension, however the notion of Hausdorff dimension which is convenient to describe subsets of $\mathbb{R}^d$ is not well adapted to a measure on $\mathbb{R}^d$.
Nevertheless one knows that the Hausdorff measure associated to a fractal subset keeps in its growth rate the signature of the Fractal dimension. Thus it is natural to seek what is the growth rate associated to the smooth pre-fractal measure.
At this stage since a soft pre-fractal measure is characterized by a smaller scale and has no upper bound for the size of its scales it is natural to explore the growth rate at infinity:

Growth rate at infinity

Definition 6.8.2. The Growth rate at infinity of a measure μ on $\mathbb{R}^d$ is the segment $[d^\infty_{f,\min}(\mu), d^\infty_{f,\max}(\mu)]$ where

$$d^\infty_{f,\min}(\mu) = \lim\inf_{r\to\infty} \frac{\mu(B(0,r))}{\ln r} \tag{6.200}$$

$$d^\infty_{f,\max}(\mu) = \lim\sup_{r\to\infty} \frac{\mu(B(0,r))}{\ln r} \tag{6.201}$$

If the pre-fractal measure is self similar ($U_n = U$, $r_n = \rho$) the growth rate at infinity is a point given by the topological pressure of U.

$$d^\infty_{f,\max}(e^{-2V}dx) = d^\infty_{f,\min}(e^{-2V}dx) = d^\infty_f(e^{-2V}dx) = d + \frac{\mathcal{P}_\rho(-2U)}{\rho} \tag{6.202}$$

Note that this definition of growth rate at infinity dimension is not invariant under a translation of U_0 (indeed under a translation by Θ_y, U_0 should be modified to $x \to U_0(x+y) - U_0(y)$ so that U^0 is well defined). Observe also that the value of $d^\infty_f(m^0_U)$ is fixed by the necessity of e^{-2U^0} to be a well defined density measure but it can be greater than the dimension of the space.
Thus $d^\infty_f(m^0_U)$ is not translation invariant and one can have $d^\infty_f(m^0_U) > d$ (so one must be careful if one tries to link it with a sort of Hausdorff fractal dimension of the pre-fractal).
For a non self similar pre fractal measure the growth rate at infinity is in an interval given by

$$d^\infty_{f,\min}(m^0_U) = d + \lim\inf_{r\to\infty} \frac{\ln \int_{T_1^d} \exp(-2U^{-n(r),0}(x))dx}{\rho(r)n(r)} \tag{6.203}$$

$$d^{\infty}_{f,\min}(m^0_U) = d + \lim\sup_{r\to\infty} \frac{\ln\int_{T_1^d}\exp(-2U^{-n(r),0}(x))dx}{\rho(r)n(r)} \tag{6.204}$$

where

$$\rho(r) = \frac{\ln r}{n(r)} \tag{6.205}$$

$$n(r) = \sup\{n \in \mathbb{N} : R_n \le r\} \tag{6.206}$$

and

$$U^{-p,k} = \sum_{n=0}^{k+p} U_n(\frac{R_p x}{R_n}) \tag{6.207}$$

6.8.4.i Growth rate at 0

One might think, that this definition of d_f which gives back a value that can be greater than d is unsatisfactory, and may be by analyzing the growth rate at 0 of the torus one might obtain a better characterization, this is the object of this sub subsection.

The natural way to define a growth rate at 0 is to consider the measure $m_{U^{-p,0}}$ on the torus T_1^d, observe that this measures are invariant if one add to each U_n a different constant c_n, then define the growth rate at 0 at the point x by the segment $[d^0_{f,x,min}, d^0_{f,x,max}]$ by for $0 < \alpha < 1$

$$d^0_{f,x} = \lim_{p\to\infty} \frac{-\ln\Big(m_{U^{-p,0}}\big(B(x, \frac{1}{R_{[p\alpha]}})\big)\Big)}{\ln R_{[p\alpha]}} \tag{6.208}$$

Then one can show that $d^0_{f,x}$ does not depend on $0 < \alpha < 1$ and

$$d^0_{f,x,\min} = d + \lim\inf_{r\to\infty} \frac{\ln\int_{T_1^d} e^{-2\left(U^{-n(r),0}(y) - U^{-n(r),0}(x)\right)}dy}{\rho(r)n(r)} \tag{6.209}$$

$$d^0_{f,x,\max} = d + \lim\sup_{r\to\infty} \frac{\ln\int_{T_1^d} e^{-2\left(U^{-n(r),0}(y) - U^{-n(r),0}(x)\right)}dy}{\rho(r)n(r)} \tag{6.210}$$

Thus the growth rate at 0 at the point 0 is the same that the growth rate at infinity at the point 0, moreover it depends on the point x and $d^0_{f,x,\max}$ can be greater than d. Thus the growth rate at 0 does suffer from the same pathology (something is hidden behind this fact, the section 6.8.5 will come back to it).

6.8.4.ii From a SPFM to a fractal measure

The purpose of this subsection is to investigate on the following problem: how to build a fractal measure on the torus from a given smooth pre fractal measure?

Completion of a self similar SPFM If the SPFM is self similar the problem is easy and there are basically two ways: The first one is to consider the sequence $(m_{U^{-p,0}})_{p\in\mathbb{N}}$ of probability measures on the torus T_1^d, where

$$m_{U^{-p,0}} = \frac{e^{-2U^{-p,0}(x)}dx}{\int_{T_1^d} e^{-2U^{-p,0}(x)}dx} \tag{6.211}$$

Since the torus is compact this sequence of measures in tight and one can extract a subsequence converging to a measure on the torus and call fractal measure the limit.
The second one is to consider the sequence $(m_{U^{-p,+\infty}})_{p\in\mathbb{N}}$ of probability measures on the cube $\mathbb{R}^d$, where

$$m_{U^{-p,+\infty}} = \frac{e^{-2U^{-p,+\infty}(x)}dx}{\int_{T_1^d} e^{-2U^{-p,0}(x)}dx} \tag{6.212}$$

This sequence of measures in tight and one can extract a subsequence converging to a measure on the each compact subset of $\mathbb{R}^d$ and call fractal measure the limit.
Unicity problems will be studied in a sequel work.

Completion of a non self similar SPFM In this case the problem is more serious because it requires an a priori choice. Indeed the first way would be to consider the sequence $m_{U^{-p,0}}$ on the torus, this sequence is tight and one can call fractal measure the limits of converging subsequences. It is easy to see that with this method the limits are not unique because the scale of order 0 is always changing. The same pathology happens if on consider the sequence of probability measures $m_{U^{-p,\infty}}$ on the unit cube $[0,1]^d$
The alternative way to avoid this pathology would be to complete the non self similar SPFM by smaller scales $(U_{-k})_{k\in\mathbb{N}^*}$ $(U_{-k}\in C^\infty(T_1^d))$ and $(r_{-k})_{k\in\mathbb{N}^*}$, $(r_{-k})_{k\in\mathbb{N}^*}$. Then write

$$\frac{1}{R_{-k}} = r_{-1}\dots r_{-k} \tag{6.213}$$

and

$$V^{-m,p} = \sum_{k=-m}^{p} U^k(\frac{x}{R_k}) \tag{6.214}$$

then consider the measure

$$m_{V^{-p,0}} = \frac{e^{-2V^{-p,0}(x)}dx}{\int_{T_1^d} e^{-2V^{-p,0}(x)}dx} \tag{6.215}$$

on the torus T_1^d or the measure

$$m_{V^{-p,+\infty}} = \frac{e^{-2V^{-p,+\infty}(x)}dx}{\int_{T_1^d} e^{-2V^{-p,0}(x)}dx} \tag{6.216}$$

on R^d. With these choices the obstacle of order $0,-1,\dots,-k$ does not change for $p\geq k$, as usual one can extract subsequences and call fractal measure the limit measure. The unicity problem is postponed to a sequel work.

6.8.5 Creation of a logarithmic pit

The growth rate at infinity of the invariant measure e^{-2V} associated to the dynamic of an IHPD is fixed by the necessity of for all x, $V(x) < \infty$, the growth rate at the origin of the signature of the multi-scale medium on the torus depends also on the starting point x. This phenomenon does not appear for a "hard fractal" such as the Sierpinski carpet because the starting point x is always "conditioned" to be outside of all the scales of obstacles.
Is something hidden behind this fact? Indeed observe that the dynamic of an IHPD does only depend on the gradient $-\nabla U_n$ of the fluctuations, and the condition $U_n(0) = 0$ has no reason to interfere with it.
Nevertheless, look a little bit closer at the mechanism of the dynamic $dy_t = d\omega_t - \nabla f(y_t)dt$ from an other point view: how to chose f to ensure that the resulting diffusion y_t will be sub-diffusive? Then the natural answer is that f must corresponds to a potential pit (and the dynamic is invariant under the change of the origin of the energies $f \to f + c$). Now refine a little bit this question, how to choose f to ensure that the mean squared displacement of y_t will be of the form $\mathbb{E}[y_t^2] \sim t^{1-\nu}$? Assume that f is radial (thus also $E_x[y_t^2]$), then write $r^2(t) = \mathbb{E}_x[y_t^2]$, then from a heuristic point of view

$$\frac{\partial r^2(t)}{\partial t} = L_f r^2 \tag{6.217}$$

is written

$$\begin{aligned} 2r\frac{\partial r}{\partial t} &= \frac{1}{2}\frac{\partial^2(r^2)}{\partial r^2} + \frac{d-1}{2r}\frac{\partial(r^2)}{\partial r} - \frac{\partial(r^2)}{\partial r}\frac{\partial f}{\partial r} \\ &= d - 2r\frac{\partial f}{\partial r} \end{aligned} \tag{6.218}$$

And using $r(t) = t^{\frac{1-\nu}{2}}$, it follows that $\partial_t r(t) = \frac{1-\nu}{2} r^{-\frac{1+\nu}{1-\nu}}$ and

$$\frac{\partial f}{\partial r} = \frac{d}{2r} - \frac{1-\nu}{2}\frac{1}{r^{\frac{1+\nu}{1-\nu}}} \tag{6.219}$$

Thus

$$f(r) = \frac{d}{2}\ln r + \frac{(1-\nu)^2}{4\nu} r^{-\frac{2\nu}{1-\nu}} + \text{constant} \tag{6.220}$$

Thus from a heuristic point of view, the potential pit corresponding to the behavior of the mean squared displacement $\mathbb{E}[y_t^2] \sim t^{1-\nu}$ is logarithmic .
Now consider a self similar IHPD, and observe that the growth rate of

$$V(x) = \sum_{n=0}^{\infty} U(\frac{x}{R^n}) \tag{6.221}$$

is like

$$\sup_{x\in B(0,r)} V \sim \text{Osc}(U)\frac{\ln r}{\ln R} \tag{6.222}$$

it has logarithmic shape, is it a coincidence? The above heuristic computation suggests no. In fact the generator of an IHPD is the h-transform (see for instance [Pin95] section 7.4) of the generator of the Brownian motion with $h = e^{-2V}$. And one knows that h-transforming a generator on a bounded open set Ω is equivalent to conditioning the behavior of its associated diffusion, for an IHPD, $\Omega = \mathbb{R}^d$ is unbounded nevertheless the h-transformation condition the diffusion to approach infinity (the boundary) at a specified sub diffusive speed.
In resume these heuristic considerations suggest that a diffusion in a fractal medium such as the Sierpinski carpet or a smooth pre fractal is sub-diffusive because through the subjacent dynamic the multi-scale medium is seen as an effective logarithmic pit.

6.8.6 Origin of the anomalous estimates on fractals

The purpose of this subsection is to answer to the following question: Why the estimates of the behavior of the Brownian on an infinitely ramified fractal (or an IHPD on a smooth pre fractal) are of the form

$$\mathbb{E}[y_t^2] \sim t^{\frac{2}{d_w}} \tag{6.223}$$

$$\mathbb{E}[\tau(0,r)] \sim r^{d_w} \tag{6.224}$$

$$\ln p(t,x,y) \sim -\big(\frac{|x-y|^{d_w}}{t}\big)^{\frac{1}{d_w-1}} \tag{6.225}$$

(the ratio between the scales is equal to ρ and $D(V_0^n) \sim \lambda^n$ with $\lambda < 1$) To do so the multi-scale homogenization technique used for the IHPD will also be used for the Sierpinski carpet on a heuristic point of view. Below the formulae giving the number of the effective scales are the same as those used for the IHPD (and the influence of the perturbation scales will be neglected). This will give three values of d_w corresponding to the forms 6.223, 6.224, 6.225 and the interesting point is to compare them.

6.8.6.i Mean squared displacement

The origin of the behavior of the mean squared displacement 6.223 is the fact that the number effective scales is fixed by the time t according to the following formula

$$n_{ef} \sim \frac{\ln t}{2\ln\rho} \tag{6.226}$$

and

$$\mathbb{E}[y_t^2] \sim t\lambda^{n_{ef}(t)} \sim t^{1+\frac{\ln\lambda}{2\ln\rho}} \tag{6.227}$$

thus

$$d_{w,1} = \frac{2}{1+\frac{\ln\lambda}{2\ln\rho}} \tag{6.228}$$

6.8.6.ii Exit times

The origin of the behavior of the exit times 6.224 is the fact that the number effective scales is fixed by the radius r according to the following formula

$$n_{ef} \sim \frac{\ln r}{\ln\rho} \tag{6.229}$$

and

$$\mathbb{E}[\tau(0,r)] \sim \frac{r^2}{\lambda^{n_{ef}}} \sim r^{2-\frac{\ln\lambda}{\ln\rho}} \tag{6.230}$$

thus

$$d_{w,2} = 2 - \frac{\ln\lambda}{\ln\rho} \tag{6.231}$$

	$\mathbb{E}_0[y_t^2]$	$\mathbb{E}_0[\tau(0,y)]$	$\ln \mathbb{P}_0[y_t \geq h]$
n_{ef}	$\frac{\ln t}{2\ln\rho}$	$\frac{\ln r}{\ln \rho}$	$\frac{\ln \frac{t}{h}}{\ln \frac{\rho}{\lambda^{\frac{1}{2}}}}$
Heuristic	$t\lambda^{n_{ef}}$	$\frac{r^2}{\lambda^{n_{ef}}}$	$-\frac{h^2}{t\lambda^{n_{ef}}}$
Anomaly	$t^{\frac{2}{d_{w,1}}}$	$r^{d_{w,2}}$	$-\left(\frac{h^{d_{w,3}}}{t}\right)^{\frac{1}{d_{w,3}-1}}$
$d_{w,i}$	$\frac{2}{1+\frac{\ln\lambda}{2\ln\rho}}$	$2-\frac{\ln\lambda}{\ln\rho}$	$1+\frac{1}{1+\frac{\ln\lambda}{\ln\rho-\frac{1}{2}\ln\lambda}}$

6.8.6.iii Transition probability densities

The origin of the behavior of the transition probability densities 6.225 is the fact that the number effective scales is fixed by the ratio $t/|x-y|$ according to the following formula

$$n_{ef} \sim \frac{\ln \frac{t}{|x-y|}}{\ln \frac{\rho}{\lambda^{\frac{1}{2}}}} \tag{6.232}$$

and

$$\begin{aligned}\ln p(t,x,y) &\sim -\frac{|x-y|^2}{t\lambda^{n_{ef}}} \sim -\frac{|x-y|^2}{t}\left(\frac{t}{|x-y|}\right)^{\frac{-\ln\lambda}{\ln\frac{\rho}{\lambda^{\frac{1}{2}}}}} \\ &= --\left(\frac{|x-y|^{d_{w,3}}}{t}\right)^{\frac{1}{d_{w,3}-1}}\end{aligned} \tag{6.233}$$

with

$$d_{w,3} = 1 + \frac{1}{1+\frac{\ln\lambda}{\ln\rho-\frac{1}{2}\ln\lambda}} \tag{6.234}$$

6.8.6.iv Comparisons

Observe that the multi-scale homogenization techniques gives back the right forms for the mean squared displacement, the exit times and the transition probability densities; it interesting to note that they are explained by the number of effective scales (on which homogenization can be considered as complete) associated to each observation. Moreover $d_{w,1}, d_{w,2}$ and $d_{w,3}$ are equal up the first order approximation in $1/\ln\rho$ nevertheless they are not equal and this is not surprising. Indeed when ρ is small the second order term in $1/(\ln\rho)^2$ can not be neglected since the perturbation scales becomes more and more dominant (and it has been shown with the IHPD that the influence of the perturbation scales is of the order of $1/(\ln\rho)^2$).

6.8.7 Uniform Harnack inequality

The proof used by Barlow-Bass to construct the reflecting Brownian motion on the Sierpinski carpet is based on an uniform Harnack inequality (it is the very core of the proof) thus it is natural to wonder what is the connection between the uniform Harnack inequality of Barlow Bass and the anomalous behavior of an IHPD? In fact there is no direct connection in the sense that the uniform Harnack inequality is in general not verified by the generator associated to an IHPD. Why is it so? The reason is simple the uniform Harnack inequality reflects an isotropy of the space seen by the diffusion, this isotropy is broken for an IHPD (since one can have $\lambda_{\max}(D(V_0^n))/\lambda_{\min}(D(V_0^n)) \to \infty$ as $n \to \infty$) this make the constant associated to the Harnack inequality explode with the number scales nevertheless it does not prevent the diffusion from being anomalous because this condition is not necessary. Where would it be useful if it was verified for an IHPD? It would be useful to control the perturbation scales, indeed the uniform Harnack inequality would allow to use the standard techniques used to control the stability of the Laplace operator because it would in a sense say that the behavior of the Green functions of the perturbed operator are close to those of the Laplace operator.

6.9 Some pictures of multi-scale media associated to an IHPD

The figure 6.5 illustrate the contour lines (6 contour lines) of

$$V(x,y) = \sum_{k=0}^{2} U(\frac{x}{\rho^k}, \frac{y}{\rho^k}) \tag{6.235}$$

with $\rho = 4$ and

$$U(x,y) = \cos(x + \pi\sin(y) + 1)^2 \sin(\pi\cos(x) - 2y + 2)\cos(\pi\sin(x) + y) \tag{6.236}$$

The figure 6.6 illustrates the same function but with 9 contour lines.
The figure 6.7 illustrates the function V with $\rho = 3$ and

$$\begin{aligned} U(x,y) =& \cos(x + \pi\sin(y))\sin(y + \pi\cos(x+1)) \\ &+ 0.4\sin(\pi\cos(x) + \pi\cos(y) + 2) + 0.4\cos(\pi(\cos(x))^2 + y + 0.5) \end{aligned} \tag{6.237}$$

The figure 6.8 illustrates the same function but with 9 contour lines.
The figure 6.9 illustrates the function

$$V(x,y) = \sum_{k=0}^{2} U_k(\frac{x}{R_k}, \frac{y}{R_k}) \tag{6.238}$$

with $R_0 = 1$, $R_1 = 3$, $R_2 = 12$ and 14 contour lines.

$$U_2(x,y) = 1.5\cos(x + 2\pi\sin(y + 0.4) + 1)\big(\cos(x+3)\big)^2\big(\sin(y+0.2)\big)^2 \tag{6.239}$$

$$U_1(x,y) = \sin(x - y + 1.4)\big(\cos(x+1.1)\big)^2\big(\cos(y+0.3)\big)^2 \tag{6.240}$$

$$U_0(x,y) = 0.7\big(\cos(2\pi\cos(x) + 2\pi\sin(y) + 1)\big)^4\big(\cos(x+3)\big)^2\big(\sin(y)\big)^2 \tag{6.241}$$

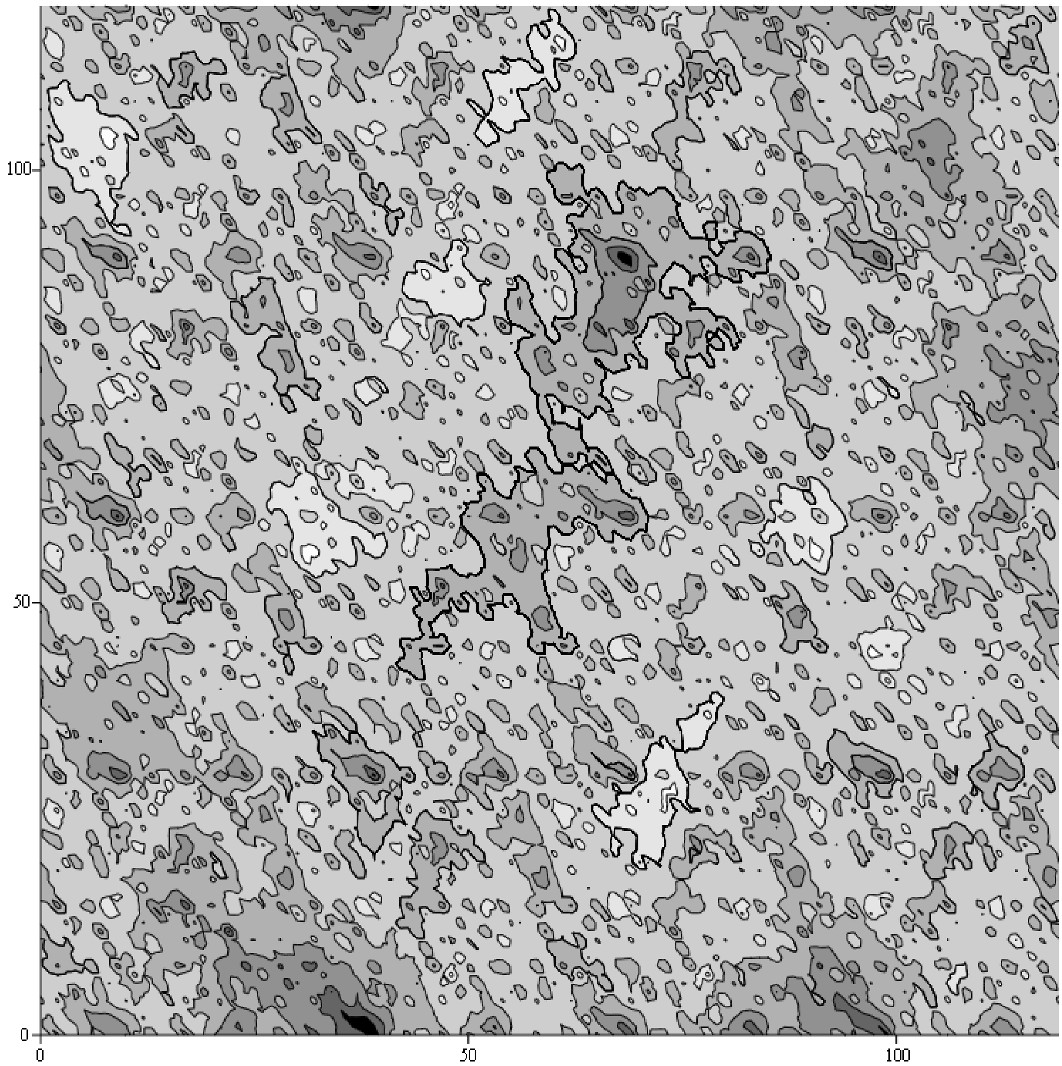

Fig. 6.5: *Smooth pre fractal medium*

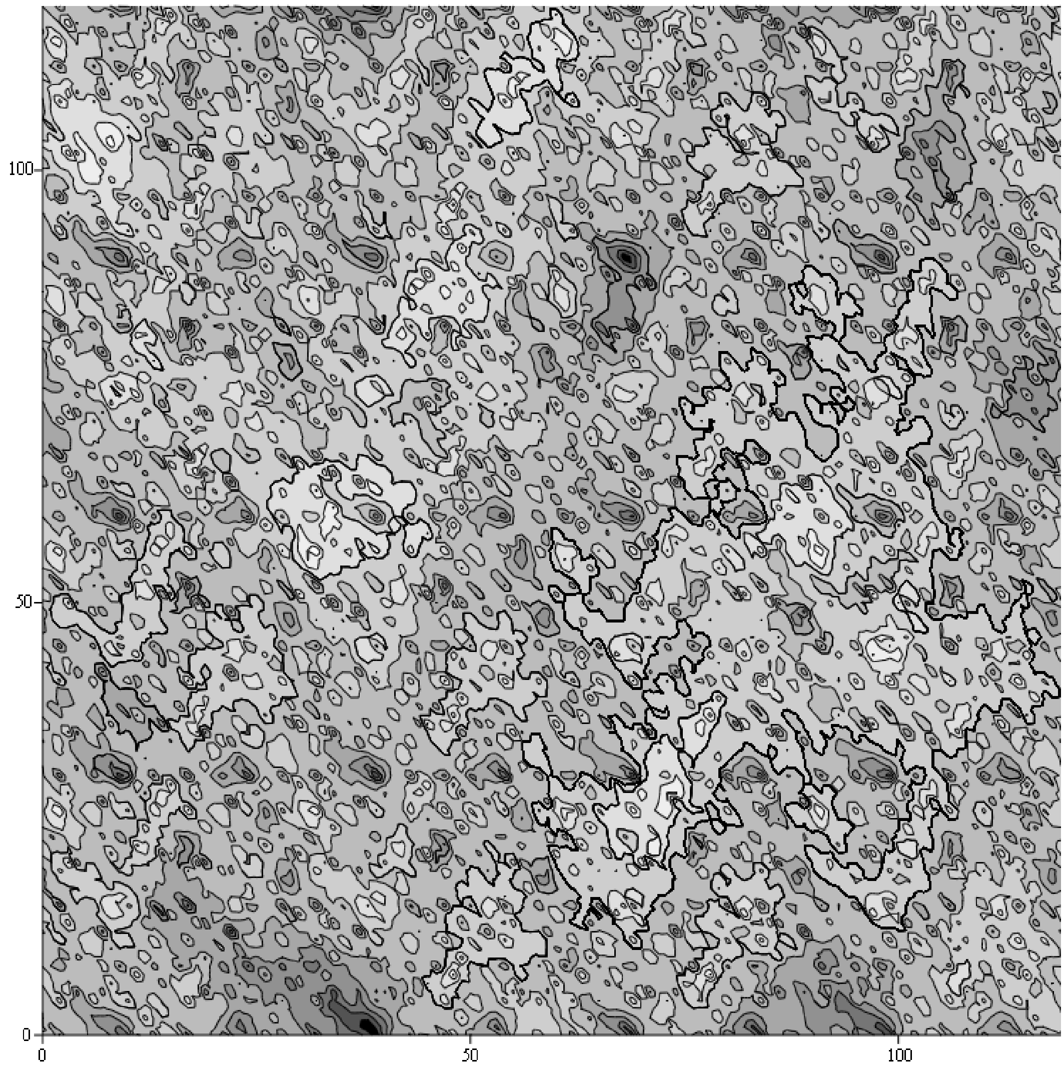

Fig. 6.6: *Smooth pre fractal medium*

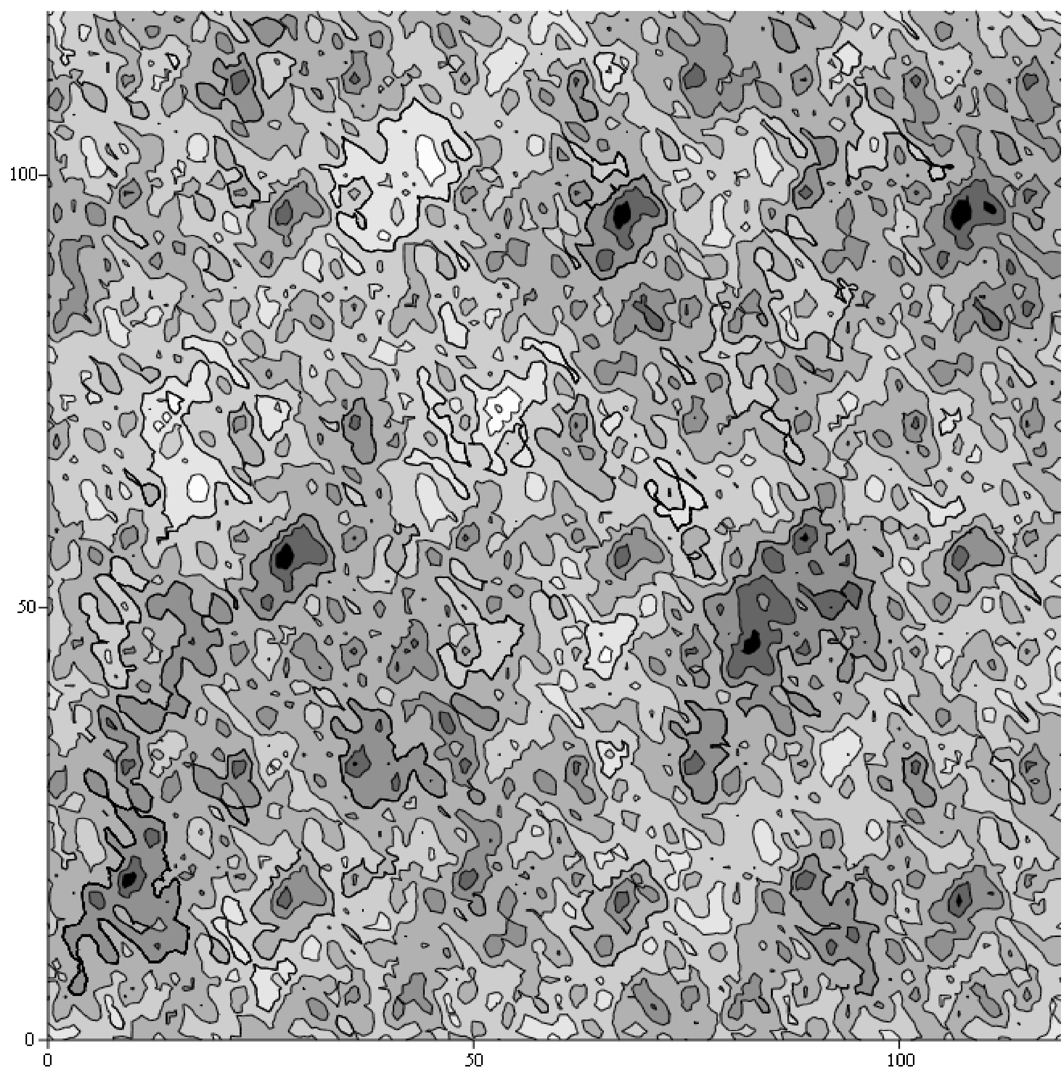

Fig. 6.7: *Smooth pre fractal medium*

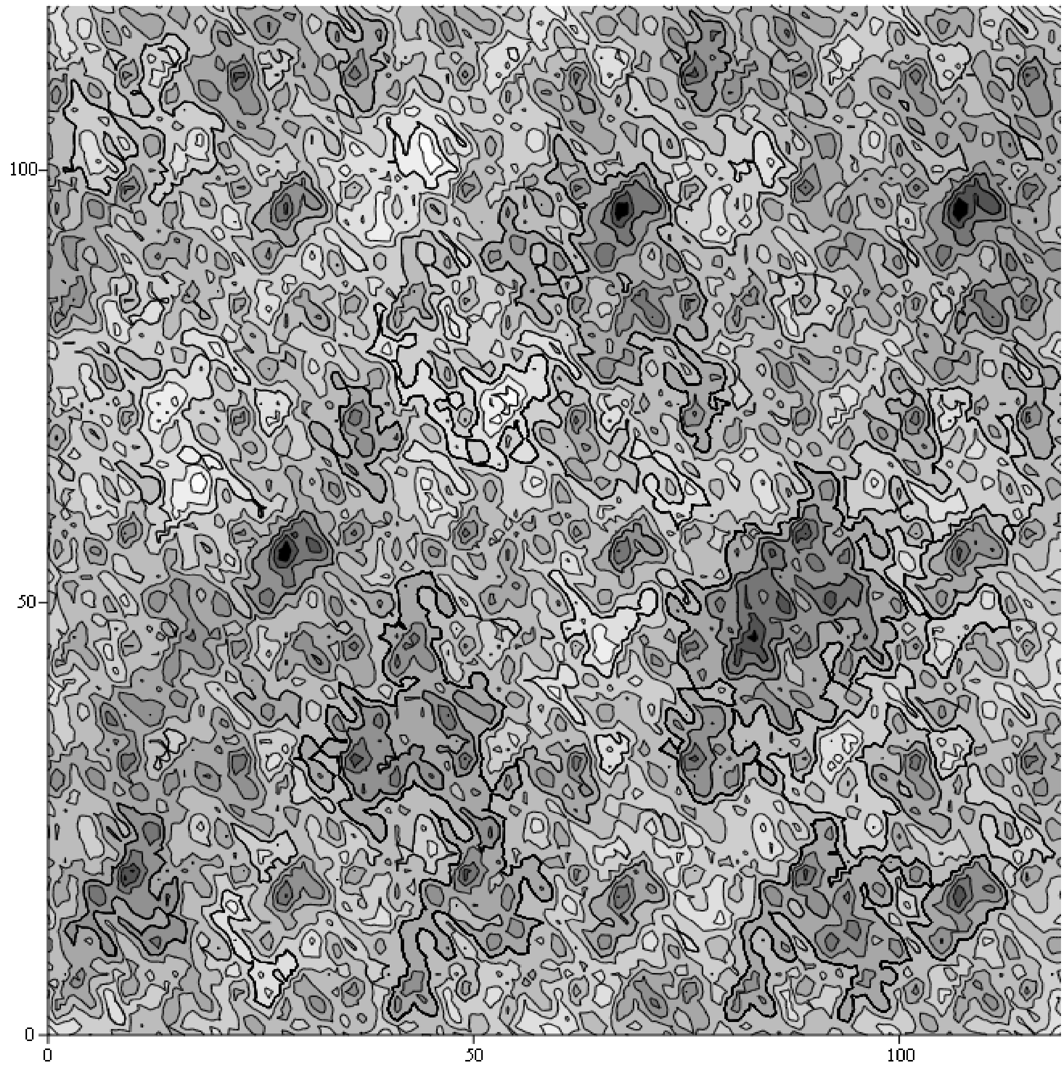

Fig. 6.8: *Smooth pre fractal medium*

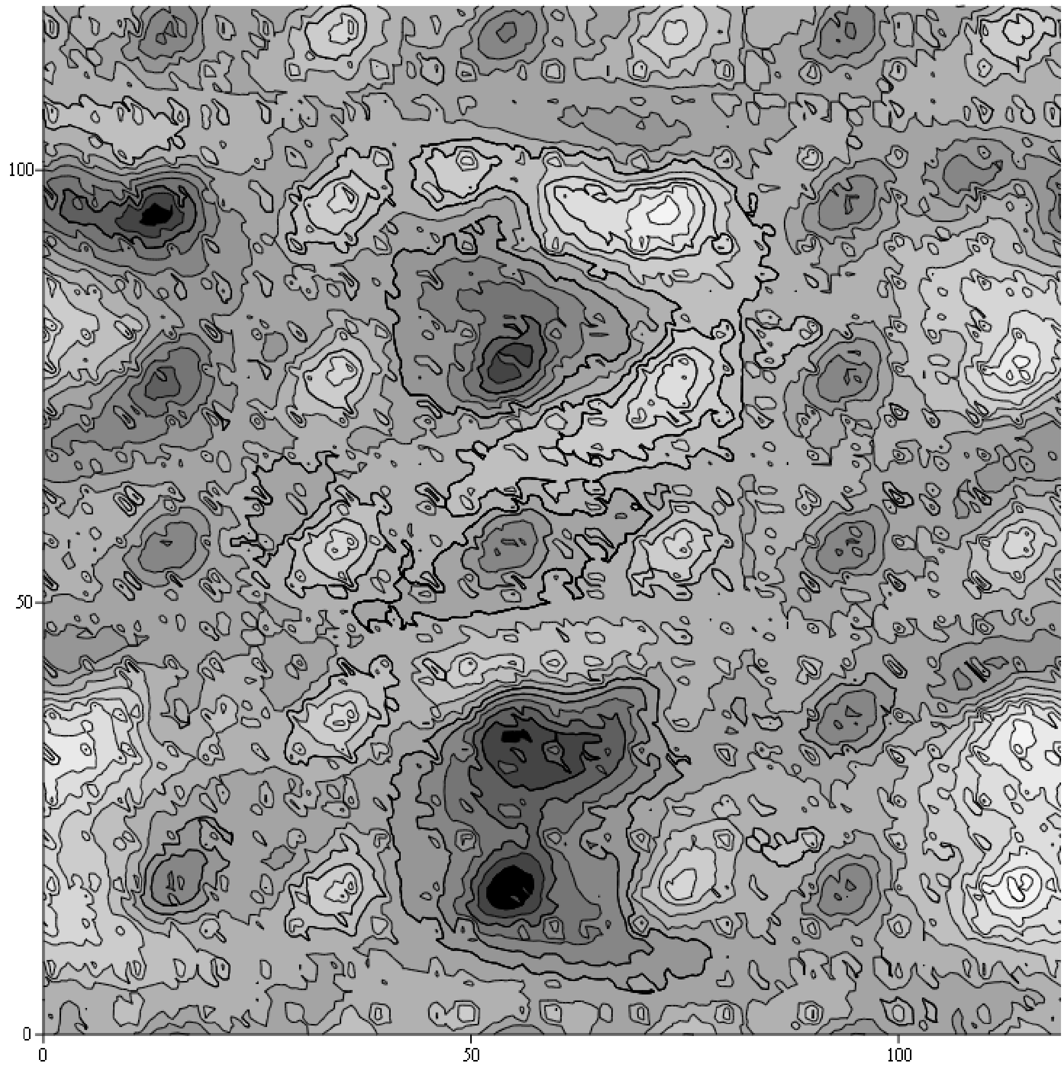

Fig. 6.9: Smooth pre fractal medium

7. SUPER-DIFFUSIVE MODEL

7.1 Infinitely homogenized eddy diffusion

7.1.1 The strategy

The strategy to analyze Infinitely Homogenized Eddy Diffusion is the same as for an Infinitely Homogenized Potential Diffusion, in the sense that the multi-scale analysis is done through effective scales, perturbation scales and drift scales. Nevertheless there are some important differences. The first and must important one is that for an IHED the multi-scale effective diffusivities $D(\Gamma^{0,n})$ must be shown to diverge towards infinity with geometric speed in order to obtain a clear super-diffusive process in fact the divergence towards infinity of the multi-scale effective diffusivities is the key to obtain a super-diffusive process.
The second one is the apparition of the diffusivity powers γ_n, in fact by the variational formulation of $D(\Gamma^{0,n})$ one can show that

$$\begin{aligned} D(\Gamma^{0,n}) &\leq \int_{T_1^d} \left|1 + 2(\Gamma^{0,n}(R_n x) - \int_{T_1^d} \Gamma^{0,n}(R_n y)\,dy)\right|^2 dx \\ &\leq (1 + 2K_0 \sum_{k=0}^{n} \gamma_n)^2 \end{aligned} \tag{7.1}$$

this inequality suggests that to obtain a geometric speed of divergence of the effective diffusivities $D(\Gamma^{0,n})$ on must have at least the same speed of divergence for the diffusivity powers γ_n; and fact for $\gamma_n = 1$, the speed of divergence of $D(\Gamma^{0,n})$ when the scales are well separated is linear (which suggests a weak form of super-diffusivity). In fact those parameters γ_n have a clear signification when the flow is compared to a real turbulent flow (will be given in a sequel section).
The third one is the fact that the generator of the diffusion is not symmetric and this has deep consequences on the method used to control the multi-scaled effective diffusivities and the influence of the perturbation scales.

Overlapping ratios For an IHPD sub diffusivity has been proved for $\rho_{\min} > \rho_0$ in the separating scales region and it has been show that when $\rho_{\min}$ is smaller than ρ_0, the ratios belongs to the overlapping region and both behaviors (normal and anomalous) are possible for the process (there are regions of anomalous behavior surrounded by regions of normal behavior in real line associated to the ratios).
Can the same phenomenon happen for an IHED?
The answer is yes, assume the IHED self-similar for simplicity ($R_n = \rho^n$, $\gamma_n = \gamma^n$, $\Gamma_n = \Gamma_0$) and choose

$$\Gamma_0 = H(x) - \gamma^p H^p(a^{-p}x) \tag{7.2}$$

where H is a skew symmetric matrix and $a \in \mathbb{N}/\{0,1\}$, $p \in \mathbb{N}^*$. And observe that for $\rho = a$, $\Gamma = \sum_{n=0}^{\infty} \Gamma_n$ is a bounded matrix which lead to normal diffusion by the Norris-Aronson estimates.
Thus as for an IHPD without a priori knowledge on the geometry of the eddies, results giving the super-diffusive nature of an IHED can exist only for $\rho_{\min} \geq \rho_0$, below this boundary both behaviors are possible and to determine whether the diffusion is super-diffusive or not, on needs more informations than the knowledge of the parameters $K_0, K_1, \lambda_{\max}, \lambda_{\min}, \gamma_n$.

7.2 The shear flow model

7.2.1 The results

Those results are proven in the chapter 11.

7.2.1.i Multi-scale effective diffusivity

Consider a IHSFD, the following theorem corresponds to the theorem 11.2.1.

Theorem 7.2.1. *assume $\gamma_{min} > 1$ and*

$$\epsilon = \frac{2^{\frac{3}{2}}}{\rho_{\min}} \frac{K_1}{\gamma_{\min} - 1} < 1 \tag{7.3}$$

then for all $p \in \mathbb{N}$

$$D(\Gamma^{0,p}) = \begin{pmatrix} 1 & 0 \\ 0 & D(\Gamma^{0,p})_{22} \end{pmatrix} \tag{7.4}$$

with

$$1 + 4(1-\epsilon)\sum_{k=0}^{p} \gamma_k^2 \leq D(\Gamma^{0,p})_{22} \leq 1 + 4(1+\epsilon)\sum_{k=0}^{p} \gamma_k^2 \tag{7.5}$$

Observe that the control on the multi-scale effective diffusivities of a IHSFD is sharper that on an IHPD, this fact is created by the introduction and divergence of the diffusivity powers γ_n.

7.2.1.ii Mean squared displacement

Bounded ratios The following theorem corresponds to the theorem 11.3.1.

Theorem 7.2.2. *assume $\gamma_{min} > 1$, $\gamma_{\max}, \rho_{\max} < \infty$,
$\rho_{min} > \rho_0(\gamma_{min}, \gamma_{max}, K_0, K_1)$ and $t > t_0(\gamma_{min}, \gamma_{max}, R_1, K_0, K_1)$ then*

$$\mathbb{E}_0[|y_t.e_2|^2] = t^{1+\nu(t)} \tag{7.6}$$

with

$$\nu(t) \leq \frac{\ln \gamma_{\max}}{\ln \rho_{min} + \ln \frac{\gamma_{\min}}{\gamma_{\max}}} + \frac{C_2}{\ln t} \tag{7.7}$$

$$\nu(t) \geq \frac{\ln \gamma_{\min}}{\ln \rho_{max} + \ln \frac{\gamma_{\max}}{\gamma_{\min}}} - \frac{C_1}{\ln t} \tag{7.8}$$

Where the constants C_1 and C_2 depends on $\rho_{\min}, \gamma_{\min}, \gamma_{\max}, \rho_{\max}, K_1, K_2$

Observe that the control on the mean squared displacement of a IHSFD is sharper that on an IHPD, this fact is created by the introduction and divergence of the diffusivity powers γ_n. Indeed for each t, the behavior of $\mathbb{E}[y_t^2]$ is dominated by a single scale.

Fast separating ratios The following theorem corresponds to the theorem 11.3.2.

Theorem 7.2.3. *assume* $\gamma_p = \gamma^p$ *and* $R_p = R_{p-1}[\frac{\rho^{p^\alpha}}{R_{p-1}}]$ *with* $\gamma, \rho > 1$ *and* $\alpha \geq 1$ *Then for* $t > t_0(\gamma_2, R_2, K_0, K_1)$

$$C_1 t \gamma^{\beta(t)} \leq \mathbb{E}_0[|y_t.e_2|^2] \leq C_2 t \gamma^{\beta(t)} \tag{7.9}$$

with

$$\beta(t) = 2(\frac{1}{2\ln\rho})^{\frac{1}{\alpha}}(\ln t)^{\frac{1}{\alpha}} \tag{7.10}$$

Where the constants C_1 *and* C_2 *depends on* $\rho, \gamma, \alpha, K_1, K_2$

Remark 7.2.1. Note that this theorem shows how the diffusion becomes more and more super-diffusive as $\alpha \downarrow 1$: the ratio between scales tends to be constant.

7.3 Links with turbulence

7.3.1 Turbulent Convection

It is clear that an infinitely homogenized eddy diffusion is a model of diffusion-convection in a incompressible turbulent flow. One knows that a turbulent flow is characterized by a large number of scales of eddies and convection rolls, the purpose of this model is to show that the presence of multi-scale eddies generates the anomalous behavior of the diffusion between appropriate time scales (or length scales) corresponding to the minimal length of the eddies and their maximal length.
Of course in a real turbulent flow each Γ^n should be time dependent and the periodicity should be replaced by a time and spacial ergodicity conditions. The study of those real turbulent flows is postponed to a sequel work.

7.3.2 Physical interpretation

Observe that this model of IHED has an interesting interpretation in the framework of fully developed turbulence (Read the subsection 4.6.4 prior to reading this one).
Here the mean velocity of the fluid is 0.
The parameters $\gamma_n \|\nabla\Gamma^n\|_\infty$ represents the amplitude of the pulsations of size R_n. Since for all scales $\|\nabla\Gamma^n\|_\infty \leq K_1$, the fact that γ_n is increasing reflects the fact that the amplitude of the pulsations increase with the scale.
The energy dissipated per unit time and unit volume in the eddies of scale n is of order of

$$\epsilon_n \propto \frac{\gamma_n^2}{R_n^4} K_2^2 \tag{7.11}$$

So to say that the energy is dissipated mainly in the small eddies is equivalent to say that $\gamma_n / R_n^2 \to 0$ as $n \to \infty$ or if $R_n = \rho^n$ and $\gamma_n = \rho^{\alpha n}$, this equivalent to say that $\alpha < 2$.
The Kolmogorov-Obukhov's law is equivalent to say that $K_2 < \infty$ for all n, $\nabla\Gamma_n(0) = 0$ and

$$\gamma_n \propto R_n^{\frac{4}{3}} \tag{7.12}$$

or if $R_n = \rho^n$ and $\gamma_n = \rho^{\alpha n}$, this equivalent to say that $\alpha = \frac{4}{3}$.

7.3.3 Links with Richardson law

Richardson's empirical law $D_\lambda \sim \lambda^{\frac{4}{3}}$ says that

$$\frac{\ln D(\Gamma^{0,n-1})}{\ln R_n} \to \frac{4}{3} \tag{7.13}$$

If $R_n \sim \rho^n$, $\gamma_n \sim \gamma^n$ and $D(\Gamma^{0,n-1}) \sim \gamma^{\beta n}$, it would say that $\gamma^\beta = \rho^{\frac{4}{3}}$. One has to be careful in the comparison with the shear flow model which is strongly anisotropic (the effective medium scale is fixed by the law of a standard Brownian motion and not the accelerated diffusion, contrary to Kolmogorov turbulence).
Now for the compatibility with the Kolmogorov law, one must have $\gamma^\beta = \gamma$, this would mean that $D(\Gamma^{0,n-1}) \sim \gamma^n$ this is quite interesting.

7.3.4 Heuristic consideration and Ansatz on the apparition of turbulence

The Navier Stokes equations

$$\frac{\partial}{\partial t} u_i = \nu \Delta u_i - u.\nabla u_i - \frac{1}{\rho}\frac{\partial}{\partial x_i} p + f_i(x,t) \tag{7.14}$$

corresponds to the convective diffusion of the velocities of the flow. Imagine the initial flow being laminar at the instant t_0 and introduce some small fluctuations or perturbations by an exterior source (ultrasonics for instance). Now those fluctuations of the flow enhance the diffusion, nevertheless although the diffusion is enhanced the smoothing term of the operator $\nu\Delta u_i$ remains unchanged, thus the fluctuations are also enhanced and observe that as those fluctuations spread over more and more scales the convective diffusion of the flow is more and more enhanced by the super-diffusive transport phenomenon enhancing the fluctuations and spreading them over more and more scales (this is a self maintained phenomenon).
Why does turbulence appear only for a sufficiently high Reynolds number? What is the link with this Ansatz? The link is quite simple. In fact the initial perturbation is increased by the local enhancement of the diffusion due to that perturbation but decreased by the smoothing term $\nu\Delta u_i$, there is a competition between those two phenomenon and one can imagine that when the kinematic viscosity is high the smoothing term wins and when it is low then the enhancing term wins. In fact the Reynolds number in the framework of an IHED corresponds to the multi-scale effective diffusivities: Assume that the flow has only n scales $\Gamma_0, \dots, \Gamma_{n-1}$ then the proper Reynolds number characterizing the flow is

$$Re = D(\Gamma^{0,n-1}) \tag{7.15}$$

7.4 Some pictures of multi-scale flow associated with an IHED

The figures given in this section illustrate the contour lines of the stream function $H(x,y)$ associated to the stream matrix

$$\Gamma^n(x_1,x_2) = \begin{pmatrix} 0 & H(x,y) \\ H(x,y) & 0 \end{pmatrix} \tag{7.16}$$

with

$$H(x,y) = \sum_{k=0}^{2} \rho^\alpha K(\frac{x}{\rho^k}, \frac{y}{\rho^k}) \tag{7.17}$$

The figure 7.1 illustrate (9 contour lines) the case $\alpha = 0.5$, $\rho = 3$ with

$$\begin{aligned} K(x,y) =& \cos(x + \pi\sin(y))\sin(y + \pi\cos(x+1)) \\ &+ 0.4\sin(\pi\cos(x) + \pi\cos(y) + 2) + 0.4\cos(\pi(\cos(x))^2 + y + 0.5) \end{aligned} \tag{7.18}$$

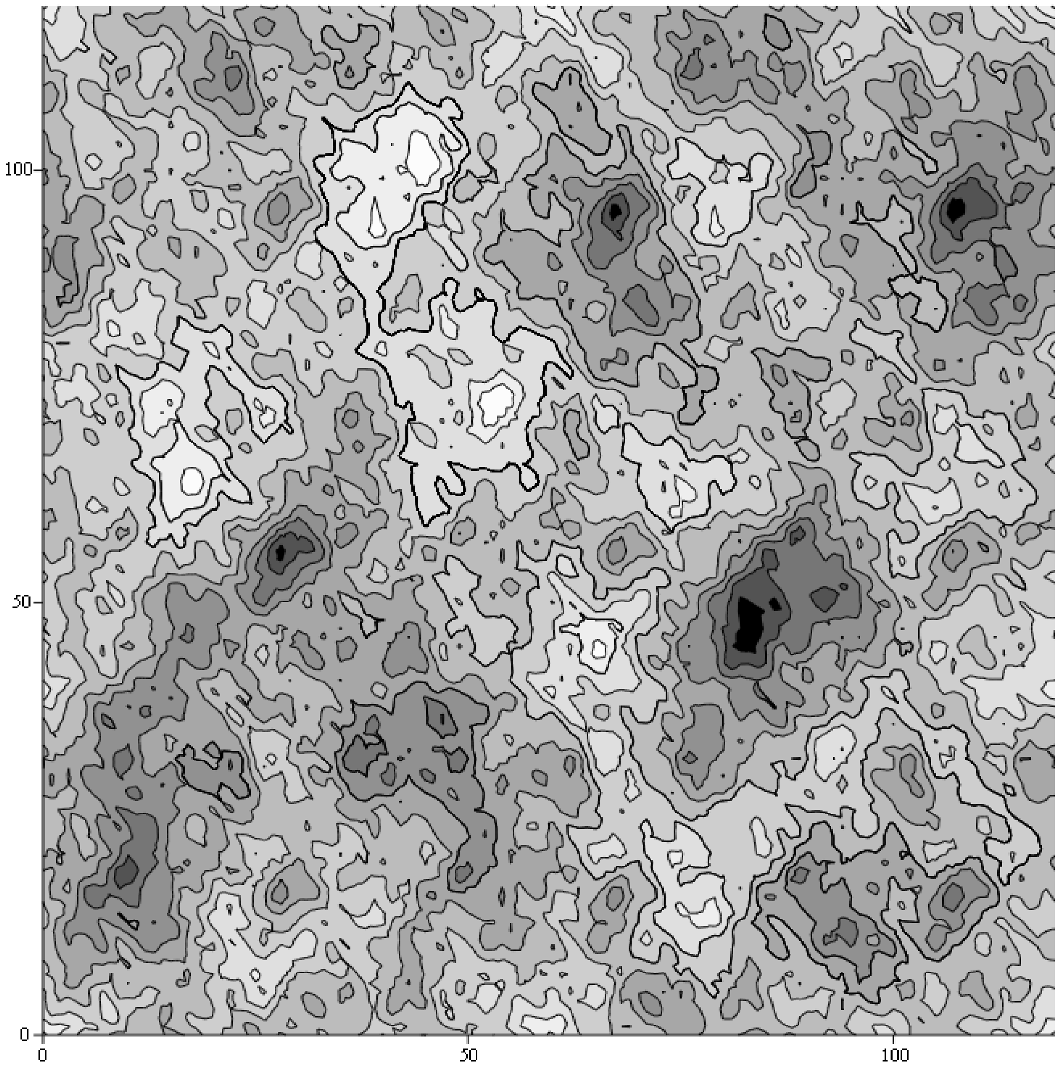

Fig. 7.1: *Multiscale flow* , $\alpha = 0.5$, $\rho = 3$

The figure 7.2 (9 contour lines) illustrate the same case but with case $\alpha = 4/3$, $\rho = 3$.
The figure 7.3 (11 contour lines) illustrate the case

$$K(x, y) = \cos(x + \pi \sin(y) + 1)^2 \sin(\pi \cos(x) - 2y + 2) \cos(\pi \sin(x) + y) \tag{7.19}$$

with case $\alpha = 0.5$, $\rho = 4$.
The figure 7.3 (11 contour lines) illustrate the same case but with $\alpha = 4/3$

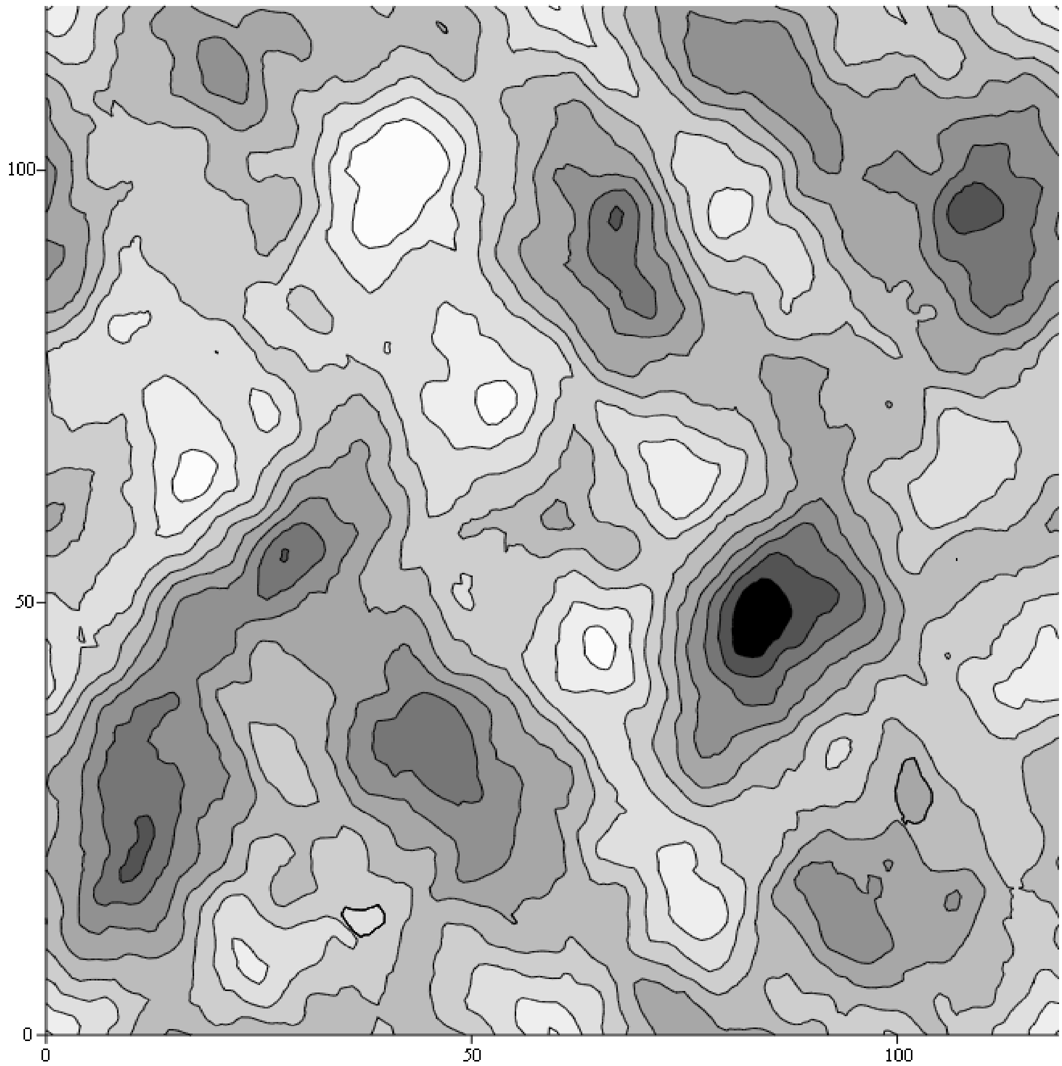

Fig. 7.2: *Multiscale flow* , $\alpha = 4/3$, $\rho = 3$

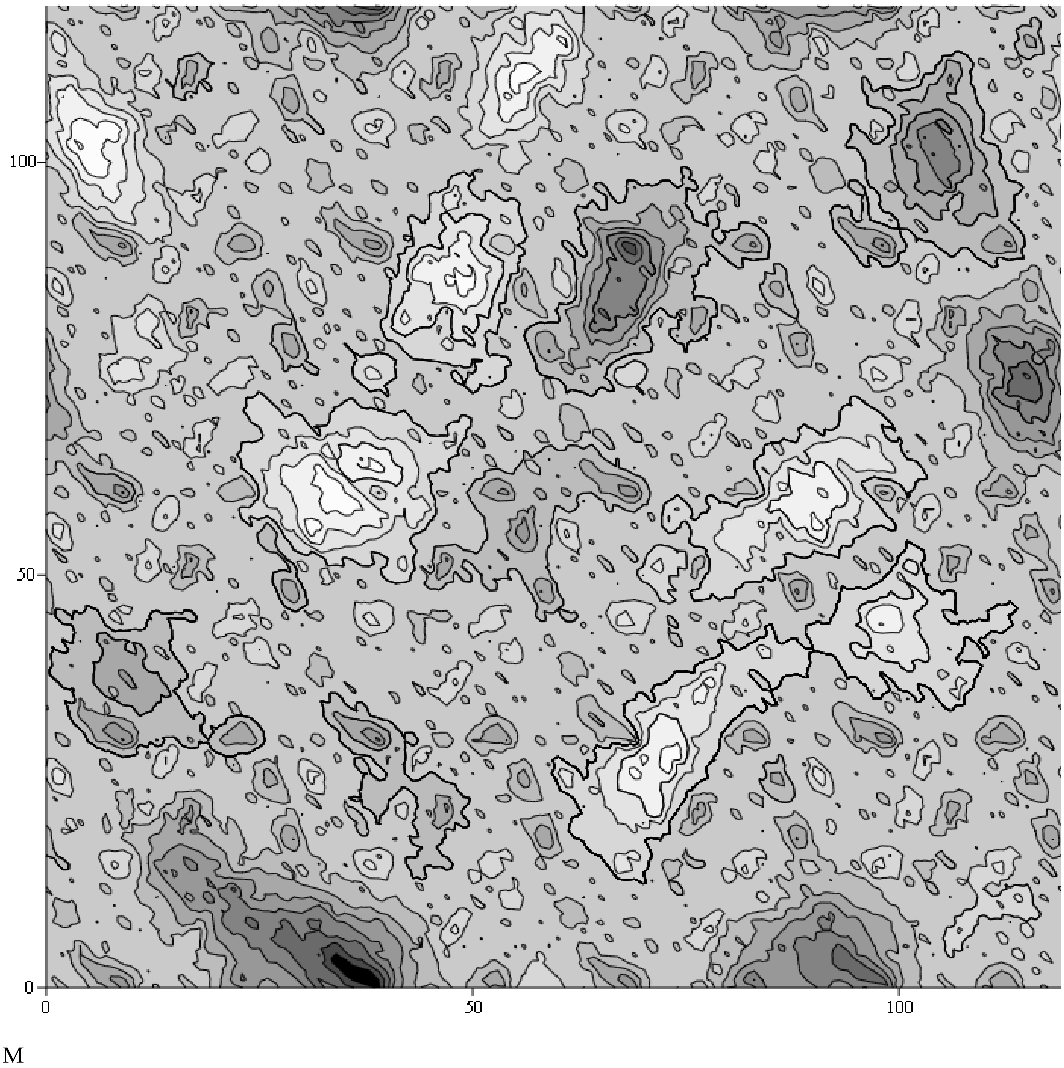

Fig. 7.3: *Multiscale flow* , $\alpha = 0.5$, $\rho = 4$

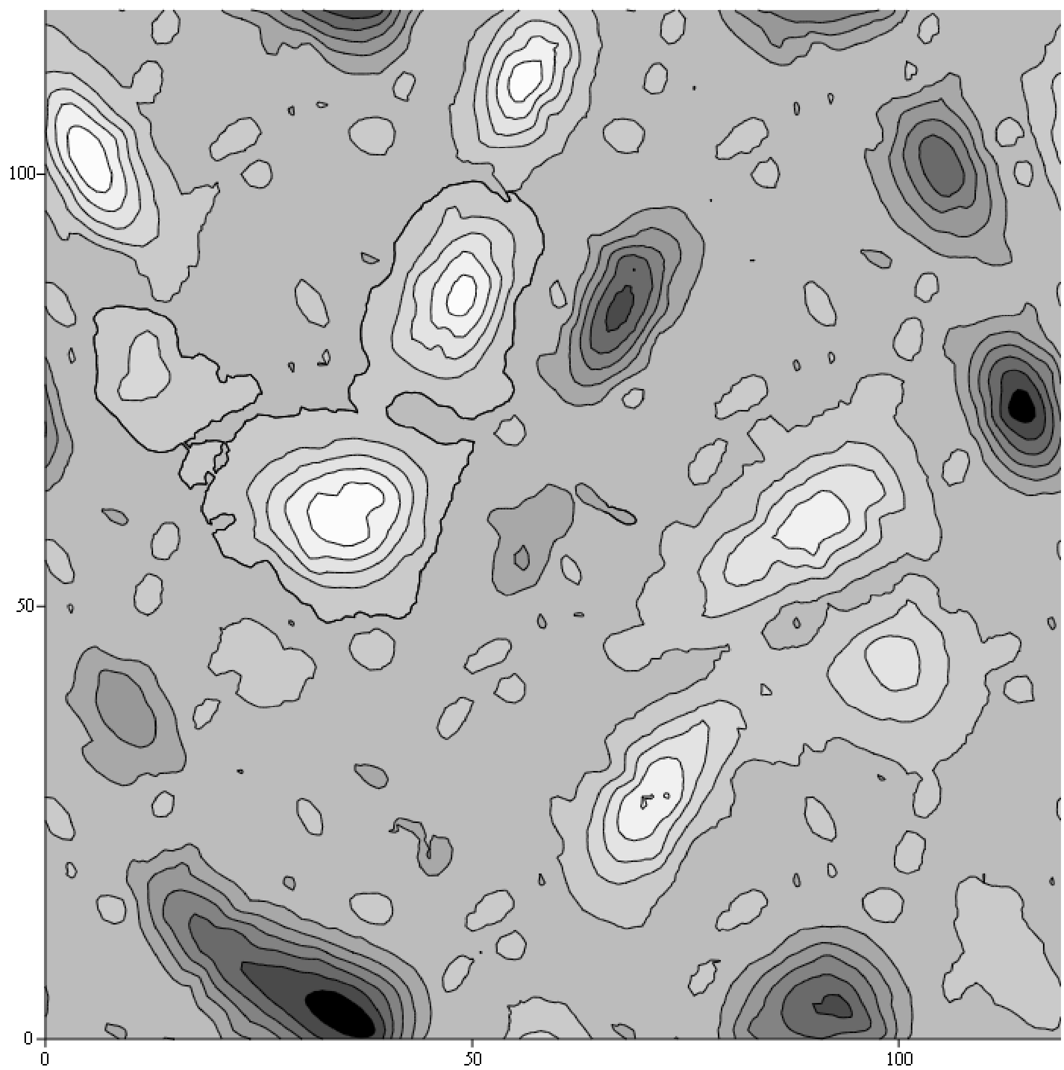

Fig. 7.4: *Multiscale flow* , $\alpha = 4/3$, $\rho = 4$

Part IV

PROOFS AND TOOLS

8. SUB DIFFUSIVITY IN DIMENSION ONE

8.1 Exact Formulas

8.1.1 Periodic potential

Let $U \in C^\infty(T_1^1)$,

Effective Diffusivity

$$D(U) = \frac{1}{\int_0^1 e^{-2U}\,dx \int_0^1 e^{2U}\,dx} \tag{8.1}$$

Cell problem For $l \in \mathbb{S}^1$, the solution of the cell problem 5.4 is equal to

$$\begin{aligned} \chi_l(x) &= l\big(x - m_{-U}([0,x])\big) \\ &= l\big(x - \frac{\int_0^x e^{2U}\,dy}{\int_0^1 e^{2U}\,dy}\big) \end{aligned} \tag{8.2}$$

and the associated quasi-linear harmonic function is

$$\begin{aligned} F_l(x) &= l.x - \chi_l(x) = m_{-U}([0,x]) \\ &= l\frac{\int_0^x e^{2U}\,dy}{\int_0^1 e^{2U}\,dy} \end{aligned} \tag{8.3}$$

Ergodicity problem For $l \in \mathbb{S}^d$, the solution of the ergodicity problem 5.10 is equal to

$$\begin{aligned} \phi_l(x) =& 2\int_0^x \Big(m_{-U}\big([0,z]\big) - m_U\big([0,z]\big)\Big)\, m_{-U}(dz) \\ &- 2m_{-U}\big([0,x]\big)\int_0^1 \Big(m_{-U}\big([0,z]\big) - m_U\big([0,z]\big)\Big)\, m_{-U}(dz) \end{aligned} \tag{8.4}$$

Exit time Write

$$\psi(x) = 2\int_0^x e^{2U(y)}\int_0^y e^{-2U(z)}\,dy\,dz \tag{8.5}$$

Since $L_U\psi = 1$ it follows by the Ito formula that

$$\mathbb{E}_0[\psi(y_{\tau(0,1)})] = \mathbb{E}_0[\tau(0,1)] \tag{8.6}$$

Observe that since F_{e_1} is harmonic with respect to L_U it follows that,

$$\mathbb{P}_0[y_{\tau(0,1)} = 1] = \mathbb{P}_0[y_{\tau(0,1)} = -1] = 1/2$$

and

$$\mathbb{E}_0[\tau(0,1)] = \int_0^1 e^{2U(y)}\int_0^y e^{-2U(z)}\,dy\,dz + \int_0^{-1} e^{2U(y)}\int_0^y e^{-2U(z)}\,dy\,dz \tag{8.7}$$

and by periodicity

$$\mathbb{E}_0[\tau(0,1)] = \int_0^1 e^{2U(y)}dy\int_0^1 e^{-2U(y)}dy = \frac{1}{D(U)} \tag{8.8}$$

8.2 Effective diffusivities

8.2.1 Self-similar case

Let y_t be a self-similar infinitely homogenized potential diffusion with ratio between scales $R \in \mathbb{N}/\{0,1\}$ and periodic potential $U \in C^\infty(T_1^1)$.
By the theorem C.1.1

Theorem 8.2.1.

$$\lim_{n\to\infty} -\frac{1}{n} \ln \left(D(V^{n-1}) \right) = \mathcal{P}_R(2U) + \mathcal{P}_R(-2U) \tag{8.9}$$

where $\mathcal{P}_R$ is the topological pressure associated to the shift s_R.
And the theorem $C.1.2$ says that $\mathcal{P}_R(2U)+\mathcal{P}_R(-2U) > 0$ if and only U does not belong to the closed subspace of $\mathcal{C}(T_1^d)$ generated the elements $T(x) - T(R^k x)$ with $T \in C(T_1^d)$ and $k \in \mathbb{N}$. Moreover it is easy to see that as $R \to \infty$

$$\mathcal{P}_R(2U) + \mathcal{P}_R(-2U) \to \ln \int_{T_1^d} e^{2U} dx + \ln \int_{T_1^d} e^{-2U} dx \tag{8.10}$$

which is strictly positive if U is not constant.

8.2.2 General case

Theorem 8.2.2. *For $\rho_{\min} > 2K_1 e^{2K_0}$*

$$\prod_{k=0}^{n-1} \frac{1}{\int_{T_1^1} e^{2U_k(x)}dx \int_{T_1^1} e^{-2U_k(x)}dx} \frac{1}{(1+\frac{2K_1 e^{2K_0}}{r_k})^2} \leq D(V^{n-1}) \tag{8.11}$$

and

$$D(V^{n-1}) \leq \prod_{k=0}^{n-1} \frac{1}{\int_{T_1^1} e^{2U_k(x)}dx \int_{T_1^1} e^{-2U_k(x)}dx} \frac{1}{(1-\frac{2K_1 e^{2K_0}}{r_k})^2} \tag{8.12}$$

Proof. Direct consequence of the corollary C.1.1 and a simple induction. □

Corollary 8.2.1. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$\lim_{n\to\infty} \frac{1}{n} \ln D(V^{n-1}) = \frac{1}{\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx} \tag{8.13}$$

Proof. Direct consequence of the theorem 8.2.2 and a simple induction. □

8.3 Exit times

Let y_t be an infinitely homogenized potential diffusion.

Let $r > 1$, write

$$n_{ef}(r) = \sup\{n \geq 0 \, : \, R_n \leq r\} \tag{8.14}$$

$n_{ef}(r)$ is the effective scale (aggregation of the scales $0,\dots,n_{ef}$) corresponding to the length r.

Theorem 8.3.1.

$$\frac{r^2}{D(V_0^{n_{ef}})}\frac{1}{C_\tau} \le \mathbb{E}_0[\tau(0,r)] \le \frac{r^2}{D(V_0^{n_{ef}})}C_\tau \tag{8.15}$$

with

$$C_\tau = 4e^{4(K_0+K_1\frac{\rho_{\min}}{\rho_{\min}-1})} \tag{8.16}$$

Proof. Observe that for $x \in B(0,r)$,

$$|V^\infty_{n_{ef}+1}| \le K_0 + K_1\frac{\rho_{\min}}{\rho_{\min}-1} \tag{8.17}$$

Indeed, $U_{n_{ef}+1}$ acts as a perturbation scale and its norm is bounded by K_0 and $V^\infty_{n_{ef}+2}$ acts as a drift scale and is bounded by $\sum_{p=n_{ef}+2}^\infty \|\nabla U_k\|_\infty r/R_k$ on $B(0,r)$.
Write

$$p_{ef} = \sup\{p \ge 1 \,:\, pR_{n_{ef}} \le r\} \tag{8.18}$$

p_{ef} corresponds to the maximum number of periods of the scale n_{ef} included in the segment $[0,r]$
Observe that

$$\mathbb{E}_0[\tau(0,r)] \ge \mathbb{E}_0[\tau(0,p_{ef}R_{n_{ef}})] \tag{8.19}$$

and

$$\mathbb{E}_0[\tau(0,r)] \le \mathbb{E}_0[\tau(0,(p_{ef}+1)R_{n_{ef}})] \tag{8.20}$$

And observe that (write y_t^k the diffusion associated to the potential V_0^k and $\mathbb{E}^k$ its expectation) by the equation 8.8 and the invariance by scaling of the effective diffusivity

$$\mathbb{E}_0^{n_{ef}}[\tau(0,pR_{n_{ef}})] = \frac{p^2R^2_{n_{ef}}}{D(V_0^{n_{ef}})} \tag{8.21}$$

It follows by the corollary 13.5.2 that

$$\mathbb{E}_0[\tau(0,r)] \ge \frac{r^2}{D(V_0^{n_{ef}})}\frac{1}{4}e^{-4(K_0+K_1\frac{\rho_{\min}}{\rho_{\min}-1})} \tag{8.22}$$

and

$$\mathbb{E}_0[\tau(0,r)] \le \frac{r^2}{D(V_0^{n_{ef}})}4e^{4(K_0+K_1\frac{\rho_{\min}}{\rho_{\min}-1})} \tag{8.23}$$

□

Corollary 8.3.1.

$$r^{2+\nu}\frac{1}{C_\tau} \le \mathbb{E}_0[\tau(0,r)] \le r^{2+\nu}C_\tau \tag{8.24}$$

with

$$\nu(r) = -\frac{1}{\ln r}\ln\left[D(V_0^{n_{ef}(r)})\right] \tag{8.25}$$

and

$$C_\tau = 4e^{4(K_0+K_1\frac{\rho_{\min}}{\rho_{\min}-1})} \tag{8.26}$$

Proof. Direct consequence of the previous theorem. □

Self-similar case

Corollary 8.3.2. *Let y_t be a self-similar infinitely homogenized potential diffusion. Then*

$$\mathbb{E}_0[\tau(0,r)] = r^{2+\nu(r)} \tag{8.27}$$

with

$$\nu(r) = \frac{\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)}{\ln \rho} + \epsilon(r) \tag{8.28}$$

with $\epsilon(r) \to 0$ as $r \to \infty$.

Proof. Direct consequence of the corollary 8.3.1 and the theorem 8.2.1. □

General case

Corollary 8.3.3. *Let y_t be an infinitely homogenized potential diffusion such that, $\rho_{\min} > 4K_1 e^{2K_0}$, $\rho_{\max} < \infty$ and $\lambda_{\max} < 1$. Then*

$$C_1 r^{2+\nu(r)} \leq \mathbb{E}_0[\tau(0,r)] \leq C_2 r^{2+\nu(r)} \tag{8.29}$$

where C_1, C_2 depends only on K_0, K_1 and $\rho_{\min}$ and

$$0 < -\frac{\ln \lambda_{\max}}{\ln \rho_{\max}} - \frac{8K_1 e^{2K_0}}{\rho_{\min} \ln \rho_{\max}} \leq \nu(r) \leq -\frac{\ln \lambda_{\min}}{\ln \rho_{\min}} + \frac{4K_1 e^{2K_0}}{\rho_{\min} \ln \rho_{\min}} \tag{8.30}$$

Remark 8.3.1. Observe that

$$\lambda_{\max} = \sup_{k \in \mathbb{N}} \frac{1}{\int_{T_1^1} e^{2U_k(x)} dx \int_{T_1^1} e^{-2U_k(x)} dx} \tag{8.31}$$

and the diffusion shows a clear anomalous behavior as soon as

$$\rho_{\min} > -8 \frac{K_1 e^{2K_0}}{\ln \lambda_{\max}} \tag{8.32}$$

Proof. Direct consequence of the corollary 8.3.1 and the theorem 8.2.2. □

Fast separation of scales case

Corollary 8.3.4. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1} [\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$C_1 r^{2+\nu(r)} \leq \mathbb{E}_0[\tau(0,r)] \leq C_2 r^{2+\nu(r)} \tag{8.33}$$

where C_1, C_2 depends only on K_0, K_1, ρ, α and

$$\nu(r) = \frac{1}{(\ln r)^{1-\frac{1}{\alpha}}} \frac{\ln \left(\int_{T_1^1} e^{2U(x)} dx \int_{T_1^1} e^{-2U(x)} dx \right)}{(\ln \rho)^{\frac{1}{\alpha}}} \tag{8.34}$$

Remark 8.3.2. Observe that this corollary shows how the behavior of the diffusion passes from weakly anomalous one to strongly anomalous as $\alpha \downarrow 1$ and the ratio between scales tends to be constant.

Proof. Direct consequence of the corollary 8.3.1 and the theorem 8.2.2. □

8.4 Massage of the harmonic functions

8.4.1 Linearity versus norm of the cell problem

Let $U, P \in C^\infty(T_1^1)$ be smooth periodic potentials. Write $V = S_R U + P$, with $R \in \mathbb{N}^*$. Write χ^V, χ^P the solutions of the cell problems associated to V and P.

Lemma 8.4.1.

$$\|\chi^V - \chi^P\|_\infty \leq 2\frac{e^{2\,\mathrm{Osc}(P)}}{R}[1 + 4\|\nabla P\|_\infty] \tag{8.35}$$

Proof. This follows from the explicit formulas for χ an application of the corollary C.1.1 and a straightforward computation. □

8.4.2 Perturbed ergodicity

Let $U, P \in C^\infty(T_1^1)$ and $T \in C^\infty(\mathbb{R}^1)$ a smooth potential with bounded gradient. Write for $R \in \mathbb{N}/\{0,1\}$, $V = S_R U + P + T$ and y_t the diffusion associated to the generator L_V. Write $W = S_R U + P$ and χ^W the solution of the cell problem associated to L_W.
Write for $\zeta > 0$

$$\phi_\zeta = 2\int_0^x \frac{e^{2V(y)}}{\int_0^1 e^{2W(y)}dy}\Big[\int_0^y \frac{e^{2(P-T)(z)}}{\int_0^1 e^{2P(z)}dz}dz(1+I) - \zeta\int_0^y \frac{e^{-2(P+T)(z)}}{\int_0^1 e^{-2P(z)}dz}dz\Big]dy \tag{8.36}$$

Lemma 8.4.2. *For $l \in \mathbb{S}^1$, and $\zeta > 0$*

$$L_V\phi_\zeta = |l - \chi_l^W|^2 - \zeta D(W) \tag{8.37}$$

Moreover if

$$R > 16e^{4\,\mathrm{Osc}(P)}(\|\nabla P\|_\infty + \|\nabla T\|_\infty)e^{2\|\nabla T\|_\infty/R}$$

then

1. *for $\zeta = 6e^{4\,\mathrm{Osc}(P)}$*

$$\sup_{\mathbb{R}} \phi_\zeta \leq 900\frac{e^{10\,\mathrm{Osc}(P)}}{R^2}e^{4\|\nabla T\|_\infty/R} \tag{8.38}$$

2. *for $\zeta = \frac{e^{-4\,\mathrm{Osc}(P)}}{6}$*

$$\inf \phi_\zeta \geq -100\frac{e^{10\,\mathrm{Osc}(P)}}{R^2}e^{4\|\nabla T\|_\infty/R} \tag{8.39}$$

Proof. Observe that by the corollary C.1.3

$$\Big|\int_0^x \frac{e^{2(W+T)(y)}}{\int_0^1 e^{2W(y)}dy}dy - \int_0^x \frac{e^{2(P+T)(y)}}{\int_0^1 e^{2P(y)}dy}dy\Big| \leq \frac{I}{R}$$

with

$$\begin{aligned} I =& 2e^{2\,\mathrm{Osc}(P)}\Big[e^{2\frac{\|\nabla T\|_\infty}{R}}\big(e^{2T(x)} + 2(\|\nabla P\|_\infty + \|\nabla T\|_\infty)\Big|\int_0^x e^{2T(y)}dy\Big|\big) \\ &+ 2\|\nabla P\|_\infty \frac{\int_0^x e^{2(P+T)(y)}dy}{\int_0^1 e^{2P(y)}dy}\Big] \end{aligned}$$

It follows that (with R is chosen so that $I_1 < R$)

$$\begin{aligned}\phi_\zeta \leq & 2\int_0^x \frac{e^{2V(y)}}{\int_0^1 e^{2W(y)}dy}\Big[\int_0^y \frac{e^{2(P-T)(z)}}{\int_0^1 e^{2P(z)}dz}dz(1+\frac{I_1}{R}) + \frac{I_2}{R}(1+\zeta)e^{-2T(y)} \\ & -\zeta(1-\frac{I_1}{R})\int_0^y \frac{e^{-2(P+T)(z)}}{\int_0^1 e^{-2P(z)}dz}dz\Big]dy\end{aligned}$$

with

$$I_1 = 8(\|\nabla P\|_\infty + \|\nabla T\|_\infty)e^{4\,\mathrm{Osc}(P)+2\|\nabla T\|_\infty/R}$$

and

$$I_2 = 2e^{2\,\mathrm{Osc}(P)2\|\nabla T\|_\infty/R}$$

This leads to

$$\phi_\zeta \leq 2\int_0^x \frac{e^{2V(y)}}{\int_0^1 e^{2W(y)}dy}\Big[\frac{I_2}{R}(1+\zeta)e^{-2T(y)} - I_3\int_0^y e^{-2T(z)}dz\Big]dy$$

with

$$I_3 = \zeta(1-\frac{I_1}{R})e^{-2\,\mathrm{Osc}(P)} - e^{2\,\mathrm{Osc}(P)}(1+\frac{I_1}{R})$$

now

$$\phi_\zeta \leq 2\int_0^x \frac{e^{2W(y)}}{\int_0^1 e^{2W(y)}dy}\Big[\frac{I_2}{R}(1+\zeta) - I_3\int_0^y e^{-2\|\nabla T\|_\infty z}dz\Big]dy$$

and the function

$$f(y) = \frac{I_2}{R}(1+\zeta) - I_3\int_0^y e^{-2\|\nabla T\|_\infty z}dz$$

is negative for $x \geq x_0$ with

$$x_0 = -\frac{1}{2\|\nabla T\|_\infty}\ln\big(1-\frac{I_2}{RI_3}(1+\zeta)2\|\nabla T\|_\infty\big)$$

chose

$$\zeta = 6e^{4\,\mathrm{Osc}(P)}$$

with the inequality

$$R > 16e^{4\,\mathrm{Osc}(P)}(\|\nabla P\|_\infty + \|\nabla T\|_\infty)e^{2\|\nabla T\|_\infty/R}$$

so that

$$\frac{I_2}{RI_3}(1+\zeta)2\|\nabla T\|_\infty < \frac{1}{2}$$

but for $0 < x < 1/2$, $-\ln(1-x) \leq 2x$. It follows that

$$\begin{aligned}\phi_\zeta &\leq 2\int_0^{x_0} \frac{e^{2W(y)}}{\int_0^1 e^{2W(y)}dy}\frac{I_2}{R}(1+\zeta) \\ &\leq 2e^{2\,\mathrm{Osc}(P)}\frac{I_2}{R^2}(1+\zeta)[1+2\frac{I_2}{I_3}(1+\zeta)]\end{aligned}$$

and using $I_3 \geq e^{4\,\mathrm{Osc}(P)}$ it follows after a straightforward computation that

$$\phi_\zeta \leq 900 \frac{e^{10\,\mathrm{Osc}(P)}}{R^2} e^{4\|\nabla T\|_\infty/R}$$

which proves the upper bound.

The proof of the lower bound is similar. First observe that

$$\begin{aligned}\phi_\zeta \geq & 2\int_0^x \frac{e^{2V(y)}}{\int_0^1 e^{2W(y)}dy}\Big[\int_0^y \frac{e^{2(P-T)(z)}}{\int_0^1 e^{2P(z)}dz}dz(1-\frac{I_1}{R}) - \frac{I_2}{R}(1+\zeta)e^{-2T(y)} \\ & - \zeta(1+\frac{I_1}{R})\int_0^y \frac{e^{-2(P+T)(z)}}{\int_0^1 e^{-2P(z)}dz}dz\Big]dy \\ \geq & 2\int_0^x \frac{e^{2V(y)}}{\int_0^1 e^{2W(y)}dy}\Big[I_3\int_0^y e^{-2T(z)}dz - \frac{I_2}{R}(1+\zeta)e^{-2T(y)}\Big]dy\end{aligned}$$

with

$$I_1 = 8(\|\nabla P\|_\infty + \|\nabla T\|_\infty)e^{4\,\mathrm{Osc}(P)+2\|\nabla T\|_\infty/R} < R/2$$

and

$$I_3 = e^{-2\,\mathrm{Osc}(P)}(1-\frac{I_1}{R}) - \zeta(1+\frac{I_1}{R})e^{2\,\mathrm{Osc}(P)}$$

choose

$$\zeta = \frac{e^{-4\,\mathrm{Osc}(P)}}{6}$$

Then

$$I_3 \geq \frac{e^{-2\,\mathrm{Osc}(P)}}{6}$$

It follows that

$$\phi_\zeta \geq 2\int_0^x \frac{e^{2W(y)}}{\int_0^1 e^{2W(y)}dy}\Big[I_3\int_0^y e^{-2\|\nabla T\|_\infty y} - \frac{I_2}{R}(1+\zeta)\Big]dy$$

and by noting that $I_3\int_0^y e^{-2\|\nabla T\|_\infty y} - \frac{I_2}{R}(1+\zeta)$ is positive for $y \geq x_0 = \frac{I_2}{R}\frac{(1+\zeta)}{I_3}$ it follows after a straightforward computation that

$$\begin{aligned}\phi_\zeta \geq & -\frac{I_2}{R^2}(1+\zeta)e^{2\,\mathrm{Osc}(P)}\big(\frac{I_2}{I_3}(1+\zeta)+1\big) \\ \geq & -100\frac{e^{10\,\mathrm{Osc}(P)}}{R^2}e^{4\|\nabla T\|_\infty/R}\end{aligned}$$

□

8.5 Anomalous mean square displacement

Let y_t be an infinitely homogenized potential diffusion.

Theorem 8.5.1. *Assume that $\rho_{\min} > 16e^{4K_0}K_1e^{2K_1}$, then for $t > \rho_{\min}^2$*

$$\mathbb{E}[y_t^2] \leq C_2 e^{8n_{per}K_0} D(V_0^{n_{flu}})t \tag{8.40}$$

with

$$n_{flu} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \leq t\} \tag{8.41}$$

and

$$n_{per} = \inf\{n \in \mathbb{N} \,:\, R_{n_{flu}-n}^2 \leq tD(V_0^{n_{flu}})\}$$

is well defined with $n_{per} \leq n_{flu}$

$$C_2 = 40e^{K_1^2} \tag{8.42}$$

Theorem 8.5.2. *Assume $\rho_{\min} > 10e^{30K_1}$ and $t > R_9$ then*

$$\mathbb{E}[y_t^2] \geq \frac{e^{-8n_{per}K_0}}{50} D(V_0^{n_{flu}})t \tag{8.43}$$

with

$$n_{flu} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \leq t\} + 1 \tag{8.44}$$

and

$$n_{per} = \inf\{n \in \mathbb{N} \,:\, R_{n_{flu}-n}^2 e^{14nK_0} 10^4 e^{4K_1} \leq tD(V_0^{n_{flu}})\}$$

is well defined and $n_{per} \leq n_{flu}$

8.5.1 Proof

Let $t > \rho_{\min}^2$, write

$$n_{flu} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \leq t\} \tag{8.45}$$

Let $n_{per} \in \mathbb{N}$ such that $n_{per} \leq n_{flu}$ choose for the aggregation of effective scales

$$U_{eff} = V_0^{n_{flu}-n_{per}}$$

for the aggregation of perturbation scales

$$P_{per} = V_{n_{flu}-n_{per}+1}^{n_{flu}}$$

and for the aggregation of drift scales

$$T_{dri} = V_{n_{flu}+1}^{\infty}$$

Now observe that the conditions of the lemma C.3.3 are satisfied by the lemmas 8.4.2 and 8.4.1 with for χ^P the solution of the cell problem associated to P_{per} and for χ^U, $\chi^{V_0^{n_{flu}}} - \chi^{P_{per}}$.

$$C_1^\chi = e^{-4n_{per}K_0} \quad C_2^\chi = e^{4n_{per}K_0} \quad C_U = 2e^{2n_{per}K_0}[1+4K_1]$$

$$R_W = R_{n_{flu}} \quad R_P = \frac{R_{n_{flu}}}{R_{n_{flu}-n_{per}}}$$

and under the assumption

$$\rho_{min} > 16e^{4K_0}K_1e^{2K_1} \tag{8.46}$$

$$\zeta_2 = 6e^{4n_{per}K_0} \quad \zeta_1 = \frac{e^{-4n_{per}K_0}}{6}$$

$$C_2^\phi = 900e^{10n_{per}K_0}e^{4K_1} \quad C_1^\phi = 100e^{10n_{per}K_0}e^{4K_1}$$

8.5.1.i Upper bound

Proof. It follows by the lemma C.3.3 that

$$\begin{aligned}\mathbb{E}[y_t^2] \leq & 8e^{8n_{per}K_0}R^2_{n_{flu}-n_{per}}[1+4K_1]^2 \\ & + 4e^{4n_{per}K_0}e^{K_1^2}\left(6e^{4n_{per}K_0}D(V_0^{n_{flu}})t + R^2_{n_{flu}-n_{per}}e^{4n_{per}K_0}\right)\end{aligned} \tag{8.47}$$

Then n_{per} is chosen so that the influence of the perturbations are less or equal to the influence of the aggregation of the effective scales:
More precisely with

$$n_{per} = \inf\{n \in \mathbb{N} \,:\, R^2_{n_{flu}-n} \leq tD(V_0^{n_{flu}})\}$$

It is easy to see that for $\rho_{\min} > e^{2K_0}$, one has $n_{per} \leq n_{flu}$ and

$$\mathbb{E}[y_t^2] \leq C_2 e^{8n_{per}K_0} D(V_0^{n_{flu}})t \tag{8.48}$$

with

$$C_2 = 40e^{K_1^2} \tag{8.49}$$

□

8.5.1.ii Lower bound

Proof. For $t > R_1^2$, choose

$$n_{flu} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \leq t\} + 1 \tag{8.50}$$

and define the aggregation of scales as for the upper bound with this new definition of n_{flu}.
Observe that

$$t\|\nabla T_{dri}\|^2_\infty \leq R^2_{n_{eff}}(2K_1/R_{n_{eff}+1})^2$$

Observe that for $\rho_{\min} > 4K_1$ It follows by the lemma C.3.3 that

$$\begin{aligned}\mathbb{E}[y_t^2] \geq & \frac{e^{-8n_{per}K_0}}{24}D(V_0^{n_{flu}})t - R^2_{n_{flu}-n_{per}}25e^{6n_{per}K_0}e^{4K_1} \\ & - 4R^2_{n_{flu}-n_{per}}[1+4K_1]^2\end{aligned} \tag{8.51}$$

Then n_{per} is chosen so that the influence of the perturbations are strictly less to the influence of the aggregation of the effective scales:
More precisely with

$$n_{per} = \inf\{n \in \mathbb{N} \,:\, R^2_{n_{flu}-n}e^{14nK_0}10^4e^{4K_1} \leq tD(V_0^{n_{flu}})\}$$

It is easy to see that for $\rho_{\min} > 10e^{30K_1}$ and $t > R_9$, such n exists with $n_{per} \leq n_{flu}$ and

$$\mathbb{E}[y_t^2] \geq \frac{e^{-8n_{per}K_0}}{50}D(V_0^{n_{flu}})t \tag{8.52}$$

□

Corollary 8.5.1. *Assume that* $\rho_{\min} > 16e^{4K_0}K_1e^{2K_1}$*, then for* $t > \rho_{\min}^2$

$$\mathbb{E}[y_t^2] \le C_2\lambda_2^{n_{flu}} t \tag{8.53}$$

where the constant C_2 *depends only on* $K_1, K_0, \lambda_{\min}, \rho_{\min}$ *and*

$$n_{flu} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \le t\} \tag{8.54}$$

$$\lambda_2 = \lambda_{\max}(1 + \frac{4K_1e^{2K_0}}{\rho_{\min}} + \frac{16K_0}{\ln\rho_{\min}}\ln(\frac{9}{8\lambda_{\min}})) \tag{8.55}$$

Proof. This is a simple application of the theorem 8.5.1. Observe that by the theorem 8.2.2, for $\rho_{\min} > 16e^{4K_0}K_1e^{2K_1}$ one has

$$D(V_0^{n-1}) \ge (\frac{8}{9}\lambda_{\min})^n \tag{8.56}$$

It follows that

$$\begin{aligned} n_{per} &\le \inf\{n \in \mathbb{N} \,:\, \rho_{\min}^{-n} \le (\frac{8}{9}\lambda_{\min})^{n_{flu}+1}\} \\ &\le (n_{flu}+1)\frac{1}{\ln\rho_{\min}}\ln(\frac{9}{8\lambda_{\min}}) + 1 \end{aligned}$$

Thus

$$\mathbb{E}[y_t^2] \le C_3 e^{n_{flu} 8\frac{K_0}{\ln\rho_{\min}}\ln(\frac{9}{8\lambda_{\min}})} D(V_0^{n_{flu}})t \tag{8.57}$$

where the constant C_3 depends only on $K_1, K_0, \lambda_{\min}, \rho_{\min}$ and it follows by the theorem 8.2.2 that

$$\mathbb{E}[y_t^2] \le C_4\lambda_2^{n_{flu}} t \tag{8.58}$$

with

$$\begin{aligned} \lambda_2 &= (\frac{9}{8\lambda_{\min}})^{\frac{8K_0}{\ln\rho_{\min}}} \frac{\lambda_{\max}}{1 - \frac{2K_1e^{2K_0}}{\rho_{\min}}} \\ &\le \lambda_{\max}(1 + \frac{4K_1e^{2K_0}}{\rho_{\min}} + \frac{16K_0}{\ln\rho_{\min}}\ln(\frac{9}{8\lambda_{\min}})) \end{aligned} \tag{8.59}$$

□

Corollary 8.5.2. *Assume* $\rho_{\min} > 10e^{30(K_1+4K_0^2)}$ *and* $t > R_9$ *then*

$$\mathbb{E}[y_t^2] \ge C_1\lambda_1^{n_{flu}} t \tag{8.60}$$

$$n_{flu} = \sup\{n \in \mathbb{N} \,:\, R_n^2 \le t\} \tag{8.61}$$

where the constant C_1 *depends only on* $K_1, K_0, \lambda_{\min}, \rho_{\min}$ *and*

$$\lambda_1 = \lambda_{\min}(1 - \frac{4K_1e^{2K_0}}{\rho_{\min}} - \frac{32K_0}{\ln\rho_{\min}}\ln(\frac{16}{15\lambda_{\min}})) \tag{8.62}$$

Proof. This is a simple application of the theorem 8.5.2. Observe that by the theorem 8.2.2, for $\rho_{\min} > 10e^{30K_1}$ one has

$$D(V_0^{n-1}) \geq (\frac{15}{16}\lambda_{\min})^n \tag{8.63}$$

It follows that

$$\begin{aligned} n_{per} &\leq \inf\{n \in \mathbb{N} : \rho_{\min}^{-n} e^{14nK_0} 10^4 e^{4K_1} \leq (\frac{15}{16}\lambda_{\min})^{n_{flu}+1}\} \\ &\leq \frac{1}{\ln \rho_{\min} - 14K_0}\big((n_{flu}+1)\ln(\frac{16}{15\lambda_{\min}}) + 12 + 4K_1\big) + 1 \end{aligned}$$

Thus

$$\mathbb{E}[y_t^2] \geq C_5 e^{-8n_{flu}\frac{\ln(\frac{16}{15\lambda_{\min}})}{\ln \rho_{\min} - 14K_0}K_0} D(V_0^{n_{flu}})t \tag{8.64}$$

where the constant C_5 depends only on $K_1, K_0, \lambda_{\min}, \rho_{\min}$ and it follows by the theorem 8.2.2 that

$$\mathbb{E}[y_t^2] \geq C_6 \lambda_1^{n_{flu}} t \tag{8.65}$$

with

$$\begin{aligned} \lambda_1 &= e^{-8\frac{\ln(\frac{16}{15\lambda_{\min}})}{\ln \rho_{\min} - 14K_0}K_0} \frac{\lambda_{\min}}{1 + \frac{2K_1 e^{2K_0}}{\rho_{\min}}} \\ &\geq \lambda_{\min}(1 - \frac{4K_1 e^{2K_0}}{\rho_{\min}} - \frac{32K_0}{\ln \rho_{\min}}\ln(\frac{16}{15\lambda_{\min}})) \end{aligned} \tag{8.66}$$

□

8.5.1.iii Bounded ratio between scales case

Theorem 8.5.3. *Assume $\lambda_{\max} < 1$, $\rho_{\min} > 10e^{-\frac{30}{\ln \lambda_{\max}}(K_1 + 4K_0^2)}$, $t > R_9$ and $\rho_{\max} < \infty$ then*

$$\mathbb{E}[y_t^2] = t^{1-\nu(t)} \tag{8.67}$$

$$\nu(t) \leq -\frac{\ln \lambda_{\min}}{2\ln \rho_{\min}} + \frac{\frac{2K_1 e^{2K_0}\ln \rho_{\min}}{\rho_{\min}} + 16K_0 \ln(\frac{16}{15\lambda_{\min}})}{(\ln \rho_{\min})^2} + \epsilon(t) \tag{8.68}$$

$$\nu(t) \geq -\frac{\ln \lambda_{\max}}{2\ln \rho_{\max}} - \frac{\frac{2K_1 e^{2K_0}\ln \rho_{\min}}{\rho_{\min}} + 8K_0 \ln(\frac{9}{8\lambda_{\min}})}{\ln \rho_{\min} \ln \rho_{\max}} - \epsilon(t) \tag{8.69}$$

where $\epsilon(t) \to 0$ as $t \to \infty$ and

$$-\frac{\ln \lambda_{\max}}{2\ln \rho_{\max}} - \frac{\frac{2K_1 e^{2K_0}\ln \rho_{\min}}{\rho_{\min}} + 8K_0 \ln(\frac{9}{8\lambda_{\min}})}{\ln \rho_{\min} \ln \rho_{\max}} > 0 \tag{8.70}$$

Remark 8.5.1. Observe that for a self-similar diffusion, $\lambda_{\max} = \lambda_{\min} = \lambda$ and $\rho_{\max} = \rho_{\min} = \rho$,

$$1 - \nu(t) \sim 1 + \frac{\ln \lambda}{2\ln \rho} \tag{8.71}$$

Proof. Straightforward by the corollaries 8.5.1 and 8.5.2 □

8.5.1.iv Fast separation between scales case

Theorem 8.5.4. *Assume that for all k, $U_k = U$ and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then

$$C_1 t^{1-\nu(t)} \leq \mathbb{E}_0[y_t^2] \leq C_2 t^{1-\nu(t)} \tag{8.72}$$

where C_1, C_2 depends only on K_0, K_1, ρ, α and

$$\nu(t) = \frac{1}{(\ln t)^{1-\frac{1}{\alpha}}} \frac{\ln\big(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\big)}{(2\ln\rho)^{\frac{1}{\alpha}}}(1+\epsilon(t)) \tag{8.73}$$

with $\epsilon(t) \to 0$ as $t \to \infty$.

Remark 8.5.2. Observe that $\mathbb{E}[y_t^2]/t \to 0$ as $t \to \infty$ but for all $1 > \beta > 0$, $\mathbb{E}[y_t^2]/t^{1-\beta} \to \infty$. Moreover this theorem shows how the behavior of the diffusion passes from a slightly anomalous one to a strongly anomalous one.

$$\mathbb{E}[y_t^2] \sim \frac{t}{\big(\int_{T_1^1} e^{2U(x)}dx \int_{T_1^1} e^{-2U(x)}dx\big)^{\frac{1}{(2\ln\rho)^{\frac{1}{\alpha}}}(\ln t)^{\frac{1}{\alpha}}}} \tag{8.74}$$

Proof. Straightforward by the theorems 8.5.1 and 8.5.2. Observe that the ratio between the number of perturbation scales with the numbers of fluctuating scales tends towards 0 as $t \to \infty$ □

8.6 Transition probability densities, upper bound

Let y_t be an inifintely homogenized potential diffusion.
For $p \in \mathbb{N}^*$ define the function

$$n_{per}(p) = \inf\{n \in \mathbb{N} : \frac{R_p}{R_{p-n}} e^{-3nK_0} \geq 2^9 K_1 e^{2(K_0+K_1)} (D(V_0^{p-1}))^{\frac{-1}{2}}\} \tag{8.75}$$

$n_{per}(p)$ corresponds to the number of perturbation scales among p fluctuating scales. Observe that for

$$\rho_{\min} \geq 2^9(1+K_1)e^{(8K_0+2K_1)} \tag{8.76}$$

this function is well defined and $1 \leq n_{per}(p) \leq p$. (one can assume $K_1 \geq 1$ without restricting the results) Then choose

$$n_{flu} = \inf\{n \in \mathbb{N} : \frac{K_1}{R_{n+1}} 2^6 e^{2n_{per}(n)K_0} (D(V_0^n))^{\frac{1}{2}} \leq \frac{h}{t}\} \tag{8.77}$$

$n_{flu} - n_{per}$ corresponds to the number of fully homogenized scales given h/t. n_{flu} well defined and greater than 1 under the assumption that

$$\frac{K_1}{R_2} e^{-2K_0} 2^6 \geq \frac{h}{t} \tag{8.78}$$

Theorem 8.6.1. *Under the assumptions 8.76, 8.78 and*

$$\max(X,Y) \leq \frac{h^2}{t(D(V_0^{n_{flu}-1}))^{\frac{1}{2}}} \tag{8.79}$$

with

$$X = K_1 e^{2K_0} 2^7 e^{2n_{per}K_0} \quad Y = 2^{14} e^{4(n_{per}+2)K_0} \ln\left[R_{n_{flu}+1}\right](D(V_0^{n_{flu}}))^{\frac{1}{2}} \tag{8.80}$$

it follows that

$$\mathbb{P}[l.y_t \geq h] \leq C e^{-\frac{h^2}{2^{11} e^{4n_{per}K_0} D(V_0^{n_{flu}})t}} \tag{8.81}$$

Proof. Let $n_{flu} \in \mathbb{N}$ and $n_{per} \in N$, $n_{per} \leq n_{flu}$ observe that by the explicit formula for the solution of the cell problem and the lemma 8.4.2 under the assumption $\rho_{min} > 16e^{4K_0}K_1e^{2K_1}$ the conditions of the lemma C.4.2 are satisfied with for the aggregation of effective scales

$$U_{eff} = V_0^{n_{flu}-n_{per}}$$

for the aggregation of perturbation scales

$$P_{per} = V_{n_{flu}-n_{per}+1}^{n_{flu}}$$

and for the aggregation of drift scales

$$T_{dri} = V_{n_{flu}+1}^{\infty}$$

$$R_W = R_{n_{flu}} \quad R_P = \frac{R_{n_{flu}}}{R_{n_{flu}-n_{per}}}$$

$$\zeta_2 = 6e^{4n_{per}K_0} \quad C_2^{\phi} = 900e^{10n_{per}K_0}e^{4K_1} \quad C^{\chi} = 1$$

Observe that with the definition of n_{flu} the left inequality in C.68 is satisfied. Moreover Then the right inequality in C.68 is satisfied if

$$\frac{K_1}{R_{n_{flu}}} 2^6 e^{2n_{per}(n_{flu})K_0}(D(V_0^{n_{flu}-1}))^{\frac{1}{2}} \leq \frac{e^{-n_{per}K_0}e^{-2K_1}}{R_{n_{flu}-n_{per}}5} D(V_0^{n_{flu}}) \tag{8.82}$$

which is implied by the definition of n_{per}.
Now observe that the inequality C.67 is satisfied if

$$2R_{n_{flu}} \leq h$$

By the definition of n_{flu}, this inequality is implied by

$$\frac{h}{2} \geq \frac{K_1 t}{h} 2^6 e^{2n_{per}(n_{flu}-1)K_0}(D(V_0^{n_{flu}-1}))^{\frac{1}{2}} \tag{8.83}$$

which follows from

$$\frac{h^2}{t(D(V_0^{n_{flu}-1}))^{\frac{1}{2}}} \geq 2K_1 e^{2K_0} 2^6 e^{2n_{per}(n_{flu})K_0} \tag{8.84}$$

Now observe that the inequality C.69 is satisfied if

$$2^{11}6e^{4n_{per}K_0}\ln\Big[\frac{1}{R_{n_{flu}-n_{per}}5e^{n_{per}K_0}e^{2K_1}}D(V_0^{n_{flu}})\frac{t}{h}\Big]\leq\frac{h^2}{D(V_0^{n_{flu}})t} \tag{8.85}$$

by the definition of n_{flu} this inequality is satisfied if

$$2^{11}6e^{4n_{per}K_0}\ln\Big[\frac{R_{n_{flu}+1}(D(V_0^{n_{flu}}))^{\frac{1}{2}}}{R_{n_{flu}-n_{per}}e^{3n_{per}K_0}K_1 2^8 e^{2K_1}}\Big]\leq\frac{h^2}{D(V_0^{n_{flu}})t} \tag{8.86}$$

and by the definition of n_{per} this inequality is satisfied if

$$2^{14}e^{4(n_{per}+1)K_0}\ln\Big[\frac{R_{n_{flu}+1}R_{n_{flu}-n_{per}-1}}{R_{n_{flu}-n_{per}}}\Big]\leq\frac{h^2}{D(V_0^{n_{flu}})t} \tag{8.87}$$

which is implied by

$$2^{14}e^{4(n_{per}+1)K_0}\ln\Big[R_{n_{flu}+1}\Big]\leq\frac{h^2}{D(V_0^{n_{flu}})t} \tag{8.88}$$

With this assumption, it follows by the lemma C.4.2 that

$$\mathbb{P}[l.y_t\geq h]\leq Ce^{-\frac{h^2}{2^{11}e^{4n_{per}K_0}D(V_0^{n_{flu}})t}} \tag{8.89}$$

□

Theorem 8.6.2. *Assume* $\rho_{\max}<\infty$, $\lambda_{\max}<1$,
$\rho_{\min}>C_{16}(K_0,K_1,\rho_{\min},\rho_{\max},\lambda_{\max})$

$$\frac{h^2}{t}\geq C_{11}(K_0,K_1,\rho_{\max},\rho_{\min})(\frac{t}{h})^{\frac{\ln\lambda_{\max}}{2\ln\rho_{\max}}+\frac{C_{12}(K_0,K_1)}{(\ln\rho_{\min})^2}} \tag{8.90}$$

and

$$\frac{t}{h}\geq C_{13}(K_0,K_1,R_2) \tag{8.91}$$

then for $l\in\mathbb{S}^d$

$$\mathbb{P}[l.y_t\geq h]\leq C_{14}e^{-C_{15}(K_0,K_1,\rho_{\max},\rho_{\min},\lambda_{\max})\frac{h^2}{t}(\frac{t}{h})^{\nu}} \tag{8.92}$$

with

$$\nu=-\frac{\ln\lambda_{\max}}{\ln\rho_{\max}}-\frac{C_6(K_0,K_1)}{\ln\rho_{\min}\ln\rho_{\max}}>0 \tag{8.93}$$

Remark 8.6.1. It is not surprising to have the condition 8.91 since even with one scale the homogenized behavior of the transition probability densities starts for $t>h$. Observe also that the condition 8.90 corresponds to the condition that the behavior of the diffusion is far from the heat kernel diagonal regime, however here since $\frac{\ln\lambda_{\max}}{2\ln\rho_{\max}}+\frac{C_{12}(K_0,K_1)}{(\ln\rho_{\min})^2}<0$ one can have $h^2/t<<1$ before reaching this regime.
Observe that the equation 8.92 is equivalent to

$$\mathbb{P}[l.y_t\geq h]\leq C_{14}e^{-C_{15}(K_0,K_1,\rho_{\max},\rho_{\min},\lambda_{\max})(\frac{h^{d_w}}{t})^{\frac{1}{d_w-1}}} \tag{8.94}$$

with $d_w=1+\frac{1}{1-\nu}$ which is the form found for a diffusion in the Sierpinski carpet. It is very interesting to notice that this particular form is due to the fact that the fluctuating scale is fixed by the ratio t/h.
Observe also that for a self-similar diffusion

$$\nu\sim-\frac{\ln\lambda}{\ln\rho} \tag{8.95}$$

Proof. This is a direct application of the theorem 8.6.1. First observe that under the assumption 8.76,

$$\begin{aligned} n_{flu} &\geq \inf\{n \in \mathbb{N} : \frac{K_1}{R_{n+1}} 2^6 (D(V_0^n))^{\frac{1}{2}} \leq \frac{h}{t}\} \\ &\geq \frac{\ln(\frac{t}{h})}{\ln \rho_{\max} + K_0} - C_1(K_1, \rho_{\max}) \end{aligned} \tag{8.96}$$

Moreover

$$n_{per}(p) \leq \frac{K_0 p}{\ln \rho_{\min} - 3K_0} + C_2(K_0, K_1, \rho_{\max}) \tag{8.97}$$

It follows that

$$\begin{aligned} e^{4n_{per}K_0} D(V_0^{n_{flu}}) &\leq C_4(K_0, K_1, \rho_{\max}) \big(\lambda_{\max}(1 + \frac{C_3(K_0, K_1)}{\rho_{\min}}) e^{4\frac{K_0^2}{\ln \rho_{\min} - 3K_0}}\big)^{n_{flu}} \\ &\leq C_5(K_0, K_1, \rho_{\max}, \rho_{\min}, \lambda_{\max}) (\frac{t}{h})^{\frac{\ln \lambda_{\max}}{\ln \rho_{\max}} + \frac{C_6(K_0, K_1)}{\ln \rho_{\min} \ln \rho_{\max}}} \end{aligned} \tag{8.98}$$

and observe that in the condition 8.79

$$(D(V_0^{n_{flu}-1}))^{\frac{1}{2}} X \leq C_7(K_0, K_1, \rho_{\max}, \rho_{\min}) (\frac{t}{h})^{\frac{\ln \lambda_{\max}}{2 \ln \rho_{\max}} + \frac{C_8(K_0, K_1)}{(\ln \rho_{\min})^2}} \tag{8.99}$$

and

$$(D(V_0^{n_{flu}-1}))^{\frac{1}{2}} Y \leq C_9(K_0, K_1, \rho_{\max}, \rho_{\min}) (\frac{t}{h})^{\frac{\ln \lambda_{\max}}{\ln \rho_{\max}} + \frac{C_{10}(K_0, K_1)}{(\ln \rho_{\min})^2}} \tag{8.100}$$

and the condition 8.79 is satisfied if

$$\frac{h^2}{t} \geq C_{11}(K_0, K_1, \rho_{\max}, \rho_{\min}) (\frac{t}{h})^{\frac{\ln \lambda_{\max}}{2 \ln \rho_{\max}} + \frac{C_{12}(K_0, K_1)}{(\ln \rho_{\min})^2}} \tag{8.101}$$

□

8.6.0.v Fast separation between scales case

Theorem 8.6.3. *Assume that for all k, $U_k = U$ (U non constant) and*

$$R_k = R_{k-1}[\frac{\rho^{k^\alpha}}{R_{k-1}}]$$

with $\rho, \alpha > 1$ then for

$$C_1(\rho, \alpha, K_0, K_1) < \frac{t}{h} < C_2(\rho, \alpha, K_0, K_1) h \tag{8.102}$$

one has

$$\mathbb{P}[l.y_t \geq h] \leq C_3 e^{-C_4(K_0, K_1, \rho, \lambda) \frac{h^2}{t} g(\frac{t}{h})} \tag{8.103}$$

with

$$g(x) = (\frac{1}{\lambda})^{(\frac{x}{\ln \rho})^{\frac{1}{\alpha}} (1 + \epsilon(x))} \tag{8.104}$$

and $\epsilon(x) \to 0$ as $x \to \infty$

Remark 8.6.2. Observe that $\frac{t}{h^2} \ln \mathbb{P}[l.y_t \geq h] \to -\infty$ as $t/h \to \infty$. Moreover this theorem shows how the behavior of the diffusion passes from a slightly anomalous one to a strongly anomalous one.

Proof. This is a simple application of the theorem 8.6.1. Observe that the ratio between the number of perturbation scales with the numbers of fluctuating scales tends towards 0 as $t/h \to \infty$ □

9. MULTI-SCALE CONTROL OF THE POTENTIAL EFFECTIVE DIFFUSIVITY

9.1 General set up

The purpose of this chapter is to compute the effective diffusivities associated to a smooth periodic pre-fractal media (see section C.2). As it has been shown in section 5.2, DEM theories and reiterated homogenization techniques have been developed to deal with such problems when the scales are well separated. The general technique used to obtain Multi-scale homogenization results for those media is to replace the solution of the cell problem by its first order approximation in the method of asymptotic expansion and use it as a test function in a variational formula. But the error made by this way is of order of the ratio between scales multiplied by a constant that tends to grow with the number of scales.
That's why this method can not be used to describe materials for which the ratio between scales is fixed in dependently from the number of scales and this is the situation of this chapter.

Indeed, here it will not be assumed that the ratio between scales goes to 0 as the number of scales grows to infinity, moreover the influence of each scale won't be assumed to be diluted on global behavior.

The proof of the main result allowing homogenization on an arbitrary large number of scales with bounded ratios is mainly based on three ideas and observations.

1. When homogenization takes place on two scales separated by a ratio R, a translation of the first one with respect to the second one does not change much the effective diffusivity (see lemma 9.3.2, the perturbation can easily be controlled).
2. The distance between the solution of the cell problem and itself translated by e_k/R is small with respect to the effective diffusivity of the medium (see lemma 9.3.3).
3. The effective diffusivity of n different scales is obtained by recurrence by adding the smaller scale to the $n-1$ bigger ones (here the point of view is technically different from the one of DEM theory where at each step a bigger scale is added to a matrix of smaller ones).

9.2 Main results

9.2.1 Smooth self-similar periodic pre-fractal

Let $R \in \mathbb{N}/\{0,1\}$ and $U \in C^\infty(T_1^d)$. Write

$$V^{n-1} = V_0^{n-1} = \sum_{k=0}^{n-1} (S_R)^k U \tag{9.1}$$

then one has the following theorem:

Theorem 9.2.1. *If $R \geq C_{1,d,U}$ then for all $n \geq 1$*

$$\lambda_{\max}\big(D(V^{n-1})\big) \leq \Big(\lambda_{\max}\big(D(U)\big)\Big)^n (1 + \frac{C_{2,d,U}}{R^{\frac{1}{2}}})^n \tag{9.2}$$

$$\lambda_{\min}\big(D(V^{n-1})\big) \geq \Big[\frac{\lambda_{\min}\big(D(U)\big)}{1 + \frac{C_{2,d,U}}{R^{\frac{1}{2}}}}\Big]^n \tag{9.3}$$

with

$$C_{1,d,U} = C_d e^{(6d+16)\,\mathrm{Osc}(U)}(1 + \|\nabla U\|_\infty)^3 \tag{9.4}$$

and

$$C_{2,d,U} = C_d e^{(3d+8)\,\mathrm{Osc}(U)}(1 + \|\nabla U\|_\infty)^{\frac{1}{2}} \tag{9.5}$$

As a first reaction to this theorem, it is interesting to deduce the following corollary

Corollary 9.2.1. *If in all the directions $l \in \mathbb{S}^d$ of the space $l.\nabla U$ is not the null function then $D(U) < 1$ and for*

$$R > \rho_{d,U} = \Big[\frac{C_{d,\mathrm{Osc}(U),\|\nabla U\|_\infty}\lambda_{\max}\big(D(U)\big)}{1 - \lambda_{\max}\big(D(U)\big)}\Big]^2 \tag{9.6}$$

$D(V^n)$ tends geometrically towards 0 with an explicit control of the speed of convergence given by the theorem 9.2.1.

This is the key leading to the sub diffusive behavior in a smooth periodic pre fractal. It is interesting also to observe that when U is isotropic that is to say the minimal and maximal eigenvalues of $D(U)$ are equal then the multi-scale effective diffusivity $D(V^n)$ behaves like $\lambda\big(D(U)\big)^n(1 + \frac{\text{error}}{R^{\frac{1}{2}}})^n$ but one must be careful this doesn't mean that $D(V^n)$ is isotropic.

In fact the theorem 9.2.1 is deduced from a more general result allowing to control the effective diffusivities when the medium is not self-similar.

9.2.2 General smooth periodic pre-fractal

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C^\infty(T_1^d)$ such that for all n, $\mathrm{Osc}(U_n) \leq K_0$ and $\|\nabla U_n\|_\infty \leq K_1$
$(r_n)_{n\in\mathbb{N}}$ a sequence of integer in $\mathbb{N}^*$ such that for all $n \geq 1$, $r_n \geq \rho_{\min}$
Write $R_n = r_0 \cdots r_n$ and

$$V^n(x) = \sum_{k=0}^{n} U_k(\frac{xR_n}{R_k})$$

Thus the scales can be non symmetric, non self-similar, the ratios may vary and all those elements characterizing the multi-scale media may be chosen at random.

Theorem 9.2.2. *If $\rho_{min} \geq C_{1,d,K_0,K_1}$ then for all $n \geq 1$*

$$\lambda_{\max}(D(V^{n-1})) \leq (1 + \frac{C_{2,d,K_0,K_1}}{\rho_{\min}^{\frac{1}{2}}})^n \prod_{k=0}^{n-1} \lambda_{\max}(D(U_k)) \tag{9.7}$$

and

$$\lambda_{\min}\big(D(V^n)\big) \geq (1+\frac{C_{2,d,K_0,K_1}}{\rho_{\min}^{\frac{1}{2}}})^{-n}\prod_{k=0}^{n-1}\lambda_{\min}\big(D(U_k)\big) \tag{9.8}$$

$$C_{1,d,K_0,K_1} = C_d e^{(6d+16)K_0}(1+K_1)^3 \tag{9.9}$$

and

$$C_{2,d,K_0,K_1} = C_d e^{(3d+8)K_0}(1+K_1)^{\frac{1}{2}} \tag{9.10}$$

The constants $C_{1,d,K_0,K_1}, C_{2,d,K_0,K_1}$ given above are not the optimal ones because clarity of presentation has been privileged. The theorem above is deduced from the theorems 9.3.2 and 9.4.1.

And those theorems are themselves deduced from more general results, that is to say, the propositions 9.3.5 and 9.4.1 which allow to control the whole matrix $D(V^{n-1})$.

It is also interesting to deduce the following corollary

Corollary 9.2.2. *if one has for all* n, $\lambda_{\max}\big(D(U_n)\big) \leq \lambda_{\max} < 1$, *then if for all* n

$$r_n > \rho_{\lambda_{\max},d,K_0,K_1} \tag{9.11}$$

then

$$C_1\lambda_1^n \leq D(V^n) \leq C_2\lambda_2^n \tag{9.12}$$

with $0 < \lambda_1 \leq \lambda_2 < 1$ *and*

$$\rho_{\lambda_{\max},d,K_0,K_1} = \Big[\frac{C(d,K_0,K_1)\lambda_{\max}}{1-\lambda_{\max}}\Big]^2 \tag{9.13}$$

9.2.3 Dimension one

In the results given above the geometric speed of convergence of $D(V^n)$ towards 0 is obtained only for ρ_{min} greater than constant $\rho_{d,K_0,K_1,\lambda_{\max}}$ characterized by the medium. Thus it is natural to wonder whether this condition is necessary (it will shown that the answer yes) and what happens below this constant.

Consider the self-similar case given in subsection 9.2.1 in dimension one. Here the theorem C.1.1 says that

Theorem 9.2.3. *in dimension one for all* $R \in \mathbb{N}/\{0,1\}$

$$\lim_{n\to\infty} -\frac{1}{n}\ln\big(D(V^n)\big) = \mathcal{P}_R(2U) + \mathcal{P}_R(-2U) \tag{9.14}$$

where $\mathcal{P}_R$ is the topological pressure associated to the shift s_R.
And the theorem $C.1.2$ says that $\mathcal{P}_R(2U)+\mathcal{P}_R(-2U) > 0$ if and only U does not belong to the closed subspace of $\mathcal{C}(T_1^d)$ generated the elements $T(x)-T(R^kx)$ with $T \in C(T_1^d)$ and $k \in \mathbb{N}$. Moreover it is easy to see that as $R \to \infty$

$$\mathcal{P}_R(2U) + \mathcal{P}_R(-2U) \to \ln\int_{T_1^d} e^{2U}dx + \ln\int_{T_1^d} e^{-2U}dx \tag{9.15}$$

which is strictly positive if U is not constant.
This is very interesting because if one take for instance $U = T - S_{81}T$ with $T \in C^\infty(T_1^1)$ one sees that for $R = 3; 9$ or 81, $D(V^n)$ remains lower bounded by a strictly positive constant whereas between these integers it can have a geometric decrease towards 0, it does suggest that with a fine tuning of the ratio between scales a diffusion on a smooth pre fractal may successively pass from a normal behavior to an anomalous behavior and the regions of anomaly in the space of ratios can be non connected and separated by regions of normal behavior.

9.2.4 Connection between cohomology and homogenization, dimension two

In higher dimensions, the constant $\rho_{d,U}$ associated to the corollary 9.2.1 appears as an upper bound to the regions of normal behavior, when U is characterized only by $\lambda_{\max}\big(D(U)\big)$, $\|\nabla U\|_\infty$ and $\mathrm{Osc}(U)$. Moreover by the Voigt Reiss's inequality

$$\lim\inf_{n\to\infty} -\frac{1}{n}\ln\Big(\lambda_{\min}\big(D(V^n)\big)\Big) \le \mathcal{P}_R(2U)+\mathcal{P}_R(-2U) \tag{9.16}$$

Thus if U belongs to the closed subspace of $\mathcal{C}(T_1^d)$ generated the elements $T(x)-T(R^k x)$, then

$$\lim_{n\to\infty}\frac{1}{n}\ln\Big(\lambda_{\min}\big(D(V^n)\big)\Big)=0 \tag{9.17}$$

And the diffusion doesn't show a clear anomaly, this suggests that regions of normality separated by regions of anomaly exists (they can be built on simple examples).

Now an interesting question arises: if $R\le\rho_{d,\alpha,U}$ and is bounded above by a region of normality (are normal region only points? Or can they be open an non void ?) then what is the mechanism behind this the geometric decrease of $D(V^n)$ towards 0, what kinds of large deviations are hidden behind this sort of transition of phase ? This question will be investigated here in dimension, two. Indeed as there is a strong connection between homogenization and cohomology that allows to obtain the following result (which corresponds to the theorem 9.3.1):

Theorem 9.2.4. *For $d=2$ one has*

$$\begin{aligned}\lambda_{\max}\big(D(U)\big)\lambda_{\min}\big(D(-U)\big) &= \lambda_{\min}\big(D(U)\big)\lambda_{\max}\big(D(-U)\big)\\ &= \frac{1}{\int_{T_1^d}\exp(2U)dx\int_{T_1^d}\exp(-2U)dx}\end{aligned} \tag{9.18}$$

from which one deduces that if $D(U)=D(-U)$ then

$$\lambda_{\max}\big(D(U)\big)=\lambda_{\min}\big(D(U)\big)=\frac{1}{\sqrt{\int_{T_1^d}\exp(2U)dx\int_{T_1^d}\exp(-2U)dx}} \tag{9.19}$$

Moreover

Theorem 9.2.5. *In the self-similar case given in the subsection 9.2.1, if $d=2$ and for all n, $D(V^n)=D(-V^n)$ then*

$$\lim_{n\to\infty}-\frac{1}{n}\ln\big(\lambda(D(V^{n-1}))\big)=\frac{\mathcal{P}_R(2U)+\mathcal{P}_R(-2U)}{2} \tag{9.20}$$

where $\mathcal{P}_R$ is the topological pressure associated to the shift s_R.
As an example of medium satisfying the condition of the previous theorem one can give the following corollary

Corollary 9.2.3. *In the self-similar case given in the subsection 9.2.1, if $d=2$ and for all n, $U_n(-x)=-U_n(x)$ then*

$$\lim_{n\to\infty}-\frac{1}{n}\ln\big(\lambda(D(V^{n-1}))\big)=\frac{\mathcal{P}_R(2U)+\mathcal{P}_R(-2U)}{2} \tag{9.21}$$

9.2.5 Perspectives

These statements show clearly that when the scales are not self-similar and non symmetric (can be chosen at random) the geometric speed of convergence of $D(V^n)$ towards 0 can be controlled without the necessity to use large deviations techniques, however it is interesting to wonder how this is translated in the theory of shifts dynamical systems. For instance note that $V^0 = U_0$ and

$$V^{n+1} = S_{r_{n+1}}(V^n + U_{n+1}) \tag{9.22}$$

and the latter inductive definition will be interesting to explore in the shift spaces, what notion will replace the pressure? What kind of large deviations might be hidden behind the behavior of the eigenvalues of the matrix $D(V^n)$ in any dimension? Those questions will be postponed to a future work.

9.3 Upper Bound

9.3.1 Cohomological Framework

Consider $U \in C^\infty(T_1^d)$ and m_U the measure 5.1 associated to U on the torus. Write $\mathcal{C} = \big(C^\infty(T_1^d)\big)^d$ the set of C^∞ vector fields on T_1^d and $H = (L^2(m_U))^d$ the completion of $\mathcal{C}$ with respect to the norm $\|.\|_H$ where for $\xi \in \mathcal{C}$

$$\|\xi\|_H^2 = \int_{T_1^d} |\xi(x)|^2 m_U(dx) \tag{9.23}$$

Thus H is a real Hilbert space equipped with the scalar product

$$(\xi, \nu)_H = \int_{T_1^d} \xi(x).\nu(x) m_U(dx) \tag{9.24}$$

Write ("pot" for potential vector fields and "sol" for solenoidal)

$$\mathcal{C}_{pot} = \Big\{\xi \in \mathcal{C} \,|\, \exists f \in C^\infty(T_1^d) \text{ with } \xi = \nabla f\Big\} \tag{9.25}$$

$$\mathcal{C}_{sol} = \Big\{\xi \in \mathcal{C} | \exists p \in \mathcal{C} \text{ with } \operatorname{div}(p) = 0 \text{ and } \xi = p \exp(2U) \int_{T_1^d} e^{-2U(x)} dx\Big\} \tag{9.26}$$

and H_{pot}, H_{sol} the closure of $\mathcal{C}_{pot}$, $\mathcal{C}_{sol}$ in H with respect to the norm $\|.\|_H$.
Observe now that the following orthogonal decomposition can easily be obtained

$$H = H_{pot} \oplus H_{sol} \tag{9.27}$$

Proof. Indeed, let $\nu \in \mathcal{C}_{pot}$ and $\xi \in \mathcal{C}_{sol}$. Then $\nu = \nabla f$ with $f \in \mathcal{C}^\infty(T_1^d)$ and $\xi = pe^{2U}$ with $p \in \mathcal{C}$ and $\operatorname{div}(p) = 0$. It follows that

$$(\nu, \xi)_H = \int_{T_1^d} \nabla f.p dx = \int_{T_1^d} f \operatorname{div}(p) dx = 0 \tag{9.28}$$

and since $\mathcal{C}_{pot}$ and $\mathcal{C}_{sol}$ are dense in H_{pot} and H_{sol}, it follows that $H_{sol} \subset H_{pot}^\perp$. Now let $\xi \in H_{pot}^\perp \cap \mathcal{C}$, then for all $f \in C^\infty(T_1^d)$

$$0 = (\nabla f, \xi)_H = \frac{\operatorname{div}(e^{-2U}\xi) f(x)\, dx}{\int_{T_1^d} e^{-2U} dx} \tag{9.29}$$

it follows that $\xi = pe^{2U} \int_{T_1^d} e^{-2U} dx$ with $p \in \mathcal{C}$ and $\operatorname{div} p = 0$. Thus $H_{pot}^\perp \subset H_{sol}$ and $H_{sol} = H_{pot}^\perp$ which proves the orthogonal decomposition. ∎

Now observe that for $l \in \mathbb{R}^d$,

$$\sqrt{{}^t l D(U) l} = \operatorname{dist}(l, H_{pot}) \tag{9.30}$$

is then norm in H of the orthogonal projection of l on H_{sol} (which is equivalent to say to projection on H_{sol} parallel to H_{pot}). Indeed this is a direct consequence of the variational formulation 5.21. Moreover if χ_l is the solution of the cell problem 5.4 Then

$$l = \nabla \chi_l + \exp(2U) p_l \tag{9.31}$$

is the orthogonal decomposition of l

9.3.2 Abstract Tools

9.3.2.i Duality

Lemma 9.3.1. *For all $\xi \in H$*

$$\operatorname{dist}(\xi, H_{pot}) = \sup_{\delta \in \mathcal{C}_{sol}} \frac{(\delta, \xi)_H}{\|\delta\|_H} \tag{9.32}$$

Proof. This is a direct consequence of the orthogonal decomposition 9.27 Indeed $\xi = \nu + \mu$ with $\nu \in H_{sol}$ and $\mu \in H_{pot}$ and

$$\operatorname{dist}(\xi, H_{sol}) = \inf_{f \in \mathcal{C}^\infty(T_1^d)} \|\xi - \nabla f\|_H = \|\xi - \mu\|_H = \|\nu\|_H$$

since $\mathcal{C}_{pot}$ is dense in H_{pot} it follows that Thus

$$\begin{aligned} \inf_{f \in \mathcal{C}^\infty(T_1^d)} \|\xi - \nabla f\|_H &= \sup_{\delta \in \mathcal{C}_{sol}} \frac{(\nu, \delta)_H}{\|\delta\|_H} \\ &= \sup_{\delta \in \mathcal{C}_{sol}} \frac{(\xi, \delta)_H}{\|\delta\|_H} \end{aligned}$$

□

Remark 9.3.1. This lemma gives the following variational formula for the effective diffusivity by taking $\xi = l \in \mathbb{R}^d$

$${}^t l D(U) l = \sup_{p \in \mathcal{C} \ \operatorname{div}(p)=0} \frac{\left(\int_{T_1^d} l.p dx\right)^2}{\int_{T_1^d} p^2 \exp(2U) dx \int_{T_1^d} \exp(-2U) dx} \tag{9.33}$$

which gives back the Voigt-Reiss's inequality 5.24 by taking $p = l$

9.3.2.ii Connections between cohomology and duality

Write

$$\mathcal{F}_{sol} = \left\{ p \in \mathcal{C} | \operatorname{div}(p) = 0 \text{ and } \int_{T_1^d} p dx = 0 \right\}$$

Write $Q(U)$ the positive, definite, symmetric matrix associated to the following variational problem. For $l \in \mathbb{S}^d$

$${}^t l Q(U) l = \inf_{p \in \mathcal{F}_{sol}} \frac{\int_{T_1^d} |l - p|^2 \exp(2U) dx}{\int_{T_1^d} \exp(2U) dx} \tag{9.34}$$

Then the following proposition is a direct consequence of the equation 9.33.

Proposition 9.3.1. *For all* $l \in \mathbb{S}^d$

$$ {}^t l D(U) l = \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx} \sup_{\xi \in \mathbb{S}^d} \frac{(l.\xi)^2}{{}^t\xi Q(U)\xi} \tag{9.35} $$

The previous proposition allows to establish a one to one correspondence between the eigenvalues of $D(U)$ and $Q(U)$. Indeed write in the increasing order $\lambda(D(U))_i$ and decreasing order $\lambda(Q(U))_i$ those eigenvalues, then the following proposition is a simple consequence of proposition 9.3.1

Proposition 9.3.2. *For all* $i \in \{1, \dots, d\}$

$$ \lambda(D(U))_i \lambda(Q(U))_i = \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx} \tag{9.36} $$

In particular

$$ \begin{aligned} \lambda_{\max}(D(U))\lambda_{\min}(Q(U)) &= \lambda_{\min}(D(U))\lambda_{\max}(Q(U)) \\ &= \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx} \end{aligned} \tag{9.37} $$

Proof. In the orthonormal basis corresponding to the diagonalization of Q observe that

$$ \frac{(l.\xi)^2}{{}^t\xi Q(U)\xi} = \frac{(l.\xi)^2}{\sum_{i=1}^d \lambda(Q(U))_i \xi_i^2} \tag{9.38} $$

And it is an easy exercise to check that the supremum of the equation 9.38 on $\mathbb{S}^d$ is reached for ξ proportional to the vector $(l_i/\lambda(Q(U))_i)$. Which gives

$$ {}^t l D(U) l = \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx} \sum_{i=1}^d \frac{l_i^2}{\lambda(Q(U))_i} \tag{9.39} $$

□

Dimension two In dimension two, the Poincaré duality establishes a simple correspondence between $Q(U)$ and $D(-U)$.

Proposition 9.3.3. *For* $d = 2$*, one has*

$$ Q(U) = {}^t P D(-U) P \tag{9.40} $$

where P *stands for the rotation matrix*

$$ P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9.41} $$

Proof. Observe that by the Poincaré duality one has

$$ \mathcal{F}_{sol} = \{P\nabla f \,:\, f \in C^\infty(T_1^d)\} \tag{9.42} $$

Then the results follows directly from the definition of $Q(U)$. □

Theorem 9.3.1. *For* $d = 2$ *one has*

$$ \begin{aligned} \lambda_{\max}\big(D(U)\big)\lambda_{\min}\big(D(-U)\big) &= \lambda_{\min}\big(D(U)\big)\lambda_{\max}\big(D(-U)\big) \\ &= \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx} \end{aligned} \tag{9.43} $$

Proof. This is a direct consequence of the propositions 9.3.3 and 9.3.2. □

Corollary 9.3.1. *For $d = 2$ if $D(U) = D(-U)$ then*

$$D(U) = \lambda I_d \tag{9.44}$$

with

$$\lambda = \frac{1}{\sqrt{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}} \tag{9.45}$$

Proof. This is a direct consequence of the theorem 9.3.1. □

9.3.2.iii Multi-scale translation and multi-scale homogenization

Let $R \in \mathbb{N}^*$, $V,\ T \in C^\infty(T_1^d)$ and write $U = S_R V + T$. Write $D(V)$, $D(T)$ and $D(U)$ the effective diffusivities associated to the homogenization on V, T and U. For $y \in T_1^d$, write $\Theta_y V$ the function $x \to V(x + y)$. (we recall that Θ_y is the translation operator by $-y$).
Then the following lemma shows that when R is large with respect to $\|\nabla T\|_\infty$, a relative translation between the two scales, does not change much the multi-scale effective diffusivity.

Lemma 9.3.2. *For all $y \in \mathbb{R}^d$,*

$$e^{-4\frac{\|\nabla T\|_\infty}{R}} D(S_R V + T) \leq D(S_R V + \Theta_y T) \leq e^{4\frac{\|\nabla T\|_\infty}{R}} D(S_R V + T) \tag{9.46}$$

Proof. The result follows from the following simple observation: ($[Ry]$ is the vector with the integral parts of $(yR)_i$ as coordinates)

$$S_R V + \Theta_y T = \Theta_{[Ry]/R}(S_R V + T) + \Theta_y T - \Theta_{[Ry]/R} T \tag{9.47}$$

Thus

$$D(S_R V + \Theta_y T) \leq e^{4\|\Theta_y T - \Theta_{[Ry]/R} T\|_\infty} D(\Theta_{[Ry]/R}(S_R V + T)) \tag{9.48}$$

and the result follows by observing that $D(\Theta_{[Ry]/R}(S_R V + T)) = D(S_R V + T)$ (the effective diffusivity is invariant under a translation of the medium) □

Now choose a sequence (U_n, r_n) of smooth functions on the torus T_1^d with uniformly bounded gradients $\|\nabla U_n\|_\infty \leq K_1$ and integers uniformly bounded from below $r_n \geq \rho_{\min}$ (for $n \geq 1$). Then the following proposition is a direct consequence of the previous lemma and a simple induction (we recall that $R_n = r_0 \dots r_n$)

Proposition 9.3.4. *For all $n \in \mathbb{N}$, $(y_0, \dots, y_n - 1) \in \mathbb{R}^{d \times n}$*

$$\begin{aligned} D(\sum_{i=0}^{n-1} \Theta_{y_n} S_{1/R_k} S_{R_n} U_n) &\leq D(\sum_{i=0}^{n-1} S_{1/R_k} S_{R_n} U_n) \prod_{k=1}^{n-1} e^{4\frac{K_1}{r_k}} \\ &\geq D(\sum_{i=0}^{n-1} S_{1/R_k} S_{R_n} U_n) \prod_{k=1}^{n-1} e^{-4\frac{K_1}{r_k}} \end{aligned} \tag{9.49}$$

9.3.3 Upper bound with two scales

Let $R \in \mathbb{N}^*$, V, $T \in C^\infty(T_1^d)$. Write for $y \in T_1^d$, $x \to \chi(x,y)$ the solution of the cell problem associated to $S_R\Theta_y V + T$.
Write χ^V the solution of the cell problem associated to V and $\chi^{D(V),T}$ the T_1^d periodic solution of the following cell problem (which corresponds to a complete homogenization on the smaller scale): for $l \in \mathbb{S}^d$

$$\nabla\big(e^{-2T}D(V)(l-\nabla\chi_l^{D(V),T})\big)=0 \tag{9.50}$$

Write $D(V,T,R=\infty)$ the effective diffusivity corresponding to multi-scale homogenization on V,T with complete separation between the scales, that is to say:

$$D(V,T,R=\infty)=\int_{x\in T_1^d} {}^t(l-\nabla\chi_l^{D(V),T}(x))D(V)(l-\nabla\chi_l^{D(V),T}(x))m^T(dx) \tag{9.51}$$

Lemma 9.3.3. *One has for $l \in \mathbb{S}^d$ and $k \in \{1,\dots,d\}$ ($\{e_1,\dots,e_d\}$ being an orthonormal basis of $\mathbb{R}^d$)*

$$\int_{T_1^d}|\nabla\chi_l(x+\frac{e_k}{R},0)-\nabla\chi_l(x,0)|^2 m^{S_RV+T}(dx)\leq {}^tlD(S_RV+T)l(e^{4\frac{\|\nabla T\|_\infty}{R}}-1) \tag{9.52}$$

Proof. Observe that (using the standard property of the solution of the cell problem)

$$\begin{aligned}
&\int_{T_1^d}|\nabla\chi_l(x+\frac{e_k}{R},0)-\nabla\chi_l(x,0)|^2 m^{S_RV+T}(dx)\\
&\quad=\int_{T_1^d}|l-\nabla\chi_l(x,0)+\nabla\chi_l(x+\frac{e_k}{R},0)-\nabla\chi_l(x,0)|^2 m^{S_RV+T}(dx)\\
&\quad-{}^tlD(S_RV+T)l\\
&\quad\leq e^{4\frac{\|\nabla T\|_\infty}{R}}\int_{T_1^d}|l-\nabla\chi_l(x+\frac{e_k}{R},0))|^2 m^{\Theta\frac{e_k}{R}(S_RV+T)}(dx)\\
&\quad-{}^tlD(S_RV+T)l
\end{aligned} \tag{9.53}$$

which leads to the result ☐

Proposition 9.3.5. *One has*

$$\begin{aligned}
D(S_RV+T)\leq& D(V,T,R=\infty)e^{24\frac{\|\nabla T\|_\infty}{R}}\\
&(1+C_d\sqrt{e^{8\frac{\|\nabla T\|_\infty}{R}}-1}e^{(3d+5)\operatorname{Osc}(V)})^2
\end{aligned} \tag{9.54}$$

Proof. Let $l \in \mathbb{S}^d$. By using the standard property of the solution of the cell problem, one has

$$\begin{aligned}
\int_{y\in T_1^d}{}^tlD(S_R\Theta_yV+T)l dy&=\int_{T_1^d\times T_1^d}(l-\nabla_x\chi_l(x,y)).l\, m^{S_R\Theta_yV+T)(x)}(dx)\,dy\\
&=I_1-I_2
\end{aligned} \tag{9.55}$$

with

$$\begin{aligned}
I_1=&\int_{T_1^d\times T_1^d}(l-\nabla_x\chi_l(x,y))(I_d-\nabla\chi_.^V(Rx+y))(l-\nabla\chi_l^{D(V),T}(x))\\
&m^{V(Rx+y)+T(x)}(dx)\,dy
\end{aligned} \tag{9.56}$$

and

$$I_2 = \int_{T_1^d\times T_1^d} (l-\nabla_x\chi_l(x,y))\nabla\chi_.^V(Rx+y)(\nabla\chi_l^{D(V),T}(x)-l)m^{V(Rx+y)+T(x)}(dx)\,dy \tag{9.57}$$

Now, by the Cauchy Schwartz inequality applied to the integration in x and y one has

$$\begin{aligned} I_1 \leq& \Big(\int_{(x,y)\in(T_1^d)^2} |l-\nabla_x\chi_l(x,y)|^2 m^{V(Rx+y)+T(x)}(dx)\,dy\Big)^{\frac{1}{2}} \\ &\times\Big(\int_{(x,y)\in(T_1^d)^2} |(I_d-\nabla\chi_.^V(Rx+y))(l-\nabla\chi_l^{D(V),T}(x))|^2 m^{V(Rx+y)+T(x)}(dx),dy\Big)^{\frac{1}{2}} \end{aligned} \tag{9.58}$$

Which leads to

$$\begin{aligned} I_1 \leq& \Big(\int_{y\in T_1^d} {}^t l D(S_R\Theta_y V+T)l\,dy\Big)^{\frac{1}{2}} \\ &\times\Big({}^t l D(V,T,R=\infty)l\Big)^{\frac{1}{2}} e^{\frac{\|\nabla T\|_\infty}{R}} \end{aligned} \tag{9.59}$$

Next, observe that

$$I_2 = J_1+J_2+J_3 \tag{9.60}$$

with

$$\begin{aligned} J_1 =& \int_{(x,y)\in T_1^d\times[0,1]^d} (l-\nabla_x\chi_l(x+\frac{y}{R},0))\nabla\chi_.^V(Rx+y)(\nabla\chi_l^{D(V),T}(x)-l) \\ &\frac{e^{-2(V(Rx+y)+T(x+\frac{y}{R}))}}{\int_{T_1^d} e^{-2(V(Rz+y)+T(z))}dz}dx\,dy \end{aligned} \tag{9.61}$$

$$\begin{aligned} J_2 =& \int_{(x,y)\in T_1^d\times[0,1]^d} (\nabla_x\chi_l(x+\frac{y}{R},0)-\nabla_x\chi_l(x,y))\nabla\chi_.^V(Rx+y)(\nabla\chi_l^{D(V),T}(x)-l) \\ &\frac{e^{-2(V(Rx+y)+T(x+\frac{y}{R}))}}{\int_{T_1^d} e^{-2(V(Rz+y)+T(z))}dz}dx\,dy \end{aligned} \tag{9.62}$$

and

$$\begin{aligned} J_3 =& \int_{(x,y)\in T_1^d\times[0,1]^d} (l-\nabla_x\chi_l(x,y))\nabla\chi_.^V(Rx+y)(\nabla\chi_l^{D(V),T}(x)-l) \\ &\frac{e^{-2(V(Rx+y)+T(x+\frac{y}{R}))}}{\int_{T_1^d} e^{-2(V(Rz+y)+T(z))}dz}(1-e^{2(T(x)-T(x+\frac{y}{R}))})dx\,dy \end{aligned} \tag{9.63}$$

Then, just as for the computation associated to I_1, by using Cauchy-Schwartz inequality one obtains:

$$\begin{aligned} |J_3| \leq& (e^{2\frac{\|\nabla T\|_\infty}{R}}-1)e^{\frac{\|\nabla T\|_\infty}{R}}\Big(\int_{y\in T_1^d} {}^t l D(S_R\Theta_y V+T)l\,dy\Big)^{\frac{1}{2}} \\ &\times\Big(\int_{x\in T_1^d}(l-\nabla\chi_l^{D(V),T}(x))(I_d-D(V))(l-\nabla\chi_l^{D(V),T}(x))dx\Big)^{\frac{1}{2}} \end{aligned} \tag{9.64}$$

and by using Voigt Reiss's inequality $D(V)\geq e^{-2\,\mathrm{Osc}(V)}$ one obtains that

$$\begin{aligned} |J_3| \leq& (e^{2\frac{\|\nabla T\|_\infty}{R}}-1)e^{\frac{\|\nabla T\|_\infty}{R}+\mathrm{Osc}(V)}\Big(\int_{y\in T_1^d} {}^t l D(S_R\Theta_y V+T)l\,dy\Big)^{\frac{1}{2}} \\ &\times\Big(D(V,T,R=\infty)\Big)^{\frac{1}{2}} \end{aligned} \tag{9.65}$$

and by noticing that for $y \in [0,1]^d$

$$\begin{aligned}
&\int_{x\in T_1^d} |\nabla_x \chi_l(x+\frac{y}{R},0)-\nabla_x\chi_l(x,y)|^2 \frac{e^{-2(V(Rx+y)+T(x))}}{\int_{T_1^d} e^{-2(V(Rz+y)+T(z))}dz} dx\,dy\\
&=\int_{x\in T_1^d} |l-\nabla_x \chi_l(x+\frac{y}{R},0)|^2 \frac{e^{-2(V(Rx+y)+T(x))}}{\int_{T_1^d} e^{-2(V(Rz+y)+T(z))}dz} dx\,dy\\
&-{}^t l D(S_R\Theta_y V+T)l\\
&\leq {}^t l D(S_R V+T) l e^{4\frac{\|\nabla T\|_\infty}{R}} - {}^t l D(S_R\Theta_y V+T)l\\
&\leq {}^t l D(S_R\Theta_y V+T)l(e^{8\frac{\|\nabla T\|_\infty}{R}}-1)
\end{aligned} \tag{9.66}$$

(in the last inequality, the lemma 9.3.2 has been used), it follows that (by the same computation associated to J_3)

$$\begin{aligned}
|J_2| \leq & \Big(\int_{y\in T_1^d} {}^t l D(S_R\Theta_y V+T)l\Big)^{\frac{1}{2}} \Big(D(V,T,R=\infty)\Big)^{\frac{1}{2}}\\
&\times (e^{8\frac{\|\nabla T\|_\infty}{R}}-1)^{\frac{1}{2}} e^{4\frac{\|\nabla T\|_\infty}{R}+\mathrm{Osc}(V)}
\end{aligned} \tag{9.67}$$

Now, observe that

$$J_1 = K_1 + K_2 \tag{9.68}$$

with

$$\begin{aligned}
K_1 =& \int_{(x,y)\in T_1^d\times[0,1]^d} (l-\nabla_x\chi_l(x+\frac{y}{R},0))\nabla\chi_.^V(Rx+y)(\nabla\chi_l^{D(V),T}(x)-l)\\
& e^{-2(V(Rx+y)+T(x+\frac{y}{R}))}\Big(\frac{1}{\int_{T_1^d} e^{-2(V(Rz+y)+T(z))}dz} - \frac{1}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz}\Big) dx\,dy
\end{aligned} \tag{9.69}$$

and

$$\begin{aligned}
K_2 =& \int_{(x,y)\in T_1^d\times[0,1]^d} (l-\nabla_x\chi_l(x+\frac{y}{R},0))\nabla\chi_.^V(Rx+y)(\nabla\chi_l^{D(V),T}(x)-l)\\
& \frac{e^{-2(V(Rx+y)+T(x+\frac{y}{R}))}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx\,dy
\end{aligned} \tag{9.70}$$

Now, as usual, one obtains that

$$\begin{aligned}
|K_1| \leq & \Big(\int_{y\in T_1^d} {}^t l D(S_R\Theta_y V+T)l\Big)^{\frac{1}{2}} \Big({}^t l D(V,T,R=\infty)l\Big)^{\frac{1}{2}}\\
&\times (e^{2\frac{\|\nabla T\|_\infty}{R}}-1) e^{6\frac{\|\nabla T\|_\infty}{R}+\mathrm{Osc}(V)}
\end{aligned} \tag{9.71}$$

and by noticing that $\nabla_y\Big(e^{-2(V(Rx+y)+T(x+\frac{y}{R}))}(l-\nabla_x\chi_l(x+\frac{y}{R},0))\Big)=0$, $\nabla\chi_.^V(Rx+y)=\nabla_y\chi_.^V(Rx+y)$ and integrating by parts in y, one obtains (writing $\partial^i([0,1]^d)=\{x\in[0,1]^d\,:\,x_i=0\}$)

$$\begin{aligned}
K_2 =& \sum_{i=1}^d \int_{x\in T_1^d, y^i\in\partial^i([0,1]^d)} \Big(e^{-2T(x+\frac{y^i+e_i}{R})}(l-\nabla_x\chi_l(x+\frac{y^i+e_i}{R},0))\\
&- e^{-2T(x+\frac{y^i}{R})}(l-\nabla_x\chi_l(x+\frac{y^i}{R},0))\Big).e_i\\
&\chi_.^V(Rx+y^i)(\nabla\chi_l^{D(V),T}(x)-l)\frac{e^{-2V(Rx+y^i)}}{\int_{T_1^d} e^{-2V(z)}dz\int_{T_1^d} e^{-2T(z)}dz} dx\,dy^i
\end{aligned} \tag{9.72}$$

Now observe that

$$K_2 = G_1 + G_2 \tag{9.73}$$

with

$$\begin{aligned} G_1 =& \sum_{i=1}^{d} \int_{x\in T_1^d, y^i\in\partial^i([0,1]^d)} \Big((e^{-2T(x+\frac{y^i+e_i}{R})} - e^{-2T(x+\frac{y^i}{R})})(l - \nabla_x \chi_l(x + \frac{y^i+e_i}{R}, 0)) \Big).e_i \\ & \chi_.^V(Rx+y^i)(\nabla \chi_l^{D(V),T}(x) - l) \frac{e^{-2V(Rx+y^i)}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx\, dy^i \end{aligned} \tag{9.74}$$

and

$$\begin{aligned} G_2 =& \sum_{i=1}^{d} \int_{x\in T_1^d, y^i\in\partial^i([0,1]^d)} \Big(-\nabla_x \chi_l(x + \frac{y^i+e_i}{R}, 0) + \nabla_x \chi_l(x + \frac{y^i}{R}, 0) \Big).e_i \\ & \chi_.^V(Rx+y^i)(\nabla \chi_l^{D(V),T}(x) - l) e^{-2T(x+\frac{y^i}{R})} \frac{e^{-2V(Rx+y^i)}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx\, dy^i \end{aligned} \tag{9.75}$$

By applying Cauchy inequality for the integration in x to G_1, one obtains

$$\begin{aligned} |G_1| \leq& (e^{\frac{2\|\nabla T\|_\infty}{R}} - 1) \sum_{i=1}^{d} \int_{y^i\in\partial^i([0,1]^d)} \\ & \Big(\int_{x\in T_1^d} \big((l - \nabla_x \chi_l(x + \frac{y^i+e_i}{R}, 0)).e_i\big)^2 \frac{e^{-2V(Rx+y^i)-2T(x+\frac{y^i+e_i}{R})}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx \Big)^{\frac{1}{2}} \\ & \Big(\int_{x\in T_1^d} \big(\chi_.^V(Rx+y^i)(\nabla \chi_l^{D(V),T}(x) - l)\big)^2 \frac{e^{-2V(Rx+y^i)-2T(x+\frac{y^i+e_i}{R})}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx \Big)^{\frac{1}{2}} dy^i \end{aligned} \tag{9.76}$$

which leads to

$$\begin{aligned} |G_1| \leq& (e^{\frac{2\|\nabla T\|_\infty}{R}} - 1) \Big({}^t l D(S_R V + T) l\Big)^{\frac{1}{2}} d \|\chi_.^V\|_\infty e^{3\,\mathrm{Osc}(V)} \\ & \Big({}^t l D(V, T, R=\infty) l\Big)^{\frac{1}{2}} e^{\frac{2\|\nabla T\|_\infty}{R}} \end{aligned} \tag{9.77}$$

Moreover by applying Cauchy inequality for the integration in x to G_2, one obtains

$$\begin{aligned} |G_2| \leq& \sum_{i=1}^{d} \int_{y^i\in\partial^i([0,1]^d)} \Big(\int_{x\in T_1^d} \big((-\nabla_x \chi_l(x + \frac{y^i+e_i}{R}, 0) \\ & + \nabla_x \chi_l(x + \frac{y^i}{R}, 0)).e_i\big)^2 \frac{e^{-2V(Rx+y^i)-2T(x+\frac{y^i}{R})}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx \Big)^{\frac{1}{2}} \\ & \Big(\int_{x\in T_1^d} \big(\chi_.^V(Rx+y^i)(\nabla \chi_l^{D(V),T}(x) - l)\big)^2 \frac{e^{-2V(Rx+y^i)-2T(x+\frac{y^i}{R})}}{\int_{T_1^d} e^{-2V(z)}dz \int_{T_1^d} e^{-2T(z)}dz} dx \Big)^{\frac{1}{2}} dy^i \end{aligned} \tag{9.78}$$

and using the same trick associated to the lemma 9.3.3 one obtains that

$$\begin{aligned} |G_2| \leq& (e^{\frac{8\|\nabla T\|_\infty}{R}} - 1)^{\frac{1}{2}} \Big({}^t l D(S_R V + T) l\Big)^{\frac{1}{2}} d \|\chi_.^V\|_\infty e^{3\,\mathrm{Osc}(V)} \\ & \Big({}^t l D(V, T, R=\infty) l\Big)^{\frac{1}{2}} e^{\frac{2\|\nabla T\|_\infty}{R}} \end{aligned} \tag{9.79}$$

In resume, by using the lemma 9.3.2 and the theorem B.2.1 ($\|\chi_.^V\|_\infty \leq C_d e^{(3d+2)\operatorname{Osc}(V)}$)
it has been obtained that for all $l \in \mathbb{S}^d$

$$\begin{aligned} {}^t lD(S_R V + T)l \leq & e^{12\frac{\|\nabla T\|_\infty}{R}} \sqrt{{}^t lD(S_R V+T)l}\sqrt{{}^t lD(V,T,R=\infty)l} \\ & (1 + C_d\sqrt{e^{8\frac{\|\nabla T\|_\infty}{R}} - 1}e^{(3d+2)\operatorname{Osc}(V)}) \end{aligned} \tag{9.80}$$

which leads to

$$\begin{aligned} {}^t lD(S_R V + T)l \leq & {}^t lD(V,T,R=\infty)le^{24\frac{\|\nabla T\|_\infty}{R}} \\ & (1 + C_d\sqrt{e^{8\frac{\|\nabla T\|_\infty}{R}} - 1}e^{(3d+5)\operatorname{Osc}(V)})^2 \end{aligned} \tag{9.81}$$

□

Corollary 9.3.2. *One has for* $R \geq C_d(1+\|\nabla T\|_\infty)e^{(6d+10)\operatorname{Osc}(V)}$

$$\begin{aligned} D(S_R V + T) \leq & D(V,T,R=\infty) \\ & (1 + C_d\sqrt{\frac{\|\nabla T\|_\infty}{R}}e^{(3d+5)\operatorname{Osc}(V)}) \end{aligned} \tag{9.82}$$

9.3.4 With an arbitrary large number of scales

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C^\infty(T_1^d)$ with for all n, $\operatorname{Osc}(U_n) \leq K_0$ and $\|\nabla U_n\|_\infty \leq K_1$
$(r_n)_{n\in\mathbb{N}}$ a sequence of integer in $\mathbb{N}^*$ with for $n \geq 1$, $r_n \geq \rho_{\min} \geq 2$
Write $R_n = r_0 \cdots r_n$ and

$$V_p^n(x) = \sum_{k=p}^{n} U_k(\frac{xR_n}{R_k})$$

Theorem 9.3.2. *There exists* $C_d > 0$ *such that if* $\rho_{\min} \geq C_d(1+K_1)e^{(6d+10)K_0}$ *then for all* $n \in \mathbb{N}$ *one has*

$$\lambda_{\max}\big(D(V_0^{n-1})\big) \leq \prod_{k=0}^{n-1} \lambda_{\max}\big(D(U_k)\big) \times \big(1 + C_d\frac{e^{(3d+5)K_0}\sqrt{K_1}}{\sqrt{\rho_{\min}}}\big)^n \tag{9.83}$$

Proof. The result follows by a simple recurrence showing that for all $p \leq n-1$

$$\lambda_{\max}\big(D(V_p^{n-1})\big) \leq \prod_{k=p}^{n-1} \lambda_{\max}\big(D(U_k)\big) \times \big(1 + C_d\frac{e^{(3d+5)K_0}\sqrt{K_1}}{\sqrt{\rho_{\min}}}\big)^{n-p} \tag{9.84}$$

This is trivially true for $p = n-1$, assume that this is true for $p = m > 0$, then apply the corollary 9.3.2 with $T = V_m^n$, $V = U_{m-1}$, $R = \frac{R_n}{R_{m-1}}$ to obtain the result by observing that

$$\frac{\|\nabla T\|_\infty}{R} \leq K_1 \sum_{k=m}^{n-1} \frac{R_{m-1}}{R_k} \leq 2/\rho_{\min} \tag{9.85}$$

□

9.4 Lower Bound

9.4.1 Cohomological framework

Let $U \in C^\infty(T_1^d)$
We write $\mathcal{C} = \left(C^\infty(T_1^d)\right)^d$ the set of C^∞ vector fields on T_1^d and $Q = L^2(m_{-U})$ the completion of $\mathcal{C}$ with respect to the norm $\|.\|_Q$ where for $\xi \in \mathcal{C}$

$$\|\xi\|_Q^2 = m_{-U}(\xi^2)$$

Thus Q is a real Hilbert space equipped with the scalar product

$$(\xi, \nu)_Q = m_{-U}(\xi.\nu)$$

Write

$$\mathcal{F}_{pot} = \left\{\xi \in \mathcal{C} \,|\, \exists f \in C^\infty(T_1^d)\; l \in \mathbb{R}^d \text{ with } \xi = \frac{\exp(-2U)}{\int_{T_1^d} e^{-2U} dx}(l + \nabla f)\right\}$$

$$\mathcal{F}_{sol} = \left\{p \in \mathcal{C} |\, \mathrm{div}(p) = 0 \text{ and } \int_{T_1^d} p dx = 0\right\}$$

and Q_{pot}, Q_{sol} the closure of $\mathcal{F}_{pot}$, $\mathcal{F}_{sol}$ in Q with respect to the norm $\|.\|_Q$ Then just as for the upper bound, the following orthogonal decomposition can easily be proved

$$Q = Q_{pot} \oplus Q_{sol} \tag{9.86}$$

Moreover by the variational formula 5.22, for $\xi \in \mathbb{R}^d$

$$\left[\frac{{}^t\xi D(U)^{-1}\xi}{\int_{T_1^d} \exp(-2U(x))dx \int_{T_1^d} \exp(2U(x))dx}\right]^{\frac{1}{2}} = \mathrm{dist}(\xi, Q_{sol}) \tag{9.87}$$

is then norm in Q of the orthogonal projection of ξ on Q_{pot} (which is equivalent to say to projection on Q_{pot} parallel to Q_{sol}). Moreover, let's remember (see sub subsection 5.1.5.iii) that the unique solution of the variational problem 5.22 is given by $p_\xi = P.\xi$ where P is the matrix

$$P = I_d - \frac{\exp(-2U)}{\int_{T_1^d} e^{-2U} dx}(I_d - \nabla\chi_.)D(U)^{-1} \tag{9.88}$$

$\chi_.$ is the solution of the cell problem associated to U and

$$\xi = P.\xi + \frac{\exp(-2U)}{\int_{T_1^d} e^{-2U} dx}(I_d - \nabla\chi_.)D(U)^{-1}\xi \tag{9.89}$$

is the orthogonal decomposition of ξ. Moreover

$$\frac{D(U)^{-1}}{\int_{T_1^d} e^{-2U} dx \int_{T_1^d} e^{2U} dx} = m_{-U}\left({}^t(I_d - P)(I_d - P)\right) \tag{9.90}$$

Representation of solenoidal vector fields

Lemma 9.4.1. *There exists a $d \times d \times d$ tensor H_{ijm} such that $H_{ijm} = -H_{jim} \in C^\infty(T_1^d)$,*

$$P_{im} = \sum_{j=1}^d \partial_j H_{ijm} \tag{9.91}$$

and

$$\|H_{ijm}\|_\infty \leq C_d e^{(3d+6)\,\mathrm{Osc}(U)}(1 + \|\nabla U\|_\infty) \tag{9.92}$$

Proof. Since for each $m \in \{1,\dots,d\}$, $P_{.,m} \in \mathcal{F}_{sol}$, by the proposition 4.1 of [JK99] there exists a skew-symmetric T_1^d-periodic smooth matrices $H_{ij1},\dots,H_{ijd}$ ($H_{ijm} = -H_{jim}$) such that for all m

$$P_{im} = \sum_{j=1}^{d} \partial_j H_{ijm} \tag{9.93}$$

Moreover writing

$$P_{.m} = \sum_{k\neq 0} p_{.m}^k e^{2i\pi(k.x)} \tag{9.94}$$

the Fourier series expansion of P, one has (see the proposition 4.1 of [JK99])

$$H_{njm} = \frac{1}{2i\pi} \sum_{k\neq 0} \frac{p_{nm}^k k_j - p_{jm}^k k_n}{k^2} e^{2i\pi(k.x)} \tag{9.95}$$

Now, observe that

$$H_{njm} = \partial_j B_{nm} - \partial_n B_{jm} \tag{9.96}$$

where B_{nm} and B_{jm} are the smooth T_1^d-periodic solutions of

$$\Delta B_{nm} = P_{nm} \qquad \Delta B_{jm} = P_{jm} \tag{9.97}$$

Now using G. Stampacchia's theorem B.1.1, it is easy to see that if B_{nm} is chosen so that $\int_{T_1^d} B_{nm}(x)dx = 0$ then $\|B_{nm}\|_\infty \leq C_d \|P_{nm}\|_\infty$. Now using the theorem B.1.2 on Gradient estimates for Poisson's equation, it is simple to obtain that

$$\|\nabla B_{nm}\|_\infty \leq C_d \|P_{nm}\|_\infty \tag{9.98}$$

Then it follows that

$$\|H_{njm}\|_\infty \leq C_d(\|P_{nm}\|_\infty + \|P_{jm}\|_\infty) \tag{9.99}$$

Which leads to the result of the lemma by using the expression of P and the theorem B.2.1 which allows to control $\|\nabla\chi\|_\infty$ □

Duality The following lemma is just a remark that will not be used for the final proof, nevertheless, it might be interesting to notice the variational formulation associated to it.

Lemma 9.4.2. *For all $\xi \in Q$,*

$$\operatorname{dist}(\xi, Q_{sol}) = \sup_{\delta\in\mathcal{F}_{pot}} \frac{(\delta,\xi)_Q}{\|\delta\|_Q} \tag{9.100}$$

Proof. This is a direct consequence of the orthogonal decomposition 9.86 and the density of $\mathcal{F}_{pot}$ in Q_{pot} □

Remark 9.4.1. This lemma gives the following variational formula for the effective diffusivity $D(U)$: for $\xi \in \mathbb{R}^d$

$$^t\xi D(U)^{-1}\xi = \sup_{l\in\mathbb{R}^d} \sup_{f\in C^\infty(T_1^d)} \frac{\left(\int_{T_1^d} \xi.(l+\nabla f)dx\right)^2}{\int_{T_1^d} |l+\nabla f|^2 m_U(dx)} \tag{9.101}$$

9.4.2 Lower bound with two scales

Let $R \in \mathbb{N}^*$, V, $T \in C^\infty(T_1^d)$, then under the notations of subsection 9.3.3 write

$$P(x,y) = I_d - \frac{\exp(-2(S_R\Theta_y V + T))}{\int_{T_1^d} \exp(-2(S_R\Theta_y V + T)(x))dx}(I_d - \nabla\chi(x,y).)D(S_R\Theta_y V + T)^{-1} \tag{9.102}$$

which is the matrix giving the elements of Q_{sol} associated to the homogenization $S_R\Theta_y V + T$. Write also P^V the element of Q_{sol} associated to V and

$$P^{D(V),T}(x) = I_d - \frac{e^{-2T(x)}}{\int_{T_1^d} e^{-2T(x)}dx} D(V)(I_d - \nabla\chi^{D(V),T}(x).)D(V,T,R=\infty)^{-1} \tag{9.103}$$

which corresponds to a complete homogenization on the smaller scale.

Proposition 9.4.1. *One has*

$$\begin{aligned} D(S_R V + T)^{-1} \leq & D(V,T,R=\infty)^{-1} \\ & \times e^{20\frac{\|\nabla T\|_\infty}{R}}\left(1 + C_d(1+\|\nabla V\|_\infty)e^{(3d+8)\,\mathrm{Osc}(V)}(e^{8\frac{\|\nabla T\|_\infty}{R}} - 1)^{\frac{1}{2}}\right)^2 \end{aligned} \tag{9.104}$$

Proof. Observe that for $\xi \in \mathbb{S}^d$ one has by the equation 9.90.

$$\begin{aligned} &\int_{y\in T_1^d} {}^t\xi D(S_R\Theta_y V + T)^{-1}\xi \, dy \\ &= \int_{(x,y)\in(T_1^d)^2} \left(\int_{T_1^d} e^{-2(S_R\Theta_y V+T)(z)}dz\right) e^{2(S_R\Theta_y V+T)(x)t}\xi^t(I_d - P(x,y))\xi \, dx\, dy \\ &\leq e^{\frac{2\|\nabla T\|_\infty}{R}} \int_{T_1^d} e^{-2V(z)}\, dz \int_{T_1^d} e^{-2T(z)}\, dz \\ &\int_{(x,y)\in(T_1^d)^2} e^{2(S_R\Theta_y V+T)(x)t}\xi^t(I_d - P(x,y))\xi \, dx\, dy \\ &\leq e^{\frac{2\|\nabla T\|_\infty}{R}}(I_1 + I_2) \end{aligned} \tag{9.105}$$

with

$$\begin{aligned} I_1 = &\int_{T_1^d} e^{-2V(z)}\, dz \int_{T_1^d} e^{-2T(z)}\, dz \int_{(x,y)\in(T_1^d)^2} e^{2(S_R\Theta_y V+T)(x)t}\xi^t(I_d - P(x,y)) \\ &(I_d - P^V(Rx+y))(I_d - P^{D(V),T}(x))\xi \, dx\, dy \end{aligned} \tag{9.106}$$

$$\begin{aligned} I_2 = &\int_{T_1^d} e^{-2V(z)}\, dz \int_{T_1^d} e^{-2T(z)}\, dz \int_{(x,y)\in(T_1^d)^2} e^{2(S_R\Theta_y V+T)(x)t}\xi^t(I_d - P(x,y)) \\ &P^V(Rx+y)(I_d - P^{D(V),T}(x))\xi \, dx\, dy \end{aligned} \tag{9.107}$$

Now as in the proof of proposition 9.3.5, by using Cauchy Schwartz inequality for the integration in x then y, one obtains that:

$$\begin{aligned} |I_1| \leq &\int_{T_1^d} e^{-2V(z)}\, dz \int_{T_1^d} e^{-2T(z)}\, dz \Big(\int_{(x,y)\in(T_1^d)^2} e^{2(S_R\Theta_y V+T)(x)}((I_d - P(x,y))\xi)^2 \\ &dx\, dy\Big)^{\frac{1}{2}} \\ &\left(\int_{(x,y)\in(T_1^d)^2} e^{2(S_R\Theta_y V+T)(x)}\big((I_d - P^V(Rx+y))(I_d - P^{D(V),T}(x))\xi\big)^2 dx\, dy\right)^{\frac{1}{2}} \\ &\leq e^{\frac{\|\nabla T\|_\infty}{R}}\left(\int_{y\in T_1^d} {}^t\xi D(S_R\Theta_y V + T)^{-1}\xi \, dy\right)^{\frac{1}{2}}\left({}^t\xi D(V,T,R=\infty)^{-1}\xi\right)^{\frac{1}{2}} \end{aligned} \tag{9.108}$$

Next observe that

$$I_2 = J_1 + J_2 \tag{9.109}$$

with

$$\begin{aligned} J_1 = \int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz \int_{(x,y)\in(T_1^d)^2} & e^{2(S_R\Theta_y V+T)(x)} \\ {}^t\xi^t(P(x+\frac{y}{R},0)-P(x,y))P^V(Rx+y)&(I_d-P^{D(V),T}(x))\xi\,dx\,dy \end{aligned} \tag{9.110}$$

and

$$\begin{aligned} J_2 = &\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz \int_{(x,y)\in(T_1^d)^2} e^{2(S_R\Theta_y V+T)(x)t}\xi^t(I_d-P(x+\frac{y}{R},0)) \\ &P^V(Rx+y)(I_d-P^{D(V),T}(x))\xi\,dx\,dy \end{aligned} \tag{9.111}$$

Now by using Cauchy-Schwartz inequality for the integration in x and y in J_1, one obtains that

$$|J_1| \leq K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \tag{9.112}$$

with

$$\begin{aligned} K_1 = &\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz \\ &\int_{(x,y)\in(T_1^d)\times[0,1]^d} e^{2(S_R\Theta_y V+T)(x)}\big(((P(x+\frac{y}{R},0)-P(x,y))\xi\big)^2\,dx\,dy \end{aligned} \tag{9.113}$$

and

$$\begin{aligned} K_2 = &\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz \\ &\int_{(x,y)\in(T_1^d)\times[0,1]^d} e^{2(S_R\Theta_y V+T)(x)}\big(P^V(Rx+y)(I_d-P^{D(V),T}(x))\xi\big)^2\,dx\,dy \\ = &\int_{T_1^d} e^{-2T(z)}\,dz \int_{x\in T_1^d} e^{2T(x)t}\xi^t(I_d-P^{D(V),T}(x)) \\ &(I_d\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{2V(z)}\,dz - D(V)^{-1})(I_d-P^{D(V),T}(x))\xi\,dx\,dy \\ \leq & e^{2\operatorname{Osc}(V)} D(V,T,R=\infty)^{-1} \end{aligned} \tag{9.114}$$

where in the last inequality,
$(I_d\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{2V(z)}\,dz - D(V)^{-1}) \leq e^{2\operatorname{Osc}(V)} D(V)^{-1}$ has been used.
Concerning K_1, observe that since $P(x,y).\xi$ minimizes the following variational formula on $p \in Q_{sol}$,

$$\int_{T_1^d} e^{2(S_R\Theta_y V+T)(x)}|\xi-p(x)|^2\,dx \tag{9.115}$$

one has for $y\in[0,1]^d$

$$\begin{aligned} \int_{x\in T_1^d} & e^{2(S_R\Theta_y V+T)(x)}\big(((P(x+\frac{y}{R},0)-P(x,y))\xi\big)^2\,dx = \\ &\int_{x\in T_1^d} e^{2(S_R\Theta_y V+T)(x)}\big(((P(x+\frac{y}{R},0)-I_d)\xi\big)^2\,dx \\ &-\int_{x\in T_1^d} e^{2(S_R\Theta_y V+T)(x)}\big(((I_d-P(x,y))\xi\big)^2\,dx \end{aligned} \tag{9.116}$$

from which it follows that

$$|K_1| \leq e^{2\frac{\|\nabla T\|_\infty}{R}} {}^t\xi D(S_R V + T)^{-1}\xi - e^{-2\frac{\|\nabla T\|_\infty}{R}} \int_{y\in[0,1]^d} {}^t\xi D(S_R \Theta_y V + T)^{-1}\xi \, dy \tag{9.117}$$

and using the lemma 9.3.2 it follows that

$$|K_1| \leq (e^{6\frac{\|\nabla T\|_\infty}{R}} - e^{-2\frac{\|\nabla T\|_\infty}{R}}) \int_{y\in[0,1]^d} {}^t\xi D(S_R \Theta_y V + T)^{-1}\xi \, dy \tag{9.118}$$

Concerning J_2, observe that

$$J_2 = G_1 + G_2 \tag{9.119}$$

with

$$\begin{aligned} G_1 = &\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz \int_{(x,y)\in T_1^d\times[0,1]^d} (e^{2(S_R\Theta_y V+T)(x)} \\ &- e^{2(V(Rx+y)+T(x+\frac{y}{R}))}) \\ &{}^t\xi^t(I_d - P(x+\frac{y}{R},0))P^V(Rx+y)(I_d - P^{D(V),T}(x))\xi\,dx\,dy \end{aligned} \tag{9.120}$$

and

$$\begin{aligned} G_2 = &\int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz \int_{(x,y)\in T_1^d\times[0,1]^d} e^{2(V(Rx+y)+T(x+\frac{y}{R}))} \\ &{}^t\xi^t(I_d - P(x+\frac{y}{R},0))P^V(Rx+y)(I_d - P^{D(V),T}(x))\xi\,dx\,dy \end{aligned} \tag{9.121}$$

As usual, by using Cauchy Schwartz inequality one obtains that

$$|G_1| \leq e^{\frac{\|\nabla T\|_\infty}{R}}(e^{2\frac{\|\nabla T\|_\infty}{R}} - 1)e^{\operatorname{Osc}(V)}\Big({}^t\xi D(S_R V + T)\xi\Big)^{\frac{1}{2}}\Big({}^t\xi D(V,T,R=\infty)\xi\Big)^{\frac{1}{2}} \tag{9.122}$$

Now observe that

$$G_2 = \int_{T_1^d} e^{-2V(z)}\,dz \int_{T_1^d} e^{-2T(z)}\,dz\; L_1 \tag{9.123}$$

with

$$\begin{aligned} L_1 = &\int_{(x,y)\in T_1^d\times[0,1]^d} e^{2(V(Rx+y)+T(x+\frac{y}{R}))} \\ &{}^t\xi^t(I_d - P(x+\frac{y}{R},0))P^V(Rx+y)(I_d - P^{D(V),T}(x))\xi\,dx\,dy \\ &= \frac{1}{\int_{T_1^d}\exp(-2(S_R V+T)(z))dz}\int_{(x,y)\in T_1^d\times[0,1]^d} {}^t\xi^t D(S_R V+T)^{-1} \\ &{}^t\big(I_d - \nabla\chi.(x+\frac{y}{R})\big)P^V(Rx+y)(I_d - P^{D(V),T}(x))\xi\,dx\,dy \end{aligned} \tag{9.124}$$

Now, let H^V_{ijk} be the $d\times d\times d$ tensor associated to P^V in the lemma 9.4.1. It follows by an integration by parts (in y) that

$$\begin{aligned} L_1 = &\frac{1}{\int_{T_1^d}\exp(-2(S_R V+T)(z))dz}\int_{(x,y)\in T_1^d\times[0,1]^d}\sum_{i,j,k=1}^d \\ &({}^t\xi^t(I_d - P^{D(V),T}(x)))_i \partial_k H^V_{j,k,i}(Rx+y) \\ &\Big((I_d - \nabla\chi.(x+\frac{y}{R}))D(S_R V+T)^{-1}\xi\Big)_j\,dx\,dy \end{aligned} \tag{9.125}$$

Thus by using the same notation as in the equation 9.72, one has

$$\begin{aligned} L_1 = & \frac{1}{\int_{T_1^d} \exp(-2(S_R V + T)(z)) dz} \sum_{i,j,k=1}^{d} \int_{(x,y^k)\in T_1^d\times\partial^k([0,1]^d)} ({}^t\xi^t(I_d - P^{D(V),T}(x)))_i \\ & H^V_{j,k,i}(Rx+y^k)\big((\nabla\chi_.(x+\frac{y^k}{R}) - \nabla\chi_.(x+\frac{y^k+e_k}{R}))D(S_RV+T)^{-1}\xi\big)_j \, dx\, dy^k \end{aligned} \tag{9.126}$$

Which leads to (using Cauchy Schwartz inequality):

$$\begin{aligned} |L_1| \le & \frac{C_d \sup_{ijk} \|H^V_{j,k,i}\|_\infty}{\int_{T_1^d} \exp(-2(S_R V + T)(z)) dz} \sum_{k=1}^{d} \int_{y^k\in\partial^k([0,1]^d)} \\ & \Big(\int_{x\in T_1^d} \big((I_d - P^{D(V),T}(x))\xi\big)^2 e^{2(V(Rx+y^k)+T(x+\frac{y^k}{R}))} dx\Big)^{\frac12} \\ & \Big(\int_{x\in T_1^d} \Big(\big((\nabla\chi_.(x+\frac{y^k}{R}) - \nabla\chi_.(x+\frac{y^k+e_k}{R})\big) \\ & D(S_RV+T)^{-1}\xi\Big)^2 e^{-2(V(Rx+y^k)+T(x+\frac{y^k}{R}))} dx\Big)^{\frac12} dy^k \end{aligned} \tag{9.127}$$

But using the same trick associated to the lemma 9.3.3 one obtains that

$$\begin{aligned} & \int_{x\in T_1^d} \Big(\big((\nabla\chi_.(x+\frac{y^k}{R}) - \nabla\chi_.(x+\frac{y^k+e_k}{R})\big)D(S_RV+T)^{-1}\xi\Big)^2 \\ & \frac{e^{-2(V(Rx+y^k)+T(x+\frac{y^k}{R}))}}{\int_{T_1^d}\exp(-2(S_RV+T)(z))dz} dx \le \xi D(S_RV+T)^{-1}\xi(e^{4\frac{\|\nabla T\|_\infty}{R}} - 1) \end{aligned} \tag{9.128}$$

and by observing that

$$\begin{aligned} & \int_{T_1^d} e^{-2(S_RV+T)(x)} dx \int_{x\in T_1^d} \big((I_d - P^{D(V),T}(x))\xi\big)^2 \\ & e^{2(V(Rx+y^k)+T(x+\frac{y^k}{R}))} dx \le e^{4\,\mathrm{Osc}(V)} {}^t\xi D(V,T,R=\infty)^{-1}\xi \end{aligned} \tag{9.129}$$

it follows that (using the lemma 9.4.1)

$$\begin{aligned} |G_2| \le & C_d e^{(3d+8)\,\mathrm{Osc}(V)}(1+\|\nabla V\|_\infty) e^{4\frac{\|\nabla T\|_\infty}{R}} (e^{4\frac{\|\nabla T\|_\infty}{R}} - 1)^{\frac12} \Big(\xi D(S_RV+T)^{-1}\xi\Big)^{\frac12} \\ & \Big({}^t\xi D(V,T,R=\infty)^{-1}\xi\Big)^{\frac12} \end{aligned} \tag{9.130}$$

In resume, summing up all the inequalities and using the lemma 9.3.2, it has obtained that

$$\begin{aligned} {}^t\xi D(S_RV+T)\xi \le & e^{10\frac{\|\nabla T\|_\infty}{R}} (1 + C_d(1+\|\nabla V\|_\infty) e^{(3d+8)\,\mathrm{Osc}(V)} (e^{8\frac{\|\nabla T\|_\infty}{R}} - 1)^{\frac12}) \\ & \Big(\xi D(S_RV+T)^{-1}\xi\Big)^{\frac12} \Big({}^t\xi D(V,T,R=\infty)^{-1}\xi\Big)^{\frac12} \end{aligned} \tag{9.131}$$

which leads to

$$\begin{aligned} D(S_RV+T)^{-1} \le & e^{20\frac{\|\nabla T\|_\infty}{R}} (1 + C_d(1+\|\nabla V\|_\infty) e^{(3d+8)\,\mathrm{Osc}(V)} (e^{8\frac{\|\nabla T\|_\infty}{R}} - 1)^{\frac12})^2 \\ & D(V,T,R=\infty)^{-1} \end{aligned} \tag{9.132}$$

□

Corollary 9.4.1. *One has for* $R \ge C_d(1+\|\nabla T\|_\infty)(1+\|\nabla V\|_\infty)^2 e^{(6d+16)\,\mathrm{Osc}(V)}$

$$\begin{aligned} D(S_RV+T) \ge & D(V,T,R=\infty) \\ & \times \frac{1}{1 + \frac{1}{\sqrt{R}} C_d (1+\|\nabla V\|_\infty) e^{(3d+8)\,\mathrm{Osc}(V)} \|\nabla T\|_\infty^{\frac12}} \end{aligned} \tag{9.133}$$

9.4.3 Lower bound with an arbitrary large number of scales

In the context of the subsection 9.3.4

Theorem 9.4.1. *There exists $C_d > 0$ such that if $\rho_{\min} \geq C_d(1+K_1)^3 e^{(6d+16)K_0}$ then for all $n \in \mathbb{N}$ one has*

$$\lambda_{\min}\big(D(V_0^{n-1})\big) \geq \prod_{k=0}^{n-1} \lambda_{\min}\big(D(U_k)\big) \times \big(1 + C_d(1+K_1)\frac{e^{(3d+8)K_0}\sqrt{K_1}}{\sqrt{\rho_{\min}}}\big)^{-n} \tag{9.134}$$

Proof. The result follows by a simple recurrence showing that for all $p \leq n-1$

$$\lambda_{\min}\big(D(V_p^{n-1})\big) \geq \prod_{k=p}^{n-1} \lambda_{\min}\big(D(U_k)\big) \times \big(1 + C_d(1+K_1)\frac{e^{(3d+8)K_0}\sqrt{K_1}}{\sqrt{\rho_{\min}}}\big)^{n-p} \tag{9.135}$$

This is trivially true for $p = n-1$, assume that this is true for $p = m > 0$, then apply the corollary 9.4.1 with $T = V_m^n$, $V = U_{m-1}$, $R = \frac{R_n}{R_{m-1}}$ to obtain the result by observing that

$$\frac{\|\nabla T\|_\infty}{R} \leq K_1 \sum_{k=m}^{n-1} \frac{R_{m-1}}{R_k} \leq 2/\rho_{\min} \tag{9.136}$$

□

9.5 Overlapping ratios

Proposition 9.5.1. *Let $U \in C^\infty(T_1^d)$ such that $\int_{T_1^d} U(x)\,dx = 0$ and $R \in \mathbb{N}/\{0,1\}$, then*

$$\mathcal{P}_R(2U) + \mathcal{P}_R(-2U) = 0 \Leftrightarrow \lim_{n\to\infty} \frac{1}{n} \|\sum_{k=0}^{n-1} S_{R^k} U\|_\infty = 0 \tag{9.137}$$

Proof. ($\Leftarrow$): This implication is easy since

$$0 \leq \mathcal{P}_R(2U) + \mathcal{P}_R(-2U) \leq \lim_{n\to\infty} \frac{4}{n} \|\sum_{k=0}^{n-1} S_{R^k} U\|_\infty \tag{9.138}$$

($\Rightarrow$): Assume $\mathcal{P}_R(2U) + \mathcal{P}_R(-2U) = 0$ then let $\epsilon > 0$. Then there exists $V_1, \ldots, V_k \in C(T_1^d)$ and $m_1, \ldots, m_k \in \mathbb{N}/\{0,1\}$, $\lambda_1, \ldots, \lambda_k \in R$ such that

$$V = \sum_{p=1}^{k} \lambda_p (V_p - S_{R^{m_p}} V_p) \tag{9.139}$$

and

$$\|U - V\|_\infty \leq \epsilon \tag{9.140}$$

Observe then that since $\sum_{p=0}^{n-1} S_{R^p} V$ remains bounded it follows that

$$\lim_{n\to\infty} \frac{1}{n} \|\sum_{k=0}^{n-1} S_{R^k} U\|_\infty \leq \epsilon \tag{9.141}$$

which leads to the proof. □

9.5.1 A simple example

Consider the function

$$f(x) = \sin(x) - \sin(81x) \tag{9.142}$$

Observe that for $R \in \mathbb{N}/\{0,1\}$ if $R \neq 3, 27, 81$ then

$$\int_0^1 \big(\sum_{k=0}^{n-1} f(R^k x)\big)^2 = n \tag{9.143}$$

it follows by the proposition 9.5.1 that

$$\mathcal{P}_R(f) + \mathcal{P}_R(f) = 0 \Leftrightarrow R = 3, 27\,\text{or}\,81 \tag{9.144}$$

and between these ratios $\mathcal{P}_R(f) + \mathcal{P}_R(f) > 0$

10. SUB DIFFUSIVE BEHAVIOR OF AN IHPD IN ALL DIMENSIONS

The purpose of this chapter is to show how from a sharp geometric control on the multi scale effective diffusivities (given in the chapter 9) one can deduce the anomalous behavior of an infinitely homogenized potential diffusion.

10.0.2 Sharp control of the effective medium

Let $R > 0$ and $U \in C^\infty(T_R^d)$
Write y_t the solution of the stochastic differential equation

$$dy_t = d\omega_t - \nabla U(y_t)dt \tag{10.1}$$

Write for $x \in \mathbb{R}^d$, $r > 0$ and $l \in \mathbb{S}^d$

$$\tau(x,r,l) = \inf\{t \geq 0 \ : \ |(y_t - x).l| = r\}$$

$$\tau(x,r) = \inf\{t \geq 0 \ : \ |y_t - x| = r\}$$

Theorem 10.0.1. *Let y_t be the solution of 10.1, then*

$$\mathbb{E}[\tau(x,r,l)] \leq C_2 \frac{r^2}{{}^t l D(U) l} + C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2 \tag{10.2}$$

$$\mathbb{E}[\tau(x,r,l)] \geq C_1 \frac{r^2}{{}^t l D(U) l} - C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2 \tag{10.3}$$

$$\mathbb{E}[\tau(x,r)] \leq C_2 \frac{r^2}{\lambda_{\max}\big(D(U)\big)} + C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2 \tag{10.4}$$

$$\mathbb{E}[\tau(x,r)] \geq C_1 \frac{r^2}{\lambda_{\max}\big(D(U)\big)} - C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2 \tag{10.5}$$

Proof. A proof will be given in the $x = 0$ case, the proof in the general case being quite similar.
Let $l \in \mathbb{S}^d$
Write χ_l the T_R^d-periodic solution of the cell problem associated to L_U with $\chi_l(0) = 0$.
Write ϕ_l the T_R^d-periodic solution of the ergodicity problem

$$L_U \phi_l = |l - \nabla \chi_l|^2 - {}^t l D(U) l \tag{10.6}$$

with $\phi_l(0) = 0$. Write $F_l(x) = l.x - \chi_l(x)$ and $\psi_l(x) = F_l^2(x) - \phi_l(x)$, observe that since $L_U F_l^2 = |l - \nabla \chi_l|^2$ it follows that

$$L_U \psi_l = {}^t l D(U) l \tag{10.7}$$

Thus by the Ito formula

$$\psi_l(y_t) = \int_0^t \nabla\psi_l(y_s)d\omega_s \; + \; {}^t l D(U) l \, t \tag{10.8}$$

Now write $M_{t,l}$ the martingale

$$M_{t,l} = \psi_l(y_t) - {}^t l D(U) l \, t \tag{10.9}$$

Notice that

$$C_1|l.x|^2 - C_2(\|\chi_l\|_\infty^2 + \|\phi_l\|_\infty) \leq \psi_l(x) \leq C_3(|l.x|^2 + \|\chi_l\|_\infty^2 + \|\phi_l\|_\infty) \tag{10.10}$$

Now by the theorems B.2.1, B.2.2 and the lemma B.2.1 one has

$$\|\chi_l\|_\infty^2 + \|\phi_l\|_\infty \leq C_d e^{(9d+13)\,\mathrm{Osc}(U)} R^2 \tag{10.11}$$

Write

$$\tau'(0,r,l) = \inf\{t \geq 0 \;:\; |\psi_l(y_t)| = r\}$$

According to the inequality 10.10 one has

$$\tau(0,r,l) \leq \tau'(0, C_3(r^2 + \|\chi_l\|_\infty^2 + \|\phi_l\|_\infty), l) \tag{10.12}$$

$$\tau(0,r,l) \geq \tau'(0, C_1 r^2 - C_2(\|\chi_l\|_\infty^2 + \|\phi_l\|_\infty), l) \tag{10.13}$$

Since $M_{t\wedge\tau'(0,r,l),l}$ is uniformly integrable (easy to prove by using the inequalities 10.12 and 10.13) one obtains

$$\mathbb{E}[\tau'(0,r,l)] = \frac{r}{{}^t l D(U) l} \tag{10.14}$$

Thus, by using the inequality 10.11 and the Voigt-Reiss' inequality $D(U) \geq e^{-2\,\mathrm{Osc}(U)}$ one obtains

$$\tau(0,r,l) \leq C_3 \frac{r^2}{{}^t l D(U) l} + C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2$$

$$\tau(0,r,l) \geq C_1 \frac{r^2}{{}^t l D(U) l} - C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2$$

Now if one defines M_t to be the martingale

$$M_t = \sum_{i=1}^d M_{t,e_i}$$

and if one uses the following stopping times:

$$\tau'(0,r) = \inf\{t \geq 0 \;:\; |\sum_{i=1}^d \psi_{e_i}(y_t)| = r\}$$

one obtains obtain as before

$$\mathbb{E}[\tau(x,r)] \leq C_2' \frac{r^2}{\sup_{|l|=1} {}^t l D(U) l} + C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2$$

$$\mathbb{E}[\tau(x,r)] \geq C_1' \frac{r^2}{\sup_{|l|=1} {}^t l D(U) l} - e^{(9d+15)\,\mathrm{Osc}(U)} R^2$$

□

10.0.3 Perturbation of the effective medium

10.0.3.i Weak stability result

Let $U, P \in C^\infty(\bar{\Omega})$, and Ω a smooth bounded open subset of $\mathbb{R}^d$. Write $\mathbb{E}^U$, $\mathbb{E}^{U+P}$ the expectations associated to the diffusions generated by L_U and L_{U+P} and $\tau(\Omega)$ the exit time from Ω. Write m_U^Ω the following probability measure on Ω:

$$m_U^\Omega(dx) = \frac{e^{-2U(x)}\,dx}{\int_\Omega e^{-2U(x)}\,dx} \tag{10.15}$$

Proposition 10.0.2.

$$\begin{aligned}\int_\Omega \mathbb{E}_x^U\big[\tau(\Omega)\big] m_{U+P}^\Omega(dx) &\leq e^{2\,\mathrm{Osc}(P)} \int_\Omega \mathbb{E}_x^{U+P}\big[\tau(\Omega)\big] m_{U+P}^\Omega(dx) \\ &\geq e^{-2\,\mathrm{Osc}(P)} \int_\Omega \mathbb{E}_x^{U+P}\big[\tau(\Omega)\big] m_{U+P}^\Omega(dx)\end{aligned} \tag{10.16}$$

Proof. This is a direct consequence of the theorem 13.5.4 by observing that

$$\mathbb{E}_x^U\big[\tau(\Omega)\big] = 2\int_\Omega G_U(x,y) e^{-2U(y)}\,dy \tag{10.17}$$

where G_U is the Green function on Ω associated to the operator $-\nabla\big(e^{-2U}\nabla\big)$ with Dirichlet conditions on the boundary. □

10.0.3.ii Strong stability conjecture I

Let $U \in C^\infty(T_1^d)$ and $P \in C^\infty(B(\bar{0},1))$. Write $\mathbb{E}^{S_R U}$, $\mathbb{E}^{S_R U+P}$ the expectations associated to the diffusions generated by $L_{S_R U}$ and $L_{S_R U+P}$ and $\tau(B(0,1))$ the exit time from the d dimensional unit ball $B(0,1)$.

Conjecture 10.0.1. *There exists $C_d > 0$ a constant depending only on the dimension such that for*

$$R > C_d e^{C_d(\mathrm{Osc}(U)+\mathrm{Osc}(P))} \qquad \mathrm{Osc}(P) < \frac{1}{C_d}\mathrm{Osc}(U) \tag{10.18}$$

and

$$\|\nabla P\|_\infty < \frac{R}{C_d} \tag{10.19}$$

one has

$$E_0^{S_R U+P}\big[\tau(B(0,1))\big] \leq C_d e^{C_d\,\mathrm{Osc}(P)} \sup_{x\in B(0,1)} E_x^{S_R U}\big[\tau(B(0,1))\big] \tag{10.20}$$

and

$$E_0^{S_R U+P}\big[\tau(B(0,1))\big] \geq C_d e^{-C_d\,\mathrm{Osc}(P)} \inf_{x\in B(0,\frac{1}{2})} E_x^{S_R U}\big[\tau(B(0,1))\big] \tag{10.21}$$

10.0.3.iii Strong stability conjecture II

Let $U, P \in C^\infty(B(\bar{0},1))$. Write $\mathbb{E}^U$, $\mathbb{E}^{U+P}$ the expectations associated to the diffusions generated by L_U and L_{U+P} and $\tau(B(0,1))$ the exit time from the d dimensional unit ball $B(0,1)$.

Conjecture 10.0.2. *The exists $C_d > 0$ a constant depending only on the dimension such that*

$$E_0^{U+P}\big[\tau(B(0,1))\big] \leq C_d e^{C_d\,\mathrm{Osc}(P)} \sup_{x\in B(0,1)} E_x^U\big[\tau(B(0,1))\big] \tag{10.22}$$

and

$$E_0^{U+P}\big[\tau(B(0,1))\big] \geq C_d e^{-C_d\,\mathrm{Osc}(P)} \inf_{x\in B(0,\frac{1}{2})} E_x^U\big[\tau(B(0,1))\big] \tag{10.23}$$

10.1 Anomaly with respect to the invariant measure

Consider y_t an infinitely homogenized potential diffusion and write $\mathbb{E}$ the expectation associated to its law. Write for $r > C(d, K_0)$ (a constant that is computed so that $n_{ef}(r) \geq 1$)

$$n_{ef}(r) = \sup\{n \geq 0 \;:\; e^{(n+1)(9d+15)K_0} R_n^2 \leq C_d^1 r^2\} < \infty \tag{10.24}$$

where $C_d^1 = C_1/(8C_d)$ and C_d and C_1 are the constants appearing in the inequality 10.5. $n_{ef}(r) + 1$ corresponds to the number of effective scales in the ball $B(0,r)$ (the scales $0, 1, \ldots, n_{ef}$ are effective scales).
Write

$$n_{per}(r) = \inf\{n \geq 0 \;:\; R_{n+1} \geq r\} - n_{ef}(r) \tag{10.25}$$

$n_{per}(r)$ corresponds to the number of perturbation scales in the ball $B(0,r)$, that is to say the scales $n_{ef}+1, \ldots, n_{ef}(r) + n_{per}(r)$ are perturbation scales and the scales $0, \ldots, n_{ef}(r) + n_{per}(r)$ are fluctuating scales. Observe that for $r > C(d, K_0)$, $n_{per} \geq 0$ and is well defined.

Proposition 10.1.1. *There $\exists C(K_0, d) > 0$ such that for $r > C(K_0, d)$, $n_{ef}(r)$ and $n_{per}(r)$ are well defined and one has*

$$\begin{aligned}\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) &\leq e^{8(2K_1 + n_{per}(r)K_0)} C_6 \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \\ &\geq e^{-16K_1 - (8n_{per}(r)+2)K_0} C_{7,d} \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)}\end{aligned} \tag{10.26}$$

Moreover

$$n_{per}(r) \leq \inf\{m \geq 0 \;:\; \frac{R_{m+n_{ef}(r)+1}}{R_{n_{ef}(r)+1}} \geq \frac{1}{\sqrt{C_d^1}} e^{(n_{ef}(r)+2)(9d+15)K_0/2}\} \tag{10.27}$$

Proof. Observe that for $p > n_{ef}(r)$

$$V^{n_{ef}(r)+1,+\infty} = V^{n_{ef}(r)+1,p} + V^{p+1,\infty}$$

But on $\overline{B(0,r)}$

$$|V^{p+1,\infty}(x)| \leq \frac{1}{R_{p+1}} \sum_{n=p+1}^{+\infty} \frac{R_{p+1}}{R_n} r \sup_n \|\nabla U^n\|_\infty \leq \frac{2r}{R_{p+1}} K_1$$

and

$$|V^{n_{ef}(r)+1,p}(x)| \leq (p - n_{ef}(r)) \sup_n \operatorname{Osc}(U^n) \leq (p - n_{ef}(r)) K_0$$

Choose $p(r) = n_{per}(r) + n_{ef}(r)$ it follows that on $\overline{B(0,r)}$

$$|V^{n_{ef}(r),\infty}(x)| \leq (2K_1 + n_{per}(r)K_0)$$

Now observe that by the theorem 10.0.1, and the definition of $n_{ef}(r)$ for $x \in B(0, r/2)$

$$\begin{aligned}\mathbb{E}_x^{V^{0,n_{ef}(r)}}\big[\tau(B(0,r))\big] &\geq \mathbb{E}_x^{V^{0,n_{ef}(r)}}\big[\tau(B(x,r/2))\big] \\ &\geq C_4 \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)}\end{aligned} \tag{10.28}$$

and for $x \in B(0,r)$

$$\mathbb{E}_x^{V^{0,n_{ef}(r)}}\big[\tau(B(0,r))\big] \leq C_5 \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \tag{10.29}$$

It follows by observing that $m_V^{B(0,r/2)} \geq C_d e^{-2K_0 n_{per}}$ the proposition 10.0.2 that

$$\begin{aligned}\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) &\leq e^{8(2K_1+n_{per}(r)K_0)} C_6 \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \\ &\geq e^{-16K_1-(8n_{per}(r)+2)K_0} C_{7,d} \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)}\end{aligned} \tag{10.30}$$

Moreover observe that

$$e^{(n_{ef}(r)+1)(9d+15)K_0} R^2_{n_{ef}(r)} \leq C_d^1 r^2 \leq e^{(n_{ef}(r)+2)(9d+15)K_0} R^2_{n_{ef}(r)+1} \tag{10.31}$$

thus

$$r \leq \frac{1}{\sqrt{C_d^1}} e^{(n_{ef}(r)+2)(9d+15)K_0/2} R_{n_{ef}(r)+1} \tag{10.32}$$

it follows that if

$$\frac{R_{p+1}}{R_{n_{ef}(r)+1}} \geq \frac{1}{\sqrt{C_d^1}} e^{(n_{ef}(r)+2)(9d+15)K_0/d} \tag{10.33}$$

then

$$R_{p+1} \geq r \tag{10.34}$$

Thus

$$n_{per}(r) \leq \inf\{m \geq 0 \,:\, \frac{R_{m+n_{ef}(r)+1}}{R_{n_{ef}(r)+1}} \geq \frac{1}{\sqrt{C_d^1}} e^{(n_{ef}(r)+2)(9d+15)K_0/2}\} \tag{10.35}$$

□

10.1.1 Anomalous hitting times with bounded ratio between scales

In this subsection assume that the soft pre-fractal associated to the IHPD has bounded ratios $\rho_{\max} < \infty$ between its different scales, then the following theorem shows the anomaly of the diffusion.

Theorem 10.1.1. *One has for $r > C_{16}$,*

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(\Omega)\big] m_V^{B(0,r)}(dx) = r^{2+\nu(r)} \tag{10.36}$$

with for $\rho_{\min} > C_{13}(d, K_0, K_1, \lambda_{\max})$

$$\nu(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}} \big(1 + \frac{C_7(d, K_0, K_1)}{\ln \rho_{\min}}\big) + \frac{1}{\ln r} C_6(d, K_1, K_0) \tag{10.37}$$

and

$$\nu(r) \geq \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}} \big(1 - \frac{C_{12}(d, K_0, K_1)}{\ln \rho_{\min}}\big) - \frac{1}{\ln r} C_{11}(d, K_1, K_0) > C_{15} > 0 \tag{10.38}$$

Where C_{15}, C_{16} depends on $d, K_0, K_1, \lambda_{\max}, \rho_{\max}$

Remark 10.1.1. Observe that for the self-similar case,

$$\nu(r) \sim \frac{\ln \frac{1}{\lambda}}{\ln \rho} \tag{10.39}$$

Proof. This is a direct application of the proposition 10.1.1, indeed according to this proposition for $r > C_{d,K_0}$

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big] m_V^{B(0,r)}(dx) = r^{2+\nu(r)} \tag{10.40}$$

with

$$\nu(r) \leq \frac{1}{\ln r}\Big[8(2K_1 + n_{per}(r)K_0) + \ln C_6 - \ln\big(\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)\big)\Big] \tag{10.41}$$

and

$$\nu(r) \geq \frac{1}{\ln r}\Big[-16K_1 - (8n_{per}(r)+2)K_0 + \ln C_{7,d} - \ln\big(\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)\big)\Big] \tag{10.42}$$

now observe that by the theorem 9.2.2 one has

$$\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big) \leq \lambda_{\max}^{n_{ef}(r)+1}\big(1 + \frac{C(d,K_0,K_1)}{\rho_{\min}^{\frac{1}{2}}}\big)^{n_{ef}(r)} \tag{10.43}$$

and

$$\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big) \geq \lambda_{\min}^{n_{ef}(r)+1}\big(1 + \frac{C(d,K_0,K_1)}{\rho_{\min}^{\frac{1}{2}}}\big)^{-n_{ef}(r)} \tag{10.44}$$

Observe that by the definition of $n_{ef}(r)$

$$n_{ef}(r) \leq 2\frac{\ln r}{2\ln \rho_{\min} + (9d+15)K_0} + C_2(d,K_0) \tag{10.45}$$

and

$$n_{ef}(r) \geq 2\frac{\ln r}{2\ln \rho_{\max} + (9d+15)K_0} - C_2(d,K_0) \tag{10.46}$$

and by the inequality 10.27,

$$n_{per}(r) \leq \frac{n_{ef}(r)(9d+15)K_0/2 + C_3(d,K_0)}{\ln \rho_{\min}} \tag{10.47}$$

It follows that for $\rho_{\min} > C_{13}(d,K_0,K_1)$

$$\begin{aligned}\nu(r) \leq & \frac{1}{\ln r}\Big[C_4(d,K_1,K_0) + n_{ef}(r)\big(\frac{4(9d+15)K_0^2}{\ln \rho_{\min}} - \ln \lambda_{\min} \\ & + \ln(1 + \frac{C(d,K_0,K_1)}{\rho_{\min}^{\frac{1}{2}}})\big)\Big] \\ \leq & \frac{1}{\ln r}\Big[C_4(d,K_1,K_0) + n_{ef}(r)\big(\frac{C_5(d,K_0,K_1)}{\ln \rho_{\min}} - \ln \lambda_{\min}\big)\Big] \\ \leq & \frac{1}{\ln r}C_6(d,K_1,K_0) + \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}\big(1 + \frac{C_7(d,K_0,K_1)}{\ln \rho_{\min}}\big)\end{aligned} \tag{10.48}$$

similarly

$$\begin{aligned}\nu(r) \geq & \frac{1}{\ln r}\Big[C_8(d,K_1,K_0)+n_{ef}(r)\big(-4\frac{(9d+15)K_0^2}{\ln \rho_{\min}}-\ln\lambda_{\max}\\ & -\ln(1+\frac{C(d,K_0,K_1)}{\rho_{\min}^{\frac{1}{2}}})\big)\Big]\\ \geq & \frac{1}{\ln r}\Big[-C_9(d,K_1,K_0)+n_{ef}(r)\big(-\frac{C_{10}(d,K_0,K_1)}{\ln\rho_{\min}}-\ln\lambda_{\max}\big)\Big]\\ \geq & -\frac{1}{\ln r}C_{11}(d,K_1,K_0)+\frac{\ln\frac{1}{\lambda_{\max}}}{\ln\rho_{\max}}\big(1-\frac{C_{12}(d,K_0,K_1)}{\ln\rho_{\min}}\big)\end{aligned} \tag{10.49}$$

□

10.1.2 Anomalous hitting times with fast separation between scales

In this subsection assume that the soft pre-fractal associated to the IHPD has fast separating ratios $R_n = R_{n-1}[\frac{\rho^{p^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ between its different scales, and $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ then the following theorem shows the weak anomaly of the diffusion.

Theorem 10.1.2.

$$\int_{B(0,r)} \mathbb{E}_x\big[\tau(B(0,r))\big]m_V^{B(0,r)}(dx) = \frac{r^2}{\lambda^{\beta(r)}} \tag{10.50}$$

with for $r > C_{16}(d,K_0,K_1)$

$$\beta(r) = \big(\frac{\ln r}{\ln \rho}\big)^{\frac{1}{\alpha}}(1+\epsilon(r)) \tag{10.51}$$

with $\epsilon(r) \to 0$ *as* $r \to \infty$

Remark 10.1.2. Observe that for this theorem shows how the diffusion becomes more and more anomalous as $\alpha \downarrow 1$

Proof. This is a direct application of the proposition 10.1.1 and for the sharp control of $D(V_0^n)$, this is a simple application of the propositions 9.3.5 and 9.4.1. □

10.2 Almost sure anomaly

This section is based on the Conjecture 10.0.2, it shows that if this conjecture is true how one can deduce the anomalous behavior of an IHPD starting from any point. Actually the conjecture 10.0.1 which is weaker than 10.0.2 is sufficient to prove the anomaly of diffusion starting from any point, however, for the clarity of the proof it has been chosen to use the Conjecture 10.0.2 (the proof based on the conjecture 10.0.1 is quite similar).

10.2.1 Anomaly of the hitting times

Consider y_t an infinitely homogenized potential diffusion and write $\mathbb{E}$ the expectation associated to its law. Write for $r > C(d,K_0)$ (a constant that is computed so that $n_{ef}(r) \geq 1$)

$$n_{ef}(r) = \sup\{n \geq 0 \;:\; e^{(n+1)(9d+15)K_0}R_n^2 \leq C_d^1 r^2\} < \infty \tag{10.52}$$

where $C_d^1 = C_1/(8C_d)$ and C_d and C_1 are the constants appearing in the inequality 10.5. $n_{ef}(r)+1$ corresponds to the number of effective scales in the ball $B(0,r)$ (the scales $0,1,\dots,n_{ef}$ are effective

scales).
Write

$$n_{per}(r) = \inf\{n \geq 0 \ : \ R_{n+1} \geq r\} - n_{ef}(r) \tag{10.53}$$

$n_{per}(r)$ corresponds to the number of perturbation scales in the ball $B(0,r)$, that is to say the scales $n_{ef}+1,\dots,n_{ef}(r)+n_{per}(r)$ are perturbation scales and the scales $0,\dots,n_{ef}(r)+n_{per}(r)$ are fluctuating scales. Observe that for $r > C(d,K_0)$, $n_{per} \geq 0$ and is well defined.

Proposition 10.2.1. *Assume that the conjecture 10.0.2 is true. Then there* $\exists C(K_0,d) > 0$ *such that for* $r > C(K_0,d)$, $n_{ef}(r)$ *and* $n_{per}(r)$ *are well defined and one has*

$$\begin{aligned} \mathbb{E}_x\big[\tau(B(x,r))\big] &\leq e^{C_d n_{per}(r)K_0} C_{20}(d,K_0,K_1)\frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \\ &\geq e^{-C_d n_{per}(r)K_0} C_{21}(d,K_0,K_1)\frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \end{aligned} \tag{10.54}$$

Moreover

$$n_{per}(r) \leq \inf\{m \geq 0 \ : \ \frac{R_{m+n_{ef}(r)+1}}{R_{n_{ef}(r)+1}} \geq \frac{1}{\sqrt{C_d^1}} e^{(n_{ef}(r)+2)(9d+15)K_0/2}\} \tag{10.55}$$

Proof. One can assume $x = 0$ without loss of generality.
Just as in the proof of the proposition 10.2.1, observe that on $\overline{B(0,r)}$

$$|V^{n_{ef}(r),\infty}(x)| \leq (2K_1 + n_{per}(r)K_0)$$

Now observe that by the theorem 10.0.1, and the definition of $n_{ef}(r)$

$$\begin{aligned} \mathbb{E}_0^{V^{0,n_{ef}(r)}}\big[\tau(B(0,r))\big] &\geq \mathbb{E}_0^{V^{0,n_{ef}(r)}}\big[\tau(B(0,r))\big] \\ &\geq C_4 \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \end{aligned} \tag{10.56}$$

and similarly

$$\mathbb{E}_0^{V^{0,n_{ef}(r)}}\big[\tau(B(0,r))\big] \leq C_5 \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \tag{10.57}$$

It follows by the conjecture 10.0.2 that

$$\begin{aligned} \mathbb{E}_0\big[\tau(B(0,r))\big] &\leq e^{C_d n_{per}(r)K_0} C_{22}(d,K_0,K_1)\frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \\ &\geq C_{23}(d,K_0,K_1) e^{-C_d n_{per}(r)K_0}\frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \end{aligned} \tag{10.58}$$

Moreover just as in the proof of the proposition 10.2.1 observe that

$$n_{per}(r) \leq \inf\{m \geq 0 \ : \ \frac{R_{m+n_{ef}(r)+1}}{R_{n_{ef}(r)+1}} \geq \frac{1}{\sqrt{C_d^1}} e^{(n_{ef}(r)+2)(9d+15)K_0/2}\} \tag{10.59}$$

□

10.2.1.i Anomaly of the expectations of the hitting times with bounded ratio between scales

For $r > C(d, K_0, K_1)$ write

$$\lambda^{ef}_{\max}(r) = (\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)\big)^{\frac{1}{n_{ef}(r)+1}} \tag{10.60}$$

$\lambda^{ef}_{\max}(r)$ will be called the geometric mean maximal eigenvalue. It reflects the following image: At a scale of order r the maximal eigenvalue of the effective medium characterized by the scales $0, \ldots, n_{ef}+1$ behaves as if those scales were totally separated and the diffusivity of each scale were characterized by the same maximal eigenvalue $\lambda^{ef}_{\max}(r)$ (all associated to the same eigenvector: whose direction does not change with the scale).
Write

$$\ln \rho_{ef}(r) = \frac{\ln r}{n_{ef}(r)} \tag{10.61}$$

$\rho_{ef}(r)$ reflects the following image: The behavior of the IHPD at the scale r is the same as a diffusion with $n_{ef(r)}$ effective scales, the maximal eigenvalue associated to each scale being $\lambda^{m}_{\max}(r)$ and the ratio between each scale being $\rho_{ef}(r)$.

Theorem 10.2.1. *Assume that the conjecture 10.0.2 is true. Then for $\rho_{\min} > C(d, K_0, K_1)$, $r > C(d, K_0, K_1, \rho_{\max})$ one has*

$$\begin{aligned} \mathbb{E}_x\big[\tau(B(x,r))\big] &\leq C_{32}(d, K_0, K_1) r^{2+\sigma(r)(1+\gamma)} \\ &\geq C_{33}(d, K_0, K_1) r^{2+\sigma(r)(1-\gamma)} \end{aligned} \tag{10.62}$$

$$\sigma(r) = \frac{\ln \frac{1}{\lambda^{ef}_{\max}(r)}}{\ln \rho_{ef}(r)}, \qquad \gamma = C_{2,d} \frac{K_0}{\ln \rho_{\min}} < 0.5 \tag{10.63}$$

$$0 < c < \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}} (1 + \frac{C(d, K_0, K_1)}{\ln \rho_{\min}})^{-1} \leq \sigma(r) \tag{10.64}$$

and

$$\sigma(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}} (1 + \frac{C(d, K_0, K_1)}{\ln \rho_{\min}}) \tag{10.65}$$

Remark 10.2.1. This theorem says that the behavior of the hitting times is fixed by the geometric mean effective diffusivity $\lambda^{ef}_{\max}(r)$ and ratio $\rho_{ef}(r)$ at the scale r; the parameter γ plays the role of an error term generated by the perturbation scales. Notice that the parameters do not depend on x and that $\mathbb{E}_x[\tau(x, 2r)]$ can be bounded by the same formulas if one modifies the constants C_{32} and C_{33}. Observe that in the self-similar case when ρ is big

$$\mathbb{E}_x[\tau(B(x,r))] \sim r^{2+\frac{\ln \frac{1}{\lambda}}{\ln \rho}} \tag{10.66}$$

Proof. By the proposition 10.2.1, just as in the proof of the theorem 10.1.1, one has

$$n_{per}(r) \leq \frac{n_{ef}(r)(9d+15)K_0/2 + C_3(d, K_0)}{\ln \rho_{\min}} \tag{10.67}$$

Then by the proposition 10.2.1

$$\begin{aligned} \mathbb{E}_x\big[\tau(B(x,r))\big] &\leq C_{30}(d, K_0, K_1) e^{\frac{C_d}{\ln \rho_{\min}} n_{ef}(r) K_0} \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \\ &\geq C_{31}(d, K_0, K_1) e^{-\frac{C_d}{\ln \rho_{\min}} n_{ef}(r) K_0} \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \end{aligned} \tag{10.68}$$

Notice that by Voigt Reiss inequality, $\lambda^m_{\max}(r) \geq e^{-2K_0}$, it follows by the definition of $\lambda^m_{\max}(r)$ that

$$\begin{aligned}\mathbb{E}_x\big[\tau(B(x,r))\big] &\leq C_{32}(d,K_0,K_1)e^{n_{ef}(r)\left(\frac{C_d}{\ln\rho_{\min}}K_0+\ln\frac{1}{\lambda^{ef}_{\max}(r)}\right)}r^2 \\ &\geq C_{33}(d,K_0,K_1)e^{n_{ef}(r)\left(-\frac{C_d}{\ln\rho_{\min}}K_0+\ln\frac{1}{\lambda^{ef}_{\max}(r)}\right)}r^2\end{aligned} \tag{10.69}$$

Thus by the definition of $\ln\rho_{ef}(r)$, it follows that

$$\begin{aligned}\mathbb{E}_x\big[\tau(B(x,r))\big] &\leq C_{32}(d,K_0,K_1)r^{2+\sigma(r)(1+\gamma)} \\ &\geq C_{33}(d,K_0,K_1)r^{2+\sigma(r)(1-\gamma)}\end{aligned} \tag{10.70}$$

$$\sigma = \frac{\ln\frac{1}{\lambda^{ef}_{\max}(r)}}{\ln\rho_{ef}(r)} \tag{10.71}$$

and

$$\gamma = C_{2,d}\frac{K_0}{\ln\rho_{\min}} \tag{10.72}$$

just as in the proof of the theorem 10.1.1, one has

$$n_{ef}(r) \leq 2\frac{\ln r}{2\ln\rho_{\min}+(9d+15)K_0}+C_2(d,K_0) \tag{10.73}$$

and

$$n_{ef}(r) \geq 2\frac{\ln r}{2\ln\rho_{\max}+(9d+15)K_0}-C_2(d,K_0) \tag{10.74}$$

Thus for $r > C(d,K_0,\rho_{\max})$

$$\ln\rho_{ef}(r) \leq \ln\rho_{\max}+(9d+15)\frac{K_0}{2}+\frac{C(d,K_0,\rho_{\min})}{\ln r} \tag{10.75}$$

$$\ln\rho_{ef}(r) \geq \ln\rho_{\min}+(9d+15)\frac{K_0}{2}-\frac{C(d,K_0,\rho_{\min})}{\ln r} \tag{10.76}$$

now observe that by the theorem 9.2.2 one has

$$\lambda^{ef}_{\max}(r) \leq \lambda_{\max}(1+\frac{C(d,K_0,K_1)}{\rho_{\min}^{\frac{1}{2}}}) \tag{10.77}$$

and

$$\lambda^{ef}_{\max}(r) \geq \lambda_{\min}(1+\frac{C(d,K_0,K_1)}{\rho_{\min}^{\frac{1}{2}}})^{-1} \tag{10.78}$$

It follows that for $\rho_{\min} > C(d,K_0,K_1)$, $r > C(d,K_0,K_1,\rho_{\max})$

$$\sigma(r) \leq \frac{\ln\frac{1}{\lambda_{\min}}}{\ln\rho_{\min}}(1+\frac{C(d,K_0,K_1)}{\ln\rho_{\min}}) \tag{10.79}$$

and

$$\sigma(r) \geq \frac{\ln\frac{1}{\lambda_{\max}}}{\ln\rho_{\max}}(1+\frac{C(d,K_0,K_1)}{\ln\rho_{\min}})^{-1} \tag{10.80}$$

□

10.2.1.ii Anomaly of the expectations of the hitting times with fast separating scales

In this subsection assume that the soft pre-fractal associated to the IHPD has fast separating ratios $R_n = R_{n-1}[\frac{\rho^{p^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ between its different scales, and $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ then the following theorem shows the weak anomaly of the diffusion.

Theorem 10.2.2. *Assume that the conjecture 10.0.2 is true then*

$$\mathbb{E}_0\big[\tau(B(0,r)\big] = \frac{r^2}{\lambda^{\beta(r)}} \tag{10.81}$$

with for $r > C_{16}(d, K_0, K_1)$

$$\beta(r) = \big(\frac{\ln r}{\ln \rho}\big)^{\frac{1}{\alpha}}(1 + \epsilon(r)) \tag{10.82}$$

with $\epsilon(r) \to 0$ *as* $r \to \infty$

Remark 10.2.2. Observe that for this theorem shows how the diffusion becomes more and more anomalous as $\alpha \downarrow 1$

Proof. This is a direct application of the proposition 10.2.1 by observing that the perturbation scale is limited to only one scale. □

10.3 Anomaly of the density of probability of transitions

From the anomaly of the hitting times one can deduce the anomaly of the density of probability of transitions by adapting a strategy used by M.T. Barlow and R. Bass for the Sierpinski Carpet. This strategy is described in details in the proof of the theorem 3.11 of [Bar98].

Below the Lemma 3.14 of [Bar98] is given (this is also the Lemma 1.1 of [BB90a]) without re producing the proof.

Lemma 10.3.1. *Let* $\xi_1, \xi_2, \dots, \xi_n$*,* V *be non-negative r.v. such that* $V \geq \sum_{i=1}^n \xi_i$*. Suppose that for some* $p \in (0,1)$*,* $a > 0$ *and* $t > 0$

$$\mathbb{P}\big(\xi_i \leq t | \sigma(\xi_1, \dots, \xi_{i-1})\big) \leq p + at$$

Then

$$\ln \mathbb{P}(V \leq t) \leq 2\big(\frac{ant}{p}\big)^{\frac{1}{2}} - n \ln \frac{1}{p}$$

Now, using the notation of the theorem 10.2.1, let y_t be a $IHPD$, the following lemma allows to control the law of the exit times.

Lemma 10.3.2. *Let* y_t *be a IHPD. Then for* $r > C(d, K_0, K_1, \rho_{\max})$ *one has*

$$\mathbb{P}_x[\tau(x,r) \leq t] \leq \frac{t}{r^{2+\sigma(r)(1+\gamma)} C_{35}(d, K_0, K_1)} + 1 - C_{36}(d, K_0, K_1) r^{-2\gamma\sigma(r)}$$

Proof. This lemma is an adaptation of the lemma 3.16 of [Bar98]. Observe that

$$\begin{aligned}\mathbb{E}_x[\tau(x,r)] &\leq t + \mathbb{E}_x[1(\tau(x,r) > t)\mathbb{E}_{y_t}[\tau(x,r) - t]] \\ &\leq t + \mathbb{P}_x[1(\tau(x,r) > t)] \sup_{y \in B(x,r)} \mathbb{E}_y[\tau(x,r)]\end{aligned}$$

But $\forall y \in B(x,r)$, $\mathbb{P}_y$ a.s. $\tau(x,r) \leq \tau(y, 2r)$
Hence by the theorem 10.2.1 for $r > C(d, K_0, K_1, \rho_{\max})$

$$\begin{aligned}C_{33}(d, K_0, K_1) r^{2+\sigma(r)(1-\gamma)} &\leq \mathbb{E}_x[\tau(x,r)] \\ &\leq t + \mathbb{P}_x[\tau(x,r) > t] C_{34}(d, K_0, K_1) r^{2+\sigma(r)(1+\gamma)}\end{aligned}$$

Thus

$$\mathbb{P}_x[\tau(x,r) \leq t] \leq \frac{t}{r^{2+\sigma(r)(1+\gamma)}C_{35}(d,K_0,K_1)} + 1 - C_{36}(d,K_0,K_1)r^{-2\gamma\sigma(r)}$$

□

The following lemma is a technical calculus, not very interesting indeed but necessary.

Lemma 10.3.3. *Let $v > 0$, $\beta_2 > \beta_1 > 1$, $a > 0$ and $\beta_1 > \frac{1+\beta_2}{2}$ Let*

$$\psi(x) = a\frac{x^{\frac{1+\beta_2}{2}}}{\sqrt{1 - v\,x^{\beta_2-\beta_1}}} - x\,\ln\frac{1}{1 - v\,x^{\beta_2-\beta_1}}$$

Then with $x_0 = (\frac{v}{2\sqrt{2}a})^{\frac{1}{\beta_1 - \frac{1+\beta_2}{2}}}$ and $x_1 = (\frac{1}{2v})^{\frac{1}{\beta_2-\beta_1}}$ one has for $x \leq x_0 \wedge x_1$

$$\psi(x) \leq -\frac{v}{2}x^{\beta_2-\beta_1+1}$$

and for $x_0 \geq 2$

$$\psi([x_0]) \leq -\frac{v}{2^{\beta_2-\beta_1+2}}(\frac{v}{2\sqrt{2}a})^{\frac{\beta_2-\beta_1+1}{\beta_1-\frac{1+\beta_2}{2}}}$$

Proof. One has for $0 \leq y \leq \frac{1}{2}$

$$y \leq \ln\frac{1}{1-y} \leq \ln 2$$

Write $f(x) = a\frac{x^{\frac{1+\beta_2}{2}}}{\sqrt{1-v\,x^{\beta_2-\beta_1}}}$ and $h(x) = x\,\ln\frac{1}{1-v\,x^{\beta_2-\beta_1}}$
Then for $0 \leq v\,x^{\beta_2-\beta_1} \leq \frac{1}{2}$

$$vx^{\frac{1+\beta_2}{2}} \leq f(x) \leq 2vx^{\frac{1+\beta_2}{2}}$$

and

$$vx^{\beta_2-\beta_1+1} \leq h(x) \leq x\ln 2$$

But

$$2vx^{\frac{1+\beta_2}{2}} \leq \frac{1}{2}vx^{\beta_2-\beta_1+1} \Leftrightarrow x^{\beta_1-\frac{1+\beta_2}{2}} \leq \frac{v}{2\sqrt{(2)}}$$
$$\Leftrightarrow x \leq x_0$$

With $x_0 = (\frac{v}{2\sqrt{2}a})^{\frac{1}{\beta_1-\frac{1+\beta_2}{2}}}$
Thus for $x \leq x_0 \wedge x_1$ with $x_1 = (\frac{1}{2v})^{\frac{1}{\beta_2-\beta_1}}$ we have

$$\psi(x) \leq -\frac{v}{2}x^{\beta_2-\beta_1+1}$$

□

Proposition 10.3.1. *Assume that the conjecture 10.0.2 is true. Then for* $\rho_{\min} > C(d, K_0, K_1)$ *and*

$$C_{40}r \le t \le C_{41}r^{2+\sigma(r)(1-3\gamma)}$$

one has

$$\ln \mathbb{P}_x[|y_t| \ge r] \le \ln \mathbb{P}_x[\tau(x,r) \le t] \le -C_{42}\frac{r^2}{t}(\frac{t}{r})^{\nu}\Big(\frac{r^{3+2\gamma\sigma(r)}}{t^2}\Big)^{\mu}$$

with

$$\nu(r) = \frac{\sigma(r)(1-\gamma)}{1+\sigma(r)(1-3\gamma)}$$

$$\mu(r) = \frac{2\sigma(r)\gamma}{1+\sigma(r)(1-3\gamma)}$$

$1 - 3\gamma > 0.5$ *and the constants* C_{40}, C_{41}, C_{42} *depend on* $d, K_0, K_1, \rho_{\max}, \rho_{\min}$.

Remark 10.3.1. Observe that the term μ is very small in comparison to ν, it acts as an error term generated by the perturbation scales, in the next theorem it will be shown that one can make this term disappear by a slight modification of ν. Actually this proposition says that for $C_1 r < t$ (homogenization has started) and $t < C_2 r^{2+\kappa}$ (the behavior is far from the heat kernel diagonal regime) one has at the first order

$$p(|y_t| > h) \sim -\frac{h^2}{t}(\frac{t}{h})^{\sigma(r)} \tag{10.83}$$

with $\sigma(r) \sim \big(\ln(1/\lambda^{ef}_{\max}(r))\big)/(\ln \rho_{ef}(r))$: at the first order the behavior of the transition probability densities is fixed by the mean effective maximal diffusivity $\lambda^{ef}_{\max}(r)$ and the mean ratio between scales $\rho_{ef}(r)$ associated to the effective scales corresponding to the length r.

Proof. Let $n \ge 1$ and $g = \frac{r}{n}$
Define the stopping times S_i $i \ge 0$ by $S_0 = 0$ and

$$S_{i+1} = \inf\{t \ge S_i \ : \ |y_t - y_{S_i}| \ge g\}$$

Write $\xi_i = S_i - S_{i-1}$ for $i \ge 1$ Let $\mathcal{F}_t$ be the filtration of y_t and let $\mathcal{G}_i = \mathcal{F}_{S_i}$ Then it follows from the lemma the lemma 10.3.2 that for $g > C(d, K_0, K_1, \rho_{\max})$

$$\begin{aligned}\mathbb{P}_x[\xi_{i+1} \le t|(G)_i] &= \mathbb{P}_{y_{S_i}}[\tau(y_{S_i}, g) \le t] \\ &\le C_{37}(d, K_0, K_1)\frac{t}{g^{2+\sigma(r)(1+\gamma)}} + 1 - C_{36}(d, K_0, K_1)g^{-2\sigma(r)\gamma}\end{aligned}$$

Since $|y_{S_i} - y_{S_{i+1}}| = g$ it follows that $\mathbb{P}_x$ a.s. $|x - y_{S_n}| \le r$
Thus

$$S_n = \sum_{i=1}^{n} \xi_i \le \tau(x,r)$$

And by the lemma 10.3.1 with

$$a = C_{37}(d, K_0, K_1)(\frac{n}{r})^{2+\sigma(r)(1+\gamma)}$$

$$p = 1 - C_{36}(d, K_0, K_1)(\frac{n}{r})^{2\sigma(r)\gamma}$$

One has

$$\ln \mathbb{P}_x[\tau(x,r) \le t] \le 2\Big(\frac{n\,t\,C_{37}(\frac{n}{r})^{2+\sigma(r)(1+\gamma)}}{1 - C_{36}(\frac{n}{r})^{2\sigma(r)\gamma}}\Big)^{\frac{1}{2}} - n \ln \frac{1}{1 - C_{36}(\frac{n}{r})^{2\sigma(r)\gamma}}$$

Now use the lemma 10.3.3 to choose n with $\beta_2 = 2+\sigma(r)(1+\gamma)$, $\beta_1 = 2+\sigma(r)(1-\gamma)$, $a = 2(tC_{37}r^{-\beta_2})$ and $v = C_{36}r^{\beta_1-\beta_2}$
It follows that

$$x_0 = \Big(\frac{C_{36}r^{\beta_1-\beta_2}}{4\sqrt{2C_{37}r^{-\beta_2}}}\Big)^{\frac{1}{\beta_1 - \frac{1+\beta_2}{2}}} = \Big(\frac{C_{36}}{4\sqrt{2C_{37}}}\Big)^{\frac{1}{\beta_1-\frac{1+\beta_2}{2}}} \frac{r^{\frac{\beta_1-\frac{\beta_2}{2}}{\beta_1-\frac{1+\beta_2}{2}}}}{t^{\frac{1}{2(\beta_1-\frac{1+\beta_2}{2})}}}$$

and

$$x_1 = (\frac{1}{C_{36}})^{\frac{1}{\beta_2-\beta_1}}\, r$$

But one needs $n \le x_0 \wedge x_1$ and $r/n > C(d, K_0, K_1, \rho_{\max})$

$$x_0 \le x_1 \Leftrightarrow \Big(\frac{C_{36}}{4\sqrt{2C_{37}}}\Big)^{\frac{1}{\beta_1-\frac{1+\beta_2}{2}}} r^{\frac{1}{2(\beta_1-\frac{1+\beta_2}{2})}} \le (\frac{1}{C_{36}})^{\frac{1}{\beta_2-\beta_1}} t^{\frac{1}{2(\beta_1-\frac{1+\beta_2}{2})}}$$
$$\Leftrightarrow r \le C_{38}(d, K_0, K_1, \rho_{\min}, \rho_{\max})t$$

Choose C_{38} so that $x_0C(d, K_0, K_1, \rho_{\max}) < r$ is also satisfied.
Moreover

$$x_0 \ge 2 \Leftrightarrow r \ge C_{39}(d, K_0, K_1, \rho_{\min}, \rho_{\max})\, t^{\frac{1}{2\beta_1-\beta_2}}$$

so for $r \ge C_{39}t^{\frac{1}{2\beta_1-\beta_2}}$ and $r \le C_{38}t$

$$\ln \mathbb{P}_x[\tau(x,r) \le t] \le -\frac{C_4r^{\beta_1-\beta_2}}{2^{2+\beta_2-\beta_1}}\Big(\frac{C_4r^{\beta_1-\beta_2}}{4\sqrt{2tC_3r^{-\beta_2}}}\Big)^{\frac{1+\beta_2-\beta_1}{\beta_1-\frac{1+\beta_2}{2}}}$$

and after some calculus and simplifications

$$\ln \mathbb{P}_x[\tau(x,r) \le t] \le -C_{40}\frac{r^2}{t}\big(\frac{t}{r}\big)^{\nu}\Big(\frac{r^{3+\beta_2-\beta_1}}{t^2}\Big)^{\mu}$$

with

$$\nu = \frac{\beta_1 - 2}{2\beta_1 - 1 - \beta_2} = \frac{\sigma(r)(1-\gamma)}{1+\sigma(r)(1-3\gamma)}$$

$$\mu = \frac{\beta_2 - \beta_1}{2\beta_1 - 1 - \beta_2} = \frac{2\sigma(r)\gamma}{1+\sigma(r)(1-3\gamma)}$$

□

Corollary 10.3.1. *Assume that the conjecture 10.0.2 is true. Then for* $\rho_{\min} > C(d, K_0, K_1)$ *and*

$$C_{40}r \le t \le C_{41}r^{2+\sigma(r)(1-3\gamma)}$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_7 \frac{r^2}{t} \big(\frac{t}{r}\big)^{\nu'}$$

with

$$\nu'(r) = \sigma(r)(1 - \frac{C(d,K_0)}{\ln \rho_{\min}})$$

and the constants C_{40}, C_{41}, C_{42} *depend on* $d, K_0, K_1, \rho_{\max}, \rho_{\min}$.

Remark 10.3.2. Notice that the second term in the expression of ν' is an error term created by perturbation scales.

Proof. Observe that if one chooses

$$\nu - \nu' = 4\mu$$

then

$$\begin{aligned}\big(\frac{t}{r}\big)^{\nu} \Big(\frac{r^{3+\beta_2-\beta_1}}{t^2}\Big)^{\mu} \geq \big(\frac{t}{r}\big)^{\nu'} &\Leftrightarrow t \geq r^{\frac{\nu-\nu'-(3+2\gamma\sigma(r))\mu}{\nu-\nu'-2\mu}} \\ &\Leftrightarrow t \geq r^{\frac{1-2\gamma\sigma(r)}{2}}\end{aligned}$$

thus in the previous theorem one can choose

$$\begin{aligned}\nu'(r) &= \frac{\sigma(r)(1-\gamma)}{1+\sigma(r)(1-3\gamma)} - 4\frac{2\sigma(r)\gamma}{1+\sigma(r)(1-3\gamma)} \\ &= \sigma(r)(1 - \frac{C(d,K_0)}{\ln \rho_{\min}})\end{aligned}$$

□

10.3.1 Anomaly of the transition probability densities with bounded ratio between scales

Corollary 10.3.2. *Assume that the conjecture 10.0.2 is true,* $\rho_{\max} < \infty$ *and* $\lambda_{\max} < 1$*. Then for* $\rho_{\min} > C(d, K_0, K_1)$ *and*

$$C_{40} r \leq t \leq C_{41} r^{2+\sigma(r)(1-3\gamma)}$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_7 \frac{r^2}{t} \big(\frac{t}{r}\big)^{\nu'}$$

with

$$0 < c < \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}} (1 - \frac{C_{50}(d,K_0)}{\ln \rho_{\min}}) \leq \nu'(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}} (1 - \frac{C_{50}(d,K_0)}{\ln \rho_{\min}}) \tag{10.84}$$

$C_{50}(d,K_0) < 0.5 \ln \rho_{\min}$ *and the constants* C_{40}, C_{41}, C_{42} *depend on* $d, K_0, K_1, \rho_{\max}, \rho_{\min}$

Remark 10.3.3. Observe that if $\rho_{\max} = \rho_{\min}$ and $\lambda_{\max} = \lambda_{\min}$ then at the first order in $1/\ln \rho_{\min}$, ν' behaves like

$$\nu' \sim \frac{\ln \frac{1}{\ln \lambda}}{\ln \rho} \tag{10.85}$$

Proof. This is a direct consequence of the corollary 10.3.1. □

10.3.2 Anomaly of the transition probability densities with fast separating scales

In this subsection assume that the soft pre-fractal associated to the IHPD has fast separating ratios $R_n = R_{n-1}[\frac{\rho^{p^\alpha}}{R_{n-1}}]$ $(\rho, \alpha > 1)$ between its different scales, and $\lambda_{\max} = \lambda_{\min} = \lambda < 1$ then the following theorem shows the weak anomaly of the diffusion.

Theorem 10.3.1. *Assume that the conjecture 10.0.2 is true then for*

$$C_{60} r \leq t \leq C_{61} r^2$$

one has

$$\ln \mathbb{P}_x[|y_t| \geq r] \leq \ln \mathbb{P}_x[\tau(x,r) \leq t] \leq -C_{63} \frac{r^2}{t} g(\frac{t}{r})$$

with

$$g(x) = (\frac{1}{\lambda})^{(\frac{x}{\ln \rho})^{\frac{1}{\alpha}}(1+\epsilon(x))} \tag{10.86}$$

where $\epsilon(x) \to 0$ as $x \to \infty$ and the constant C_{60} to C_{63} depends on $\rho, \alpha, K_0, K_1, d$.

Remark 10.3.4. Observe that $\frac{t}{h^2} \ln \mathbb{P}[l.y_t \geq h] \to -\infty$ as $t/h \to \infty$. Moreover this theorem shows how the behavior of the diffusion passes from a slightly anomalous one to a strongly anomalous one.

Proof. Straightforward by observing that the ratio between the number of perturbation scales with the numbers of fluctuating scales tends towards 0 as $t/r \to \infty$ □

11. SUPER DIFFUSIVITY IN THE SHEAR FLOW MODEL

11.1 Explicit formulas

Let J be smooth periodic 2×2 skew-symmetric matrix

$$J(x_1,x_2)=\begin{pmatrix}0 & j(x_1)\\ -j(x_1) & 0\end{pmatrix} \tag{11.1}$$

with $j\in C^\infty(T_1^1)$, $j(0)=0$. write L_J its associated diffusion operator

$$L_J=\frac{1}{2}\Delta+\nabla J.\nabla \tag{11.2}$$

11.1.1 Cell problem

The T_1^2 periodic solution χ_l ($l\in\mathbb{S}^2$) of the cell problem $L_J(\chi_l-l.x)=0$ is ($\chi_l(0)=0$)

$$\chi_l(x_1,x_2)=2l_2\Big[\int_0^{x_1} j(y)dy-x_1\int_0^1 j(y)dy\Big] \tag{11.3}$$

its associated harmonic functions is

$$F_l(x)=l.x-\chi_l(x)=l.x-2l_2\Big[\int_0^{x_1} j(y)dy-x_1\int_0^1 j(y)dy\Big] \tag{11.4}$$

11.1.2 Effective Diffusivity

The solution of the cell problem allows to compute the effective diffusivity

$${}^t lD(J)l=\int_{T_1^2}|l-\nabla\chi_l(x)|^2dx \tag{11.5}$$

and it follows that

$$D(J)=\begin{pmatrix}1 & 0\\ 0 & 1+4\operatorname{Var}(j)\end{pmatrix} \tag{11.6}$$

11.2 Multi-scale effective diffusivity

Consider a IHSFD,

Theorem 11.2.1. *assume $\gamma_{min}>1$ and*

$$\epsilon=\frac{2^{\frac{3}{2}}}{\rho_{\min}}\frac{K_1}{\gamma_{\min}-1}<1 \tag{11.7}$$

then for all $p\in\mathbb{N}$

$$D(\Gamma^{0,p})=\begin{pmatrix}1 & 0\\ 0 & D(\Gamma^{0,p})_{22}\end{pmatrix} \tag{11.8}$$

with

$$1 + 4(1-\epsilon)\sum_{k=0}^{p}\gamma_k^2 \leq D(\Gamma^{0,p})_{22} \leq 1 + 4(1+\epsilon)\sum_{k=0}^{p}\gamma_k^2 \tag{11.9}$$

Proof. Observe that

$$\mathrm{Var}(H^p) = \int_0^1 \big((H^{p-1}(R_p x) - \int_0^1 H^{p-1}(R_p y)dy) + \gamma_p(h_p(x) - \int_0^1 h_p(y)dy)\big)^2 dx \tag{11.10}$$

It follows that

$$\big|\,\mathrm{Var}(H^p) - \mathrm{Var}(H^{p-1}) - \gamma_p^2\big| \leq 2\gamma_p|E| \tag{11.11}$$

with (Cov designate the covariance: $\mathrm{Cov}(f,g) = \int_0^1 (f(x) - \int_0^1 f(y)dy)(g(x) - \int_0^1 g(y)dy)dx$)

$$E = \mathrm{Cov}(S_{R_p}H^{p-1}, h_p) \tag{11.12}$$

By the corollary C.1.1,

$$\begin{aligned} |E| &\leq \frac{\|\nabla h_p\|_\infty R_{p-1}}{R_p}\int_0^1 |S_{R_p}H^{p-1}(x)|dx \\ &\leq \frac{K_1}{r_p}\sqrt{\mathrm{Var}(H^{p-1})} \end{aligned} \tag{11.13}$$

Now it will be shown by induction that

$$(1-\epsilon)\sum_{k=0}^{p}\gamma_k^2 \leq \mathrm{Var}(H^p) \leq (1+\epsilon)\sum_{k=0}^{p}\gamma_k^2 \tag{11.14}$$

This is trivially true for $p = 0$. Assume that 11.14 is true for $p \in \mathbb{N}$.
Then observe that

$$\begin{aligned} \sqrt{\mathrm{Var}(H^p)} &\leq (1+\epsilon)^{\frac{1}{2}}\gamma_{p+1}\Big(\sum_{k=0}^{p}(\frac{\gamma_k}{\gamma_{p+1}})^2\Big)^{\frac{1}{2}} \\ &\leq (1+\epsilon)^{\frac{1}{2}}\gamma_{p+1}\Big(\sum_{k=1}^{\infty}(\frac{1}{\gamma_{\min}^k})^2\Big)^{\frac{1}{2}} \\ &\leq (1+\epsilon)^{\frac{1}{2}}\gamma_{p+1}\Big(\frac{1}{\gamma_{\min}^2 - 1}\Big)^{\frac{1}{2}} \\ &\leq (1+\epsilon)^{\frac{1}{2}}\gamma_{p+1}\frac{1}{\gamma_{\min} - 1} \end{aligned} \tag{11.15}$$

and

$$\left\{\begin{array}{l} \epsilon \geq \frac{2K_1}{\rho_{\min}}(1+\epsilon)^{\frac{1}{2}}\frac{1}{\gamma_{\min}-1} \\ \epsilon < 1 \end{array}\right. \Leftarrow \epsilon = \frac{2K_1}{\rho_{\min}}\sqrt{2}\frac{1}{\gamma_{\min}-1} < 1 \tag{11.16}$$

then by the inequalities 11.13 and 11.11 it follows that

$$\big|\,\mathrm{Var}(H^{p+1}) - \mathrm{Var}(H^p) - \gamma_{p+1}^2\big| \leq \epsilon\gamma_{p+1}^2 \tag{11.17}$$

which proves the induction. □

11.3 Mean square displacement

Lemma 11.3.1. *If $\rho_{\min} > \gamma_{\max}$ and $\gamma_{\min} > 1$ then for all $x \in \mathbb{R}$*

$$\Big|\int_0^x \big(H^p(y) - \frac{1}{R_p}\int_0^{R_p} H^p(z)dz\big)dy\Big| \leq R_p K_0 \gamma_p \frac{\gamma_{\min}}{\gamma_{\min}-1} \tag{11.18}$$

$$\big|H^{p+1,\infty}(x)\big| \leq K_1 \frac{\gamma_{p+1}}{R_{p+1}} \frac{\rho_{\min}}{\rho_{\min}-\gamma_{\max}}|x| \tag{11.19}$$

$$\big|\partial_1 H^{p+1,\infty}(x)\big| \leq K_1 \frac{\gamma_{p+1}}{R_{p+1}} \frac{\rho_{\min}}{\rho_{\min}-\gamma_{\max}} \tag{11.20}$$

Proof. Straightforward computation □

Lemma 11.3.2. *Assume $\gamma_{min} > 1$ and*

$$\epsilon = \frac{2^{\frac{3}{2}}}{\rho_{\min}} \frac{K_1}{\gamma_{\min}-1} < \frac{1}{2} \tag{11.21}$$

For $p \in \mathbb{N}^*$, and $t > 0$

$$\mathbb{E}[\int_0^t (H^p(b_s) - \kappa^p)^2\, ds] \leq 5t\gamma_p^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 + 4K_0^2\gamma_p^2 R_p^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 \tag{11.22}$$

and

$$\mathbb{E}[\int_0^t (H^p(b_s) - \kappa^p)^2\, ds] \geq 2t\gamma_p^2 - 4K_0^2\gamma_p^2 R_p^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 \tag{11.23}$$

Proof. Write

$$I_p = \mathbb{E}[\int_0^t (H^p(b_s) - \kappa^p)^2\, ds]$$

and

$$f(x) = \int_0^x \big(H^p(y) - \kappa^p\big)^2 - \mathrm{Var}(H^p)x \tag{11.24}$$

$$g(x) = 2\Big(\int_0^x f(y)\, dy - \frac{x}{R_p}\int_0^{R_p} f(y)\, dy\Big) \tag{11.25}$$

Observe that g is periodic of period R_p and by the Ito formula

$$I_p = \mathrm{Var}(H^p)t + \mathbb{E}[g(b_t)] \tag{11.26}$$

Now the result follows by the theorem 11.2.1 and by observing that

$$\|g\|_\infty \leq 4K_0^2\gamma_p^2 R_p^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 \tag{11.27}$$

□

Lemma 11.3.3. *For $p \in N^*$, and $t > 0$*

$$\begin{aligned}\big|\mathbb{E}[\int_0^t (H^{p-1}(b_s) - \kappa^{p-1})(H^{p,p}(b_s) - \kappa^p)\, ds]\big| \leq \\ K_0\gamma_{p-1}\gamma_p \frac{\gamma_{\min}}{\gamma_{\min}-1}\big(8K_0 R_{p-1}\sqrt{t} + t\frac{K_1}{r_p}\big)\end{aligned} \tag{11.28}$$

Proof. For $p \in N^*$, write

$$J_{p,2} = \mathbb{E}[\int_0^t (H^{p-1}(b_s) - \kappa^{p-1})(H^{p,p}(b_s) - \kappa^p)\,ds] \tag{11.29}$$

write for $x \in \mathbb{R}$,

$$g(x) = \int_0^x (H^{p-1}(y) - \kappa^{p-1})(H^{p,p}(y) - \kappa^p)dy \tag{11.30}$$

Observe that by Ito formula

$$2\mathbb{E}[\int_0^{b_s} g(y)dy] = J_{p,2} \tag{11.31}$$

But by the corollary C.1.4

$$|g(x)| \leq 2K_0\gamma_{p-1}\frac{\gamma_{\min}}{\gamma_{\min}-1}\big(4\gamma_p K_0 R_{p-1} + |x| K_1 \frac{\gamma_p}{r_p}\big) \tag{11.32}$$

It follows that

$$|J_{p,2}| \leq K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\big(8K_0 R_{p-1}\sqrt{t} + t\frac{K_1}{r_p}\big) \tag{11.33}$$

□

Lemma 11.3.4. *Let $f, G \in C^\infty(T^1_R)$ such that $\int_0^R f(y)dy = 0$ and $\int_0^R G(y)dy = 0$, with $R \in \mathbb{N}^*$, let $r \in \mathbb{N}^*$ such that $R/r \in \mathbb{N}^*$ and $t > 0$*

$$\Big|\mathbb{E}\big[G(b_t)\int_0^t \partial_1 f(rb_s)\,ds\big]\Big| \leq \|f\|_{L^2(T^1_R)}\|G\|_{L^2(T^1_R)}\frac{15}{rR^{\frac{1}{2}}t^{\frac{1}{4}}} \tag{11.34}$$

Proof. write

$$I = \mathbb{E}\big[G(b_t)\int_0^t \partial_1 f(rb_s)\,ds\big] \tag{11.35}$$

write f_k and G_k the Fourier decomposition of f and G.

$$f(x) = \sum_{k\in\mathbb{Z}} f_k e^{ik\frac{2\pi}{R}x} \tag{11.36}$$

Note that

$$\|f\|^2_{L^2(T^1_R)} = R\sum_{k\in\mathbb{Z}} |f_k|^2 \tag{11.37}$$

Write for $k, m \in \mathbb{Z}$

$$J_{k,m} = \int_0^t \mathbb{E}\big[e^{ikr\frac{2\pi}{R}b_s}e^{im\frac{2\pi}{R}b_t}\big]\,ds \tag{11.38}$$

By a straightforward computation

$$\begin{aligned} J_{k,m} &= \int_0^t e^{-(\frac{2\pi}{R})^2\frac{(kr+m)^2}{2}s-(\frac{2\pi}{R})^2\frac{m^2}{2}(t-s)} \\ &= 2e^{-(\frac{2\pi}{R})^2\frac{m^2}{2}t}\frac{1-e^{-(\frac{2\pi}{R})^2(\frac{(kr)^2}{2}+krm)t}}{(\frac{2\pi}{R})^2(\frac{(kr)^2}{2}+krm)} \end{aligned} \tag{11.39}$$

(in last fraction in the above equation, if the denominator is equal to 0, consider it as a limit to obtain the exact value t)
Now

$$I = \sum_{k,m \in \mathbb{Z}^2} J_{km} ik \frac{2\pi}{R} f_k G_m \tag{11.40}$$

Thus since $f_0 = G_0 = 0$

$$|I| \leq 2 \sum_{m \in \mathbb{Z}^*} e^{-(\frac{2\pi}{R})^2 \frac{m^2}{4} t} |G_m| \frac{2\pi}{R} \big(\sum_{k \in \mathbb{Z}} |f_k|^2 \big)^{\frac{1}{2}} J_m^{\frac{1}{2}} \tag{11.41}$$

with

$$\begin{aligned} J_m &= \sum_{k \in \mathbb{Z}^*} \Big(\frac{1 - e^{-(\frac{2\pi}{R})^2 (\frac{(kr)^2}{2} + krm) t}}{(\frac{kr^2}{2} + rm)(\frac{2\pi}{R})^2} \Big)^2 e^{-(\frac{2\pi}{R})^2 \frac{m^2}{2} t} \\ &\leq \frac{t}{4} (\frac{2m}{r})^2 e^{-(\frac{2\pi}{R})^2 \frac{m^2}{2} t} + \sum_{k \in \mathbb{Z}^*} \frac{4}{r^2 k^2 (\frac{2\pi}{R})^2} \\ &\leq \frac{R^2}{2r^2} \end{aligned} \tag{11.42}$$

Thus

$$\begin{aligned} |I| &\leq \|f\|_{L^2(T_R^1)} \|G\|_{L^2(T_R^1)} \frac{10}{Rr} \big(\sum_{k \in \mathbb{Z}^*} e^{-(\frac{2\pi}{R})^2 \frac{m^2}{2} t} \big)^{\frac{1}{2}} \\ &\leq \|f\|_{L^2(T_R^1)} \|G\|_{L^2(T_R^1)} \frac{30}{rRt^{\frac{1}{4}}} (\frac{R}{2\pi})^{\frac{1}{2}} \end{aligned} \tag{11.43}$$

□

Corollary 11.3.1.

$$\begin{aligned} \Big| \mathbb{E} \big[\int_0^t \partial_1 H^{p-1}(\omega_s.e_1) ds \int_0^{b_t} (H^{p,p}(y) - \kappa^{p,p}) dy \big] \Big| \leq \\ 15 K_0^2 \frac{\gamma_{\min}}{\gamma_{\min} - 1} \gamma_{p-1} \gamma_p \frac{R_p^{\frac{5}{2}}}{r_p t^{\frac{1}{4}}} \end{aligned} \tag{11.44}$$

Proof. Straightforward by the lemma 11.3.4. □

Lemma 11.3.5. *Assume $\gamma_{min} > 1$ and*

$$\rho_{\min} \geq 8(K_0 \gamma_{\max} \gamma_{\min} + 1) \frac{K_1}{\gamma_{\min} - 1} \tag{11.45}$$

then with

$$I_p = \mathbb{E} \Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1) ds \big)^2 \Big] \tag{11.46}$$

one has

$$\begin{aligned}I_p \leq\ & t\big(5\gamma_{p-1}^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 + 8K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\frac{K_1}{r_p}\big)\\ &+ t^2K_1^2\frac{\gamma_p^2}{R_p^2}\\ &+ \sqrt{t}68\gamma_{p-1}\gamma_pR_{p-1}K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\\ &+ \frac{1}{t^{\frac{1}{4}}}60K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_p}\\ &+ 4K_0^2\gamma_{p-1}^2R_{p-1}^2\end{aligned} \tag{11.47}$$

and

$$\begin{aligned}I_p \geq\ & t\gamma_{p-1}^2\\ &- \sqrt{t}68\gamma_{p-1}\gamma_pR_{p-1}K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\\ &- \frac{1}{t^{\frac{1}{4}}}60K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_p}\\ &- 4K_0^2\gamma_{p-1}^2R_{p-1}^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2\end{aligned} \tag{11.48}$$

Proof. Observe that

$$I_p = I_{p-1} + \mathbb{E}\Big[\big(\int_0^t \partial_1 H^{p,p}(\omega_s.e_1)ds\big)^2\Big] + 2J_p \tag{11.49}$$

with

$$J_p = \mathbb{E}\Big[\big(\int_0^t \partial_1 H^{p,p}(\omega_s.e_1)ds\big)\big(\int_0^t \partial_1 H^{p-1}(\omega_s.e_1)ds\big)\Big] \tag{11.50}$$

Now by the Ito formula ($b_t = \omega_t.e_1$)

$$\int_0^t \partial_1 H^{p,p}(\omega_s.e_1)ds = 2\int_0^{b_t}(H^{p,p}(y)-\kappa^{p,p})dy - 2\int_0^t(H^{p,p}(b_s)-\kappa^{p,p})db_s \tag{11.51}$$

and

$$\int_0^t \partial_1 H^{p-1}(\omega_s.e_1)ds = 2\int_0^{b_t}(H^{p-1}(y)-\kappa^{p-1})dy - 2\int_0^t(H^{p-1}(b_s)-\kappa^{p-1})db_s \tag{11.52}$$

It follows by the lemma 11.3.1 that,

$$|J_p| \leq \gamma_{p-1}R_{p-1}K_0\frac{\gamma_{\min}}{\gamma_{\min}-1}2K_0\gamma_p\sqrt{t} + 4|J_{p,2}| + 2|J_{p,3}| \tag{11.53}$$

with

$$J_{p,2} = \mathbb{E}[\int_0^t (H^{p-1}(b_s)-\kappa^{p-1})(H^{p,p}(b_s)-\kappa^p)\,ds] \tag{11.54}$$

and

$$J_{p,3} = \mathbb{E}[\int_0^t \partial_1 H^{p-1}(\omega_s.e_1)ds\int_0^{b_t}(H^{p,p}(y)-\kappa^{p,p})dy] \tag{11.55}$$

It follows by the lemma 11.3.2, the lemma 11.3.3 and the corollary 11.3.1 that

$$\begin{aligned} I_p \leq\ & 5t\gamma_{p-1}^2\Big(\frac{\gamma_{\min}}{\gamma_{\min}-1}\Big)^2 + 4K_0^2\gamma_{p-1}^2R_{p-1}^2\Big(\frac{\gamma_{\min}}{\gamma_{\min}-1}\Big)^2 \\ & + K_1^2\frac{\gamma_p^2}{R_p^2}t^2 \\ & + 2\gamma_{p-1}R_{p-1}K_0\frac{\gamma_{\min}}{\gamma_{\min}-1}2K_0\gamma_p\sqrt{t} \\ & + 8K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\big(8K_0R_{p-1}\sqrt{t}+t\frac{K_1}{r_p}\big) \\ & + 60K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_pt^{\frac{1}{4}}} \end{aligned} \tag{11.56}$$

Thus

$$\begin{aligned} I_p \leq\ & t\big(5\gamma_{p-1}^2\Big(\frac{\gamma_{\min}}{\gamma_{\min}-1}\Big)^2 + 8K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\frac{K_1}{r_p}\big) \\ & + t^2K_1^2\frac{\gamma_p^2}{R_p^2} \\ & + \sqrt{t}68\gamma_{p-1}\gamma_pR_{p-1}K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1} \\ & + \frac{1}{t^{\frac{1}{4}}}60K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_p} \\ & + 4K_0^2\gamma_{p-1}^2R_{p-1}^2 \end{aligned} \tag{11.57}$$

Similarly by using the lemma 11.3.2, the lemma 11.3.3 and the corollary 11.3.1 one obtains that

$$\begin{aligned} I_p \geq\ & 2t\gamma_{p-1}^2 - 4K_0^2\gamma_{p-1}^2R_{p-1}^2\Big(\frac{\gamma_{\min}}{\gamma_{\min}-1}\Big)^2 \\ & - 2\gamma_{p-1}R_{p-1}K_0\frac{\gamma_{\min}}{\gamma_{\min}-1}2K_0\gamma_p\sqrt{t} \\ & - 8K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\big(8K_0R_{p-1}\sqrt{t}+t\frac{K_1}{r_p}\big) \\ & - 60K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_pt^{\frac{1}{4}}} \end{aligned} \tag{11.58}$$

Thus

$$\begin{aligned} I_p \geq\ & t\big(2\gamma_{p-1}^2 - 8K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\frac{K_1}{r_p}\big) \\ & - \sqrt{t}68\gamma_{p-1}\gamma_pR_{p-1}K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1} \\ & - \frac{1}{t^{\frac{1}{4}}}60K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_p} \\ & - 4K_0^2\gamma_{p-1}^2R_{p-1}^2\Big(\frac{\gamma_{\min}}{\gamma_{\min}-1}\Big)^2 \end{aligned} \tag{11.59}$$

Then the results follows by observing that

$$\rho_{\min} \geq 8K_0K_1\gamma_{\max}\frac{\gamma_{\min}}{\gamma_{\min}-1} \tag{11.60}$$

implies

$$\gamma_{p-1}^2 \geq 8K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\frac{K_1}{r_p} \tag{11.61}$$

□

Lemma 11.3.6. *Assume $\gamma_{min} > 1$ and*

$$2^{15}(1+K_0)^3(1+K_1)^3\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{\max}^4 \leq \rho_{\min} \tag{11.62}$$

then for

$$t > 240\frac{R_1^2}{r_1^{\frac{4}{5}}}K_0^{\frac{8}{5}}(\frac{\gamma_{\min}}{\gamma_{\min}-1})^{\frac{4}{5}}(\frac{\gamma_1}{\gamma_0})^{\frac{4}{5}}$$

one has

$$t\frac{\gamma_{p-1}^2}{4} \leq \mathbb{E}\big[(y_t.e_2)^2\big]t \leq \gamma_m^2 30(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2(1+K_1)^2 \tag{11.63}$$

with

$$p(t) = \sup\{p\in\mathbb{N}\,:\,t \geq 240\frac{R_p^2}{r_p^{\frac{4}{5}}}K_0^{\frac{8}{5}}(\frac{\gamma_{\min}}{\gamma_{\min}-1})^{\frac{4}{5}}(\frac{\gamma_p}{\gamma_{p-1}})^{\frac{4}{5}}\} \tag{11.64}$$

and

$$m(t) = \inf\{p\in\mathbb{N}\,:\,t\leq R_p^2\} \tag{11.65}$$

Proof. Since

$$y_t.e_2 = \omega_t.e_2 + \int_0^t \partial_1 h(\omega_s.e_1)ds \tag{11.66}$$

It follows that (by the independence of $\omega_t.e_2$ with $\omega_t.e_1$),

$$\mathbb{E}\big[(y_t.e_2)^2\big] = t + \mathbb{E}\Big[\big(\int_0^t \partial_1 h(\omega_s.e_1)ds\big)^2\Big] \tag{11.67}$$

thus for all $p\in\mathbb{N}^*$

$$\begin{aligned}\mathbb{E}\big[(y_t.e_2)^2\big] =& t + \mathbb{E}\Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)^2\Big] + \mathbb{E}\Big[\big(\int_0^t \partial_1 H^{p+1,\infty}(\omega_s.e_1)ds\big)^2\Big]\\ &+ 2\mathbb{E}\Big[\big(\int_0^t \partial_1 H^{p+1,\infty}(\omega_s.e_1)ds\big)\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)\Big]\end{aligned} \tag{11.68}$$

Thus by bounding the term with $\partial_1 H^{p+1,\infty}$ as a drift scale (see lemma 11.3.1) it follows that

$$\begin{aligned}\mathbb{E}\big[(y_t.e_2)^2\big] &\geq t + \frac{1}{2}\mathbb{E}\Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)^2\Big] - \mathbb{E}\Big[\big(\int_0^t \partial_1 H^{p+1,\infty}(\omega_s.e_1)ds\big)^2\Big]\\ &\geq t + \frac{1}{2}\mathbb{E}\Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)^2\Big] - t^2\big(K_1\frac{\gamma_{p+1}}{R_{p+1}}\frac{\rho_{\min}}{\rho_{\min}-\gamma_{\max}}\big)^2\end{aligned} \tag{11.69}$$

and

$$\begin{aligned}\mathbb{E}\big[(y_t.e_2)^2\big] &\leq t + 2\mathbb{E}\Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)^2\Big] + 2\mathbb{E}\Big[\big(\int_0^t \partial_1 H^{p+1,\infty}(\omega_s.e_1)ds\big)^2\Big] \\ &\leq t + 2\mathbb{E}\Big[\big(\int_0^t \partial_1 H^p(\omega_s.e_1)ds\big)^2\Big] + 2t^2\big(K_1 \frac{\gamma_{p+1}}{R_{p+1}}\frac{\rho_{\min}}{\rho_{\min}-\gamma_{\max}}\big)^2\end{aligned} \tag{11.70}$$

Then it follows by the lemma 11.3.5 that if $\gamma_{min} > 1$ and

$$\rho_{\min} \geq 8(K_0\gamma_{\max}\gamma_{\min}+1)\frac{K_1}{\gamma_{\min}-1} \tag{11.71}$$

one has

$$\begin{aligned}\mathbb{E}\big[(y_t.e_2)^2\big] \leq\ & t\big(1 + 10\gamma_{p-1}^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 + 16K_0\gamma_{p-1}\gamma_p\frac{\gamma_{\min}}{\gamma_{\min}-1}\frac{K_1}{r_p}\big) \\ & + t^2\big(K_1^2 2\frac{\gamma_p^2}{R_p^2} + 2\big(K_1\frac{\gamma_{p+1}}{R_{p+1}}\frac{\rho_{\min}}{\rho_{\min}-\gamma_{\max}}\big)^2\big) \\ & + \sqrt{t}132\gamma_{p-1}\gamma_p R_{p-1}K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1} \\ & + \frac{1}{t^{\frac{1}{4}}}120K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_p} \\ & + 8K_0^2\gamma_{p-1}^2R_{p-1}^2\end{aligned} \tag{11.72}$$

and

$$\begin{aligned}\mathbb{E}\big[(y_t.e_2)^2\big] \geq\ & t(\frac{\gamma_{p-1}^2}{2}+1) \\ & - t^2\big(K_1\frac{\gamma_{p+1}}{R_{p+1}}\frac{\rho_{\min}}{\rho_{\min}-\gamma_{\max}}\big)^2 \\ & - \sqrt{t}34\gamma_{p-1}\gamma_p R_{p-1}K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1} \\ & - \frac{1}{t^{\frac{1}{4}}}30K_0^2\frac{\gamma_{\min}}{\gamma_{\min}-1}\gamma_{p-1}\gamma_p\frac{R_p^{\frac{5}{2}}}{r_p} \\ & - 2K_0^2\gamma_{p-1}^2R_{p-1}^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2\end{aligned} \tag{11.73}$$

Now observe that in the equation 11.73 the first member is greater than 8 times the second one if

$$R_{p+1}^2\frac{\gamma_{p-1}^2}{16\gamma_{p+1}^2K_1^2}\big(1-\frac{\gamma_{\max}}{\rho_{\min}}\big)^2 \geq t \tag{11.74}$$

In the equation 11.73 the first member is greater than 8 times the third one if

$$t \geq 2^{19}\frac{\gamma_p^2}{\gamma_{p-1}^2}R_{p-1}^2K_0^4(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 \tag{11.75}$$

In the equation 11.73 the first member is greater than 8 times the fourth one if

$$t \geq 240\frac{R_p^2}{r_p^{\frac{4}{5}}}K_0^{\frac{8}{5}}(\frac{\gamma_{\min}}{\gamma_{\min}-1})^{\frac{4}{5}}(\frac{\gamma_p}{\gamma_{p-1}})^{\frac{4}{5}} \tag{11.76}$$

In the equation 11.73 the first member is greater than 8 times the fifth one if

$$t \geq 32K_0^2R_{p-1}^2(\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 \tag{11.77}$$

Then choose

$$p(t) = \sup\{p \in \mathbb{N} \,:\, t \geq 240\frac{R_p^2}{r_p^{\frac{4}{5}}} K_0^{\frac{8}{5}} (\frac{\gamma_{\min}}{\gamma_{\min}-1})^{\frac{4}{5}} (\frac{\gamma_p}{\gamma_{p-1}})^{\frac{4}{5}}\} \tag{11.78}$$

Observe that with this choice

$$t < 240\frac{R_{p+1}^2}{r_{p+1}^{\frac{4}{5}}} K_0^{\frac{8}{5}} (\frac{\gamma_{\min}}{\gamma_{\min}-1})^{\frac{4}{5}} (\frac{\gamma_{p+1}}{\gamma_p})^{\frac{4}{5}} \tag{11.79}$$

and assume that

$$2^{15}(1+K_0)^3(1+K_1)^3 \frac{\gamma_{\min}}{\gamma_{\min}-1} \gamma_{\max}^4 \leq \rho_{\min} \tag{11.80}$$

Then a straightforward computation shows that 11.78, 11.79 and 11.80 implies the conditions 11.74, 11.75, 11.76 and 11.77. It follows that

$$t\frac{\gamma_{p-1}^2}{4} \leq \mathbb{E}\big[(y_t.e_2)^2\big] \tag{11.81}$$

Similarly with the choice

$$p(t) = \inf\{p \in \mathbb{N} \,:\, t \leq R_p^2\} \tag{11.82}$$

it follows under the assumption 11.80 that

$$\mathbb{E}\big[(y_t.e_2)^2\big] \leq t\gamma_p^2 30 (\frac{\gamma_{\min}}{\gamma_{\min}-1})^2 (1+K_1)^2 \tag{11.83}$$

□

Theorem 11.3.1. *assume* $\gamma_{min} > 1$, $\gamma_{\max}, \rho_{\max} < \infty$, $\rho_{min} > \rho_0(\gamma_{min}, \gamma_{max}, K_0, K_1)$ *and* $t > t_0(\gamma_{min}, \gamma_{max}, R_1, K_0, K_1)$ *then*

$$\mathbb{E}_0[|y_t.e_2|^2] = t^{1+\nu(t)} \tag{11.84}$$

with

$$\nu(t) \leq \frac{\ln \gamma_{\max}}{\ln \rho_{min} + \ln \frac{\gamma_{\min}}{\gamma_{\max}}} + \frac{C_2}{\ln t} \tag{11.85}$$

$$\nu(t) \geq \frac{\ln \gamma_{\min}}{\ln \rho_{max} + \ln \frac{\gamma_{\max}}{\gamma_{\min}}} - \frac{C_1}{\ln t} \tag{11.86}$$

Where the constants C_1 *and* C_2 *depends on* $\rho_{\min}, \gamma_{\min}, \gamma_{\max}, \rho_{\max}, K_1, K_2$

Remark 11.3.1. Note that for $\gamma_{\min} = \gamma_{\max}$ and $\rho_{\min} = \rho_{\max} = \rho$

$$|\nu(t) - \frac{\ln \gamma}{\ln \rho}| \leq \frac{C(\gamma,\rho)}{\ln t} \tag{11.87}$$

Proof. Straightforward by the lemma 11.3.6. □

Theorem 11.3.2. *assume* $\gamma_p = \gamma^p$ *and* $R_p = R_{p-1}[\frac{\rho^{p^\alpha}}{R_{p-1}}]$ *with* $\gamma, \rho > 1$ *and* $\alpha \geq 1$ *Then for* $t > t_0(\gamma_2, R_2, K_0, K_1)$

$$C_1 t \gamma^{\beta(t)} \leq \mathbb{E}_0[|y_t.e_2|^2] \leq C_2 t\gamma^{\beta(t)} \tag{11.88}$$

with

$$\beta(t) = 2(\frac{1}{2\ln \rho})^{\frac{1}{\alpha}} (\ln t)^{\frac{1}{\alpha}} \tag{11.89}$$

Where the constants C_1 *and* C_2 *depends on* $\rho, \gamma, \alpha, K_1, K_2$

Remark 11.3.2. Note that this theorem shows how the diffusion becomes more and more super diffusive as $\alpha \downarrow 1$: the ratio between scales tends to be constant.

Proof. Straightforward by the lemma 11.3.6. □

12. RATE OF CONVERGENCE TOWARDS THE LIMIT PROCESS

12.1 Sharp upper bound for the rate of convergence towards the asymptotic process

12.1.1 Cluster expansion

Lemma 12.1.1. *Let M_t be a continuous square integrable $\mathcal{F}_t$ adapted martingale such that $M_0 = 0$ and for $\lambda, t > 0$, $\mathbb{E}[e^{\lambda M_t}] < \infty$.*
Assume that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t_2 > t_1 \geq 0$

$$\mathbb{E}[\int_{t_1}^{t_2} d < M, M >_s |\mathcal{F}_{t_1}] \leq \int_0^{t_2-t_1} f(s)ds \quad a.s. \tag{12.1}$$

1. *Then for all $\lambda \in \mathbb{R}$ and all $q > 1$*

$$\begin{aligned}\mathbb{E}[\exp(\lambda M_t)] \leq [1 + \sum_{n=1}^{+\infty} (\frac{q^2}{2(q-1)}\lambda^2)^n \\ \int_{u_i > 0} 1(u_1 + \cdots + u_n < t) f(u_1) \cdots f(u_n) du_1 \cdots du_n]^{\frac{1}{q}}\end{aligned} \tag{12.2}$$

2. *If $f(s) = f_1$ for $s < t_0$ and $f(s) = f_2$ for $s \geq t_0$ with $t_0 > 0$ and $0 \leq f_2 \leq f_1$ then with $a = \frac{f_2}{f_1}$ and $\mu = \frac{t}{t_0}$*

$$\mathbb{E}[\exp(\lambda M_t)] \leq \Big[1 + \sum_{n=1}^{+\infty} \frac{(\frac{q^2}{2(q-1)}\lambda^2 f_1 t_0)^n}{n!} \sum_{0 \leq m \leq n \wedge \mu} (\mu - m)^n C_n^m (a-1)^m\Big]^{\frac{1}{q}} \tag{12.3}$$

Proof. (a) Note that for $c > 0$

$$\mathbb{E}[\exp(\lambda M_t)] = \mathbb{E}[\exp(\lambda M_t - c\lambda^2 < M, M >_t) \exp(c\lambda^2 < M, M >_t)]$$

Then by Hölder inequality

$$\mathbb{E}[\exp(\lambda M_t)] \leq \mathbb{E}[\exp(p\lambda M_t - cp\lambda^2 < M, M >_t)]^{\frac{1}{p}} E[\exp(cq\lambda^2 < M, M >_t)]^{\frac{1}{q}}$$

with $1/p + 1/q = 1$; choose $c = p/2$ then by the Ito formula $\exp(p\lambda M_t - (p^2/2)\lambda^2 < M, M >_t)$ is a positive local martingale equal to 1 at $t = 0$ thus it is a sur-martingale and

$$\mathbb{E}[\exp(p\lambda M_t - (p^2/2)\lambda^2 < M, M >_t)] \leq 1$$

Thus for all $q > 1$

$$\mathbb{E}[\exp(\lambda M_t)] \leq \mathbb{E}[\exp(\frac{q^2}{2(q-1)}\lambda^2 < M, M >_t)]^{\frac{1}{q}}$$

Write $X_t = \exp(h_q\lambda^2 < M, M >_t)$ with $h_q = \frac{q^2}{2(q-1)}$
Then

$$X_t = \sum_{n=0}^{+\infty} \frac{(h_q\lambda^2 < M, M >_t)^n}{n!}$$

$$X_t = 1 + \sum_{n=1}^{+\infty} (h_q\lambda^2)^n \int 1(0 < t_1 < \cdots < t_n < t) d < M, M >_{t_1} \cdots d < M, M >_{t_n}$$

Write

$$W_n = \int 1(0 < t_1 < \cdots < t_n < t) d < M, M >_{t_1} \cdots d < M, M >_{t_n}$$

Now it will be proved by induction that

$$\mathbb{E}[W_n] \leq \int\limits_{u_i > 0} 1(0 < u_1 + \cdots + u_n < t) f(u_1) \cdots f(u_n) du_1 \cdots du_n$$

First of all,

$$\begin{aligned}\mathbb{E}[W_n] =& \mathbb{E}[\int 1(0 < t_1 < \cdots < t_{n-1} < t) \mathbb{E}[\int\limits_{t_{n-1}}^{t} d < M, M >_{t_n} |\mathcal{F}_{t_{n-1}}] \\ & d < M, M >_{t_1} \cdots d < M, M >_{t_{n-1}}]\end{aligned}$$

this leads to

$$\begin{aligned}\mathbb{E}[W_n] \leq & \mathbb{E}[\int 1(0 < t_1 < \cdots < t_{n-1} < t) \int\limits_{0}^{t - t_{n-1}} f(t_n) dt_n d < M, M >_{t_1} \\ & \cdots d < M, M >_{t_{n-1}}]\end{aligned}$$

$$\begin{aligned}\mathbb{E}[W_n] \leq & \mathbb{E}[\int 1(0 < t_1 < \cdots < t_{n-1} < t_n < t) d < M, M >_{t_1} \\ & \cdots d < M, M >_{t_{n-1}} f(t_n - t_{n-1}) dt_n]\end{aligned}$$

and the following inequality completes the first step of the induction

$$\begin{aligned}\mathbb{E}[W_n] \leq & \mathbb{E}[\int 1(0 < t_1 < \cdots < t_{n-1} < t) 1(0 < u_n < t - t_{n-1}) \\ & d < M, M >_{t_1} \cdots d < M, M >_{t_{n-1}} f(u_n) du_n]\end{aligned}$$

The mechanism is now clear: if

$$\begin{aligned}&\mathbb{E}[W_n] \leq \mathbb{E}[\int\limits_{u_i > 0} 1(0 < t_1 < \cdots < t_k < t_{k+1} < t) 1(0 < u_{k+2} + \cdots + u_n < t - t_{k+1}) \\ & d < M, M >_{t_1} \cdots d < M, M >_{t_k} f(u_{k+2}) \cdots f(u_n) du_{k+2} \cdots du_n]\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}[W_n] \leq \mathbb{E}[\int\limits_{u_i > 0} 1(0 < t_1 < \cdots < t_k < t) 1(0 < u_{k+2} + \cdots + u_n < t - t_k) d < M, M >_{t_1} \\ & \cdots d < M, M >_{t_k} \mathbb{E}[\int\limits_{t_k}^{t - u_{k+2} - \cdots - u_n} d < M, M >_{t_{k+1}} |F_{t_k}] f(u_{k+2}) \cdots f(u_n) du_{k+2} \cdots du_n]\end{aligned}$$

$$\mathbb{E}[W_n] \leq \mathbb{E}[\int_{u_i>0} 1(0<t_1<\cdots<t_k<t)1(0<u_{k+2}+\cdots+u_n<t-t_k)d<M,M>_{t_1}$$
$$\cdots d<M,M>_{t_k} \int_0^{t-t_k-u_{k+2}-\cdots-u_n} f(t_{k+1})dt_{k+1}f(u_{k+2})\cdots f(u_n)du_{k+2}\cdots du_n]$$

$$\mathbb{E}[W_n] \leq \mathbb{E}[\int_{u_i>0} 1(0<t_1<\cdots<t_k<t)1(0<u_{k+1}+\cdots+u_n<t-t_k)$$
$$d<M,M>_{t_1}\cdots d<M,M>_{t_k} f(u_{k+1})\cdots f(u_n)du_{k+1}\cdots du_n]$$

And the following inequality is deduced from this induction

$$\mathbb{E}[W_n] \leq \int_{u_i>0} 1(0<u_1+\cdots+u_n<t)f(u_1)\cdots f(u_n)du_1\cdots du_n$$

Thus

$$\mathbb{E}[\exp(\lambda M_t)] \leq [1+\sum_{n=1}^{+\infty}(h_q\lambda^2)^n \int_{u_i>0} 1(0<u_1+\cdots$$
$$+u_n<t)f(u_1)\cdots f(u_n)du_1\cdots du_n]^{\frac{1}{q}}$$

This proves the first part of the lemma, now the second part will be proven.

(b) Assume that $f(s)=f_1$ for $s<t_0$ and $f(s)=f_2$ for $s\geq t_0$ with $t_0>0$ and $0\leq f_2\leq f_1$ Then write $a=\frac{f_2}{f_1}$ and $\mu=\frac{t}{t_0}$ and

$$H_n = \int_{u_i>0} 1(0<u_1+\cdots+u_n<t)f(u_1)\cdots f(u_n)du_1\cdots du_n$$

Now, by writing $g(z)=1(z<1)+a1(z\geq 1)$ and $u_i=z_it_0$

$$H_n = (f_1t_0)^n \int_{z_i>0} 1(0<z_1+\cdots+z_n<\mu)g(z_1)\cdots g(z_n)dz_1\cdots dz_n$$

Define

$$G_n = \int_{z_i>0} 1(0<z_1+\cdots+z_n<\mu)g(z_1)\cdots g(z_n)dz_1\cdots dz_n$$

$$G_n = \int_{z_i>0} 1(0<z_1+\cdots+z_n<\mu)\prod_{k=1}^{n}(1(z_k<1)+a1(z_k\geq 1))dz_1\cdots dz_n$$

$$G_n = \sum_{p=0}^{n} a^pC_n^p \int_{z_i>0} 1(0<z_1+\cdots+z_n<\mu)1(z_1>0)$$
$$\cdots 1(z_p>1)1(z_{p+1}<1)\cdots 1(z_n<1)dz_1\cdots dz_n$$

Write

$$G_n^p = \int_{z_i>0} 1(0 < z_1 + \cdots + z_n < \mu)1(z_1 > 1)\cdots 1(z_p > 1)1(z_{p+1} < 1)$$
$$\cdots 1(z_n < 1)dz_1 \cdots dz_n$$
$$= \int_{z_i>0} 1(0 < z_1 + \cdots + z_n < \mu - p)1(z_{p+1} < 1)\cdots 1(z_n < 1)dz_1 \cdots dz_n$$
$$= \int_{z_i>0} (\mu - p - z_2 - \cdots - z_n)1(0 < z_2 + \cdots + z_n < \mu - p)1(z_{p+1} < 1)$$
$$\cdots 1(z_n < 1)dz_2 \cdots dz_n$$

And by induction if $(k \leq p)$

$$G_n^p = \int_{z_i>0} \frac{(\mu - p - z_k - \cdots - z_n)^{k-1}}{(k-1)!} 1(0 < z_k + \cdots + z_n < \mu - p)1(z_{p+1} < 1)$$
$$\cdots 1(z_n < 1)dz_k \cdots dz_n$$

Then

$$G_n^p = \int_{z_i>0} \frac{(\mu - p - z_{k+1} - \cdots - z_n)^k}{(k)!} 1(0 < z_{k+1} +$$
$$\cdots + z_n < \mu - p)1(z_{p+1} < 1)\cdots 1(z_n < 1)dz_{k+1} \cdots dz_n$$

Thus

$$G_n^p = \int_{z_i>0} \frac{(\mu - p - z_{p+1} - \cdots - z_n)^p}{(p)!} 1(0 < z_{p+1} +$$
$$\cdots + z_n < \mu - p)1(z_{p+1} < 1)\cdots 1(z_n < 1)dz_{p+1} \cdots dz_n$$

Now it will be shown by induction that

$$G_n^p = \sum_{k=0}^{n-p} v_{n-p}^k \frac{(\mu - p - k)^n}{n!} 1(p + k < \mu)$$

Where the sequence v_j^k is defined by

$$\forall j \geq 0 \; , v_j^0 = 1$$

$$\forall k \geq 1 \; v_0^k = 0$$

$$\forall j \geq 0 \; , k \geq 0 \; , v_{j+1}^{k+1} = v_j^{k+1} - v_j^k$$

indeed

$$G_n^p = \int_{z_i>0} \Big(\frac{(\mu - p - z_{p+2} - \cdots - z_n)^{p+1}}{(p+1)!} - 1(z_{p+2} + \cdots + z_n < \mu - p - 1)$$
$$\frac{(\mu - p - 1 - z_{p+2} - \cdots - z_n)^{p+1}}{(p+1)!} \Big)$$
$$1(0 < z_{p+2} + \cdots + z_n < \mu - p)1(z_{p+2} < 1)\cdots 1(z_n < 1)dz_{p+2} \cdots dz_n$$

If

$$G_n^p = \int_{z_i>0} \frac{(\mu - p - z_{p+j+1} - \cdots - z_n)^{p+j}}{(p+j)!} 1(0 < z_{p+j+1} + \cdots + z_n < \mu - p) 1(z_{p+j+1} < 1) \cdots 1(z_n < 1) dz_{p+j+1} \cdots dz_n + \\ \vdots \\ + \int_{z_i>0} v_j^k \frac{(\mu - p - k - z_{p+j+1} - \cdots - z_n)^{p+j}}{(p+j)!} 1(0 < z_{p+j+1} + \cdots + z_n < \mu - p - k) 1(z_{p+j+1} < 1) \cdots 1(z_n < 1) dz_{p+j+1} \cdots dz_n + \\ \vdots \\ + \int_{z_i>0} v_j^j \frac{(\mu - p - j - z_{p+j+1} - \cdots - z_n)^{p+j}}{(p+j)!} 1(0 < z_{p+j+1} + \cdots + z_n < \mu - p - j) 1(z_{p+j+1} < 1) \cdots 1(z_n < 1) dz_{p+j+1} \cdots dz_n$$

Then

$$G_n^p = \int_{z_i>0} \frac{(\mu - p - z_{p+j+2} - \cdots - z_n)^{p+j+1}}{(p+j+1)!} 1(0 < z_{p+j+2} + \cdots + z_n < \mu - p) 1(z_{p+j+2} < 1) \cdots 1(z_n < 1) dz_{p+j+2} \cdots dz_n \\ + \sum_{k=1}^{j+1} \int_{z_i>0} (v_j^k - v_j^{k-1}) \frac{(\mu - p - k - z_{p+j+2} - \cdots - z_n)^{p+j+1}}{(p+j+1)!} 1(0 < z_{p+j+2} + \cdots + z_n < \mu - p - k) 1(z_{p+j+2} < 1) \cdots 1(z_n < 1) dz_{p+j+2} \cdots dz_n$$

Which proves the induction.
Thus

$$G_n^p = \frac{(\mu - p)^n}{n!} 1(p < \mu) - (n-p) \frac{(\mu - p - 1)^n}{n!} 1(p + 1 < \mu) \\ + \frac{(n-p)(n-p-1)}{2} \frac{(\mu - p - 2)^n}{n!} 1(p + 2 < \mu) \\ + v_{n-p}^k \frac{(\mu - p - k)^n}{n!} 1(p + k < \mu) + \cdots + v_{n-p}^{n-p} \frac{(\mu - n)^n}{n!} 1(n < \mu)$$

Now it will be shown by a double induction that $\forall j \geq k$

$$v_j^k = C_j^k (-1)^k$$

$v_0^0 = 1$ so this is true for j=0
Assume that it is true for $j \leq j_0$ Observe that $v_{j_0+1}^0 = 1$, now assume that the statement is true in $j_0 + 1$ for $k \leq k_0 < j_0 + 1$ then

$$v_{j_0+1}^{k_0+1} = v_{j_0}^{k_0+1} - v_{j_0}^{k_0} \\ = (-1)^{k_0+1} [C_{j_0}^{k_0+1} + C_{j_0}^{k_0}] \\ = (-1)^{k_0+1} \frac{(j_0)!}{(k_0)!(j_0 - k_0 - 1)!} [\frac{1}{k_0 + 1} + \frac{1}{j_0 - k_0}] \\ = (-1)^{k_0+1} C_{j_0+1}^{k_0+1}$$

Which proves the induction
Thus

$$G_n = \sum_{p=0}^{n} a^p C_n^p \sum_{k=0}^{n-p} C_{n-p}^k (-1)^k \frac{(\mu - p - k)^n}{n!} 1(p + k < \mu)$$

Write $m = k + p$, then

$$G_n = \frac{1}{n!} \sum_{0 \leq m \leq \mu \wedge n} (\mu - m)^n \sum_{0 \leq p \leq m} a^p C_n^p C_{n-p}^{m-p} (-1)^(m - p)$$

But

$$C_n^p C_{n-p}^{m-p} = \frac{n!}{p!(m-p)!(n-m)!} = C_n^m C_m^p$$

Thus

$$\begin{aligned} G_n &= \frac{1}{n!} \sum_{0 \leq m \leq \mu \wedge n} C_n^m (\mu - m)^n \sum_{0 \leq p \leq m} a^p C_m^p (-1)^(m + p) \\ &= \frac{1}{n!} \sum_{0 \leq m \leq \mu \wedge n} C_n^m (\mu - m)^n (a-1)^m \end{aligned}$$

In short, is has been obtained that

$$\mathbb{E}[\exp(\lambda M_t)] \leq \Big[1 + \sum_{n=1}^{+\infty} \frac{(h_q \lambda^2 f_1 t_0)^n}{n!} \sum_{0 \leq m \leq n \wedge \mu} (\mu - m)^n C_n^m (a-1)^m \Big]^{\frac{1}{q}}$$

□

12.1.2 A combinatorial lemma

The following lemma is quite technical and will be useful to exploit the previous one

Lemma 12.1.2. *Put* $-\frac{1}{e} < x < 0$
Write for $n \in \mathbb{N}$*,*

$$I_n = \sum_{0 \leq m \leq n} \frac{x^m}{m!} (n - m)^m$$

Write u_p *the sequence defined by* $u_0 = 0$ *and* $u_{p+1} = \exp(-x u_p)$
Then u_p *is increasing and converges to* y_0 *the smallest positive solution of* $y \exp(xy) = 1$ *and* $\forall p \in \mathbb{N}^*$, $\forall n \in \mathbb{N}$

$$I_n \leq \Big(\frac{1}{u_p(x)} \Big)^n \frac{1}{1 - u_p \exp(x u_p)}$$

Proof. Write

$$y_1 = \inf \big\{ y > 0 \ : \ y \exp(|x| y) = 1 \big\}$$

(notice that $0 < y_1 < 1$) and consider for $-y_1 < y < y_1$ the function

$$f \ : \ y \to \frac{1}{1 - y \exp(xy)}$$

Observe that for $y \in (-y_1, y_1)$

$$\begin{aligned} f(y) &= \sum_{n=0}^{+\infty} (y \exp(xy))^n \\ &= \sum_{n=0}^{+\infty} y^n \sum_{m=0}^{+\infty} \frac{(nxy)^m}{m!} \end{aligned}$$

and since

$$\sum_{0\leq n,m\leq+\infty} y^n \frac{(n|x|y)^m}{m!} = \frac{1}{1-y\exp(|x|y)} < \infty$$

with a normal convergence of the series, the order of the limits can be changed. So

$$\begin{aligned} f(y) &= \sum_{m=0}^{+\infty} \frac{(nxy)^m}{m!} \sum_{n=0}^{+\infty} n^m y^n \\ &= \sum_{m=0}^{+\infty} \frac{x^m}{m!} \sum_{n=m}^{+\infty} (n-m)^m y^n \\ &= \sum_{n=0}^{+\infty} y^n \sum_{m=0}^{n} (n-m)^m \frac{x^m}{m!} \\ &= \sum_{n=0}^{+\infty} y^n I_n \end{aligned}$$

and deduce that $\forall n \in \mathbb{N},\ I_n = \frac{f^{(n)}(0)}{n!}$ Now, consider the function

$$\begin{aligned} g\ :\ y &\rightarrow y\exp(xy) \\ \mathbb{R} &\rightarrow \mathbb{R} \end{aligned}$$

Observe that,

$$g'(y) = \exp(xy)[1+xy]$$

$$g'(y) = 0 \Leftrightarrow y = -\frac{1}{x} = y_2$$

Thus, g is increasing from $-\infty$ to y_2 then decreasing and $g(y_2) = -\frac{\exp(-1)}{x}$ Thus, if $-\frac{1}{e} < x < 0$; then $y_0 = \inf\left\{y > 0\ :\ g(y) = 1\right\}$ does exist.
Moreover $\forall y \in]-y_1, y_0[,\ \forall n,\ f^{(n)}(y) \geq 0$
Deduce from the classical theorem of Taylor expansion that the series $\sum_{n=0}^{+\infty} y^n \frac{f^{(n)}(0)}{n!}$ converges towards f for $y \in]-y_1, y_0[$
So for $y \in]-y_1, y_0[$

$$\sum_{n=0}^{\infty} y^n I_n = \frac{1}{1-y\exp(xy)}$$

and this sum is finite for $y \in]0, y_0[$
Deduce that $\forall y \in]0, y_0[\ \forall n \in \mathbb{N}\ I_n = \left(\frac{1}{y}\right)^n \frac{1}{1-y\exp(xy)}$
On the other hand if one consider the sequence $u_0 = 0,\ u_{p+1} = \exp(-xu_p)$ then it is an exercise to show that u_p is increasing and will converge towards y_0
So $\forall p \in \mathbb{N}^*,\ \forall n \in \mathbb{N}$

$$I_n \leq \left(\frac{1}{u_p(x)}\right)^n \frac{1}{1-u_p\exp(xu_p)}$$

□

12.1.2.i The main theorem: sharp control of the Laplace transform

Theorem 12.1.1. *Consider M_t a continuous square integrable $\mathcal{F}_t$ adapted martingale such that $M_0 = 0$ and for $\lambda, t > 0$, $\mathbb{E}[e^{\lambda M_t}] < \infty$.*
Assume that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t_2 > t_1 \geq 0$ one has a.s.

$$\mathbb{E}[\int_{t_1}^{t_2} d < M, M >_s |\mathcal{F}_{t_1}] \leq \int_0^{t_2 - t_1} f(s) ds$$

With $f(s) = f_1$ for $s < t_0$ and $f(s) = f_2$ for $s \geq t_0$ with $t_0 > 0$ and $0 < f_2 < f_1$.
Then

1. *for all*

$$0 < |\lambda| < \frac{1}{\big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}} \tag{12.4}$$

one has

$$\mathbb{E}[\exp(\lambda M_t)] \leq e^{3(1-1/g(\lambda))} \exp(\frac{g(\lambda)}{2}\lambda^2 f_2 t) \tag{12.5}$$

with $g(\lambda) = \frac{1}{1-\lambda^2(f_1-f_2)t_0 e}$ which verify $1 \leq g \leq 2$

2. *for all*

$$0 < \nu < \frac{1}{2e(f_1 - f_2)t_0} \tag{12.6}$$

one has

$$\mathbb{E}[\exp(\nu < M, M >_t)] \leq \exp(\nu f_2 t) \frac{\exp\big(\nu t_0 (f_1 - f_2)\big)}{((f_1 - f_2)\nu t_0)^2} \tag{12.7}$$

observe that $g \to 1$ when $\lambda \to 0$ so the control is very sharp.

Proof. Write $a = \frac{f_2}{f_1}$, $\mu = \frac{t}{t_0}$ and $h_q = \frac{q^2}{2(q-1)}$
According to the lemma 12.1.1 by the Hölder inequality for $v > 1$ and $1/v + 1/\tilde{v} = 1$

$$\begin{aligned}
\mathbb{E}[\exp(\lambda M_t)] &\leq \mathbb{E}[\exp(h_q \lambda^2 < M, M >_t)]^{\frac{1}{q}} \\
&\leq \mathbb{E}[\exp\big(h_q \lambda^2 (< M, M >_{[\mu]t_0} + \int_{[\mu]t_0}^t d < M, M >_s)\big)]^{\frac{1}{q}} \\
&\leq \mathbb{E}[\exp(h_q v \lambda^2 < M, M >_{[\mu]t_0})]^{\frac{1}{qv}} \mathbb{E}[\exp\big(h_q \tilde{v} \lambda^2 \int_{[\mu]t_0}^t d < M, M >_s \big)]^{\frac{1}{q\tilde{v}}}
\end{aligned}$$

Now

$$\mathbb{E}[\exp\big(\tilde{v} h_q \lambda^2 \int_{[\mu]t_0}^t d < M, M >_s \big)] \leq \exp\big(\tilde{v} h_q \lambda^2 (t - [\mu]t_0) f_1\big)$$

and make $v \to 1$ to obtain

$$\mathbb{E}[\exp(\lambda M_t)] \leq \mathbb{E}[\exp(h_q \lambda^2 < M, M >_{[\mu]t_0})]^{\frac{1}{q}} \exp\big(\frac{h_q}{q} \lambda^2 (t - [\mu]t_0) f_1\big)$$

And by the lemma 12.1.1

$$\mathbb{E}[\exp(h_q\lambda^2 < M,M >_{[\mu]t_0})] \leq \sum_{n=0}^{+\infty}\frac{(h_q\lambda^2 f_1 t_0)^n}{n!}\sum_{0\leq m\leq n\wedge[\mu]}([\mu]-m)^n C_n^m(a-1)^m$$

Thus

$$\begin{aligned}\mathbb{E}[\exp(h_q\lambda^2 < M,M >_{[\mu]t_0})] &\leq \sum_{m=0}^{[\mu]}\frac{([\mu]-m)^m}{m!}(a-1)^m\sum_{m\leq n}\frac{(h_q\lambda^2 f_1 t_0)^n}{(n-m)!}([\mu]-m)^{n-m}\\ &\leq \exp(h_q\lambda^2 f_1 t_0[\mu])\sum_{0\leq m\leq[\mu]}\frac{([\mu]-m)^m(h_q(a-1)\lambda^2 f_1 t_0)^m}{m!}([\mu]-m)^{n-m}\end{aligned}$$

But according to the lemma 12.1.2 for

$$-\frac{1}{e} < h_q(a-1)\lambda^2 f_1 t_0 < 0 \Leftrightarrow 0 < |\lambda| < \frac{1}{\big(eh_q(f_1-f_2)t_0\big)^{\frac{1}{2}}}$$

one has

$$\sum_{0\leq m\leq[\mu]}\frac{([\mu]-m)^m(h_q(a-1)\lambda^2 f_1 t_0)^m}{m!}([\mu]-m)^{n-m} \leq (\frac{1}{u_p(y)})^{\mu}\frac{1}{1-u_p(y)\exp(yu_p(y))}$$

With $y = h_q(a-1)\lambda^2 f_1 t_0$ and $u_p(x) = u_2(x) = \exp(-x)$
Then by using $\exp(-y)-1 \geq -y$ and $-\frac{1}{e} < y < 0$

$$\begin{aligned}\sum_{0\leq m\leq[\mu]}\frac{([\mu]-m)^m y^m}{m!}([\mu]-m)^{n-m} &\leq \frac{\exp(y[\mu])}{1-\exp\big(y(\exp(-y)-1)\big)}\\ &\leq \frac{\exp(y[\mu])}{1-\exp\big(-y^2)\big)}\\ &\leq \frac{\exp(y[\mu])}{y^2}\end{aligned}$$

So

$$\mathbb{E}[\exp(h_q\lambda^2 < M,M >_{[\mu]t_0})] \leq \exp(h_q\lambda^2 f_1 t_0[\mu])\frac{\exp(h_q(a-1)\lambda^2 f_1 t_0[\mu])}{(h_q(a-1)\lambda^2 f_1 t_0)^2}$$

and

$$\mathbb{E}[\exp(h_q\lambda^2 < M,M >_{[\mu]t_0})]^{\frac{1}{q}} \leq \frac{\exp(\frac{h_q}{q}\lambda^2 f_2 t_0[\mu])}{(h_q(1-a)\lambda^2 f_1 t_0)^{\frac{2}{q}}}$$

Thus

$$\mathbb{E}[\exp(\lambda M_t)] \leq \frac{\exp(\frac{h_q}{q}\lambda^2 f_2 t_0[\mu])}{(h_q(1-a)\lambda^2 f_1 t_0)^{\frac{2}{q}}}\exp\big(\frac{h_q}{q}\lambda^2(t-[\mu]t_0)f_1\big)$$

and

$$\begin{aligned}\mathbb{E}[\exp(\lambda M_t)] &\leq \frac{\exp(\frac{h_q}{q}\lambda^2 f_2 t)}{(h_q(f_1-f_2)\lambda^2 t_0)^{\frac{2}{q}}}\exp\big(\frac{h_q}{q}\lambda^2 t_0(\frac{t}{t_0}-[\mu])(f_1-f_2)\big)\\ &\leq \frac{\exp(\frac{h_q}{q}\lambda^2 f_2 t)}{(h_q(f_1-f_2)\lambda^2 t_0)^{\frac{2}{q}}}\exp\big(\frac{h_q}{q}\lambda^2 t_0(f_1-f_2)\big)\end{aligned}$$

Thus it has been obtained that for all $q > 1$ and all

$$0 < |\lambda| < \frac{1}{\big(e\frac{q^2}{2(q-1)}(f_1 - f_2)t_0\big)^{\frac{1}{2}}} \tag{12.8}$$

one has

$$\mathbb{E}[\exp(\lambda M_t)] \leq \frac{\exp(\frac{q}{2(q-1)}\lambda^2 f_2 t)}{(\frac{q^2}{2(q-1)}(f_1 - f_2)\lambda^2 t_0)^{\frac{2}{q}}} \exp\big(\frac{q}{2(q-1)}\lambda^2 t_0(f_1 - f_2)\big) \tag{12.9}$$

Now if

$$0 < |\lambda| < \frac{1}{\big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}} \tag{12.10}$$

chose

$$q = \frac{1}{\lambda^2(f_1 - f_2)t_0 e} \tag{12.11}$$

then by a straightforward computation $q > 2$ and the inequality 12.8 is satisfied. It follows after an easy computation that

$$\mathbb{E}[\exp(\lambda M_t)] \leq e^{3(1-1/g(\lambda))} \exp(\frac{g(\lambda)}{2}\lambda^2 f_2 t) \tag{12.12}$$

with $g(\lambda) = \frac{1}{1-\lambda^2(f_1-f_2)t_0 e}$ which verify $1 \leq g \leq 2$.

Now consider

$$0 < \nu < \frac{1}{2e(f_1 - f_2)t_0} \tag{12.13}$$

for $q > 1$ chose λ so that $\nu = \lambda^2 h_q$.
Observe also that the inequality 12.8 is satisfied, thus

$$\mathbb{E}[\exp(h_q\lambda^2 < M, M >_t)]^{\frac{1}{q}} \leq \frac{\exp(\frac{h_q}{q}\lambda^2 f_2 t)}{(h_q(f_1 - f_2)\lambda^2 t_0)^{\frac{2}{q}}} \exp\big(\frac{h_q}{q}\lambda^2 t_0(f_1 - f_2)\big)$$

From which one deduces that

$$\mathbb{E}[\exp(\nu < M, M >_t)] \leq \frac{\exp(\nu f_2 t)}{((f_1 - f_2)\nu t_0)^2} \exp\big(\nu t_0(f_1 - f_2)\big)$$

Which proves the theorem. □

12.1.3 Application to bound from above the tail estimate of a martingale

Corollary 12.1.1. *Let M_t be the martingale given in theorem 12.1.1.*
Write $C_1 = \big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}/f_2$. Then for

$$r = \frac{C_1 x}{t} < 1 \tag{12.14}$$

one has

$$\mathbb{P}(M_t \geq x) \leq e^{\frac{3}{2}r^2} \exp\big(-(1 - r^2)\frac{x^2}{2f_2 t}\big) \tag{12.15}$$

Note that $0 < g_1(r) \leq 1$ and that g_1 converges towards 1 with the speed r^2 as $r \to 0$, so this upper bound gives an estimate the speed of convergence towards equilibrium of the behavior of the martingale. Note that the homogenization behavior starts when $r < 1$ and converges towards equilibrium as $x/t \to 0$ with the speed given above.

Proof. For $\lambda > 0$ and $x > 0$

$$\mathbb{P}(M_t \geq x) \leq \mathbb{E}[\exp(\lambda(M_t - x))] \tag{12.16}$$

Thus according to the theorem 12.1.1, for

$$0 < |\lambda| < \frac{1}{\big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}} \tag{12.17}$$

one has

$$\mathbb{P}(M_t \geq x) \leq e^{3(1-1/g(\lambda))} \exp(\frac{g(\lambda)}{2}\lambda^2 f_2 t - \lambda x) \tag{12.18}$$

where g is given in 12.1.1.
Now choose $\lambda = x/(f_2 t)$ and write $r = C_1 x/t$ with $C_1 = \big(2e(f_1 - f_2)t_0\big)^{\frac{1}{2}}/f_2$. Then it follows that for

$$r = \frac{C_1 x}{t} < 1 \tag{12.19}$$

one has

$$\mathbb{P}(M_t \geq x) \leq e^{\frac{3}{2}r^2} \exp\big(-g_1(r)\frac{x^2}{2f_2 t}\big) \tag{12.20}$$

with

$$g_1(r) = 1 - \frac{r^2}{2 - r^2} \tag{12.21}$$

□

12.1.4 Application to the upper bound estimate of the transition probability densities of a diffusion

Consider y_t is a diffusion on $\mathbb{R}^d$ such that for $t > 0$

$$y_t = x + \chi(t) + M_t \tag{12.22}$$

where $\chi(t)$ is a uniformly (in t) bounded random vector process ($\|\chi\|_\infty \leq C_\chi$) and M_t is a continuous square integrable $\mathcal{F}_t$ adapted martingale such that $M_0 = 0$ and for $\lambda, t > 0$, $\mathbb{E}[e^{\lambda M_t}] < \infty$.
Assume also that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $l \in \mathbb{R}^d$ with $|l| = 1$ for all $t_2 > t_1 \geq 0$ one has a.s.

$$\mathbb{E}[\int_{t_1}^{t_2} d < M.l, M.l >_s |\mathcal{F}_{t_1}] \leq \int_0^{t_2 - t_1} f(s)ds$$

With $f(s) = f_1$ for $s < t_0$ and $f(s) = {}^t l D l < f_1$ for $s \geq t_0$ with $t_0 > 0$ and $0 < f_2 < f_1$.
where D is a positive definite symmetric matrix.

Assume also that the diffusion y_t has symmetric Markovian probability densities $p(t,x,y)$ with respect to the measure $m(dy)$ such that for all $x,y \in \mathbb{R}^d$ and $t > 0$

$$p(t,x,y) \leq \frac{C_2}{t^{\frac{d}{2}}} \tag{12.23}$$

and for $\delta > 0$

$$\mathbb{P}_x(|y_t - x| \geq \delta) \leq C_3 e^{-C_4 \frac{\delta^2}{t}} \tag{12.24}$$

where C_2, C_3, C_4 are constants.

Theorem 12.1.2. *Assume that y_t is the diffusion described above. Then with $k_1 = \big(2e(f_1 - \lambda_{\min}(D))t_0\big)^{\frac{1}{2}}/\lambda_{\min}(D)$ and $k_2 = 30 + 10d\lambda_{\max}(D)(1 + C_4)$*

$$20k_1|x-y| < t, \qquad k_2 < \frac{|x-y|}{\sqrt{t}}, \qquad |x-y| > 4C_\chi \tag{12.25}$$

one has

$$p(t,x,y) \leq \frac{E_1}{t^{\frac{d}{2}}} \exp\big(-(1-E)\frac{|y-x-2C_\chi|^2}{2D(e_{y-x})t}\big) \tag{12.26}$$

with

$$E_1 = C_2\Big(\frac{e^{3/2}}{2\lambda_{\min}(D)C_4} + 2^d C_3\Big) \tag{12.27}$$

and

$$E = 8\Big(\frac{k_1|x-y|}{t}\Big)^2 + 2\frac{\sqrt{t}}{|x-y|} \leq \frac{1}{10} \tag{12.28}$$

Remark 12.1.1. Note that $E \to 0$ as $\frac{|x-y|}{t} + \frac{\sqrt{t}}{|x-y|} \to 0$, this gives an estimate on the rate of convergence towards equilibrium. The exact homogenized behavior appears in the asymptotic regime $|x-y|/t \to 0$ and $|x-y|^2/t \to \infty$.
It is interesting to note that the homogenization regime begins as soon as the time t is of order of the distance $x-y$ (which must be at least of the order of C_χ).
The condition $k_2\sqrt{t} < |x-y|$ is a natural one in the sense that if it says that the behavior of the diffusion is not too close to the center of the Gaussian, however with $20k_1|x-y| < t$ the large deviation regime is replaced by a homogenized regime. Note also that if one is only interested in the behavior of the diffusion in the direction $y-x$ all that is needed is that $\chi.e_{y-x}$ is upper bounded (χ may have a greater generality than the solution of the cell problem).

Proof. Observe that for $t > 0$, $x,y \in \mathbb{R}^d$ and $0 < q < 1$

$$p(t,x,y) = \int_{\mathbb{R}^d} p(tq,x,z)p(t(1-q),z,y)m(dz) \tag{12.29}$$

So for $A_\delta = \{z \in \mathbb{R}^d : (z-x).e_{y-x} \geq (1-\delta)|x-y|\}$ (where e_{y-x} is the unit vector in the direction $y-x$)

$$\begin{aligned} p(t,x,y) = &\int_{A_\delta} p(tq,x,z)p(t(1-q),z,y)m(dz) \\ &+ \int_{A_\delta^c} p(tq,x,z)p(t(1-q),z,y)m(dz) \\ &\leq \frac{C_2}{t^{\frac{d}{2}}}\Big[\frac{1}{(1-q)^{\frac{d}{2}}}\mathbb{P}_x(y_{tq}.e_{y-x} \geq |x-y|(1-\delta)) \\ &+ \frac{1}{q^{\frac{d}{2}}}\mathbb{P}_y(|y_{t(1-q)}| \geq \delta|x-y|)\Big] \end{aligned} \tag{12.30}$$

Now by the corollary 12.1.1, for $r < 1$ with $r = \frac{C_1 \rho}{qt}$, $\rho = |x-y|(1-\delta) - C_\chi$, and $C_1 = \big(2e(f_1 - D(e_{x-y}))t_0\big)^{\frac{1}{2}}/D(e_{x-y})$ one has

$$\mathbb{P}_x(y_{tq}.e_{y-x} \geq |x-y|(1-\delta))) \leq e^{\frac{3}{2}r^2} \exp\big(-(1-r^2)\frac{\rho^2}{2D(e_{x-y})tq}\big) \tag{12.31}$$

Then choose

$$\delta = \exp(-\frac{|x-y|}{dD(e_{x-y})\sqrt{t}}) \tag{12.32}$$

write $C_5 = dD(e_{x-y})\ln(4D(e_{x-y})C_4)$ and assume that
$|x-y|/\sqrt{t} > \max(C_5, 3dD(e_{x-y}))$ then one has $\delta < 1/10$ and one can put

$$1 - q = 2DC_4\delta < \frac{1}{2} \tag{12.33}$$

this equation associated to 12.24 imply

$$\mathbb{P}_y(|y_{t(1-q)}| \geq \delta|x-y|) \leq C_3 \exp(-\frac{|x-y|^2}{2D(e_{x-y})t}) \tag{12.34}$$

Moreover equations 12.32 and 12.33 imply

$$\frac{1}{(1-q)^{\frac{d}{2}}} \leq \frac{1}{(2D(e_{x-y})C_4)} \exp\big(-\frac{|x-y|^2}{2D(e_{x-y})t}\frac{\sqrt{t}}{|x-y|}\big) \tag{12.35}$$

Note also that for $|x-y| > 4C_\chi$

$$\rho \leq (|x-y| - 2C_\chi)(1-\delta) \tag{12.36}$$

Observe that

$$r = \frac{C_1\rho}{qt} \leq \frac{2C_1|x-y|}{t} \tag{12.37}$$

Thus for $2C_1|x-y|/t < 1$ one has by the equations 12.30, 12.31, 12.34 and 12.35

$$p(t,x,y) \leq \frac{E_1}{t^{\frac{d}{2}}} \exp\big(-(1-E_2)\frac{|y-x-2C\chi|^2}{2D(e_{y-x})t}\big) \tag{12.38}$$

with

$$E_1 = C_2(\frac{e^{3/2}}{2\lambda_{\min}(D)C_4} + 2^d C_3) \tag{12.39}$$

and

$$E_2 = 8(C_1\frac{|x-y|}{t})^2 + \frac{\sqrt{t}}{|x-y|} + 2\exp(-\frac{|x-y|}{d\sqrt{t}\lambda_{\max}(D)}) \tag{12.40}$$

and the result follows after a simple computation □

12.1.5 Application to homogenization in periodic media

The theorem 12.1.2 can be used to give estimates on the rate of convergence to equilibrium of a diffusion in a periodic media as soon as a cell problem is well defined.
As an example, it will be applied here to obtain estimates on the rate of convergence towards equilibrium of periodic potential form diffusions.

$$dy_t = d\omega_t - \nabla U(y_t)dt \tag{12.41}$$

where $U \in C^1(T_1^d)$ $(U(0) = 0)$

Corollary 12.1.2. *Consider $p(t,x,y)$ the transition density probabilities of the diffusion 12.41 with respect to the measure m_U. then for*

$$20k_1|x-y| < t, \qquad k_2 < \frac{|x-y|}{\sqrt{t}}, \qquad |x-y| > 4C_\chi \tag{12.42}$$

one has

$$p(t,x,y) \leq \frac{E_1}{t^{\frac{d}{2}}} \exp\big(-(1-E)\frac{|y-x-2C_\chi|^2}{2D(e_{y-x})t}\big) \tag{12.43}$$

where k_1, k_2, C_χ, E_1 are constants depending only on d and $\mathrm{Osc}(U)$. *Moreover*

$$E = 8(\frac{k_1|x-y|}{t})^2 + 2\frac{\sqrt{t}}{|x-y|} \leq \frac{1}{10} \tag{12.44}$$

Remark 12.1.2. Since the constants appearing in this corollary doesn't depend on $\|\nabla U\|_\infty$ but only on $\mathrm{Osc}(U)$ it is an easy task to extend this result to case where U is only bounded (left to the reader, see for instance the theorem 1.2 of [CQHZ98]).
Note also that this control gives the rate of convergence towards equilibrium for the upper bound and allows to complete the image associated to the different regimes:

1. **Large deviation regime**: for $|x-y| >> t$ the paths of the diffusion concentrate on the geodesics and

$$\ln p(t,x,y) \sim -\frac{|x-y|^2}{2t} \tag{12.45}$$

2. **Homogenization regime**: for $1 << |x-y| << t$ and $|x-y|^2 >> t$, homogenization takes place and

$$p(t,x,y) \sim \frac{1}{t^{\frac{d}{2}}} \exp(-\frac{|x-y|^2}{2D(e_{y-x})t}) \tag{12.46}$$

3. **Heat kernel diagonal regime**: for $|x-y|^2 << t$, the behavior is fixed by the diagonal of the heat kernel and

$$p(t,x,y) \sim \frac{C_0(x)}{t^{\frac{d}{2}}} \tag{12.47}$$

Note that $|x-y| << 1$ and $|x-y| << t$ imply $|x-y|^2 << t$ thus all the regimes are here.

Remark 12.1.3. Note also that one can consider a wider class of periodic diffusions such as the one 5.131 considered by J.R. Norris, combine the theorem 12.1.2 to the generalized Aronson type estimates obtained by J.R. Norris [Nor97] (see subsection 5.3.2) in order to obtain estimates on the rate of convergence towards equilibrium of the diffusions associated to those operators. This application is left to the reader.

Proof. Indeed this diffusion has symmetric probability densities with respect to the measure m_U (defined in 5.1) and the following Aronson type upper bound is available (see 3.28).

$$p_t(x,y)e^{-2U(y)} \geq e^{-Z}\frac{1}{t^{\frac{d}{2}}}\exp\big(-Z\frac{|x-y|^2}{t}\big) \tag{12.48}$$

with

$$Z = Ce^{10(4+d)\,\mathrm{Osc}(U)} \tag{12.49}$$

it follows directly that the conditions 12.23 and 12.24 are satisfied with constants C_2, C_3, C_4 depending only on d and $\mathrm{Osc}(U)$. Now write χ_l the solution of the cell problem 5.4 associated to this diffusion, it follows that for $l \in \mathbb{R}^d$ ($|l| = 1$)

$$l.y_t = x + \chi_l(y_t) - \chi_l(x) + \int_0^t (l - \nabla\chi_l)d\omega_s \tag{12.50}$$

which is the form given in 12.22 (by the theorem B.2.1, $\|\chi_l\|_\infty \leq C_\chi < \infty$ where C_χ does only depend on d and $\mathrm{Osc}(U)$). The martingale is

$$l.M_t = \int_0^t (l - \nabla\chi_l)d\omega_s \tag{12.51}$$

Its bracket is equal to

$$< l.M, l.M >_t = \int_0^t |l - \nabla\chi_l(y_s)|^2 ds \tag{12.52}$$

and since $U \in C^1(T_1^d)$, according to theorem B.2.1

$$\|l - \nabla\chi_l\|_\infty^2 = f_1 < \infty \tag{12.53}$$

where f_1 is a constant depending only on $d, \mathrm{Osc}(U)$ and $\|\nabla U\|_\infty$. Moreover consider ϕ the solution of the ergodicity equation B.39. Since $\|\phi\|_\infty$ is bounded by a constant C_ϕ depending only on d and $\mathrm{Osc}\, U$ (see theorem B.2.2) it follows from the Ito formula

$$\mathbb{E}[< l.M, l.M >_t] = \mathbb{E}\big[\phi(y_t) - \phi(x)\big] + t\,{}^t l D(U) l \tag{12.54}$$

thus the martingale satisfies the conditions of the theorem 12.1.2 with

$$f_2 = {}^t l D(U) l \tag{12.55}$$

and

$$t_0 = \frac{C_\phi}{f_1 - \lambda_{\min}(D)} \tag{12.56}$$

Now one can use the theorem 12.1.2 to obtain a sharp control on the heat kernel. It is very important to note that all the constants appearing in that theorem only depend and d and $\mathrm{Osc}(U)$ except may be $k_1 = \big(2e(f_1 - \lambda_{\min}(D))t_0\big)^{\frac{1}{2}}/\lambda_{\min}(D)$ in which f_1 appears. This is where the trick operates, indeed $(f_1 - \lambda_{\min}(D))t_0 = C_\phi$ which is a constant depending only on $\mathrm{Osc}(U)$ and d. Thus in reality all the constants only depends on the dimension and on $\mathrm{Osc}(U)$. which proves the corollary ☐

12.2 Lower bound: Sharp estimate of the rate of convergence towards the asymptotic process

Consider the diffusion y_t on $\mathbb{R}^d$ associated to the generator ($U \in C^\infty(T_1^d)$)

$$L = \frac{1}{2}\Delta - \nabla U.\nabla \tag{12.57}$$

Theorem 12.2.1. *For $l \in \mathbb{S}^d$, $\lambda > C_6(d, \mathrm{Osc}(U))$ and*

$$C_7(d, \mathrm{Osc}(U))\lambda < t \tag{12.58}$$

one has

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{4\sqrt{2\pi}} \int_X^\infty e^{-z^2/2} dz \tag{12.59}$$

with

$$X = \frac{\lambda}{\sqrt{{}^t l D(U) l t}}(1 + E) \tag{12.60}$$

and

$$E = \frac{C_8(d, \mathrm{Osc}(U))}{\lambda} + C_5(d, \mathrm{Osc}(U))\sqrt{\frac{\lambda}{t}} \leq \frac{1}{10} \tag{12.61}$$

Remark 12.2.1. Observe that all the constants appearing above only depends on d and $\mathrm{Osc}(U)$, thus it is easy to extend this result to the case when U is only bounded and periodic (left to the reader). Note also that $E \to 0$ as $1/\lambda + \sqrt{\frac{\lambda}{t}} \to 0$ giving the rate of convergence towards equilibrium. Note also that one can consider a wider class of periodic diffusions such as the one 5.131 considered by J.R. Norris (this extension is left to the reader)

Proof. write for ($l \in \mathbb{S}^d$ where $\mathbb{S}^d$ is the sphere of $\mathbb{R}^d$ of center 0 and radius 1)

$$F_l(x) = l.x - \chi_l(x) \tag{12.62}$$

where χ_l is the solution of the cell problem B.34 ($\chi_l(0) = 0$) and write ϕ_l the solution of the ergodicity problem B.39 ($\phi_l(0) = 0$).
Write $(\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t)$ the probability space associated to the diffusion y_t (note that $\mathcal{F}_t$ can be chosen as the filtration generated by the Brownian motion ω_t appearing in the stochastic differential equation associated to y).
Note that $F_l(y_t)$ is a $(\mathbb{P}, \mathcal{F}_t)$-continuous local martingale vanishing at 0 such that (Ito calculus)

$$< F_l, F_l >_t = tD(l) + \phi_l(y_t) + M_t \tag{12.63}$$

with $M_t = -\int_0^t \nabla\phi_l(y_s) d\omega_s$.
Now, since ϕ_l is bounded and $M_t/t \to 0$ a.s. (see for instance [RY91], chapter V, exercise 1.16) it follows that $< F_l, F_l >_\infty = \infty$ a.s.
Write

$$T_t = \inf\{s :< F_l, F_l >_s > t\} \tag{12.64}$$

Now apply the Dambis, Dumbins-Schartz representation theorem (see for instance [RY91] theorem 1.6 of chapter V) to see that $B_t = F_l(y_{T_t})$ is a $(\mathcal{F}_{T_t})$-Brownian motion and $F_l(y_t) = B_{<F_l, F_l>_t}$.

Let $\lambda > 0$, observe that

$$\begin{aligned}\mathbb{P}[F_l(y_t) \geq \lambda] &= \mathbb{P}[B_{<F_l,F_l>_t} \geq \lambda] \\ &= \mathbb{P}[B_{D(l)t} + E_t \geq \lambda]\end{aligned} \tag{12.65}$$

with $E_t = B_{<F_l,F_l>_t} - B_{D(l)t}$. Thus after an easy computation, for $\mu > 0$

$$\begin{aligned}\mathbb{P}[F_l(y_t) \geq \lambda] &\geq \mathbb{P}[B_{D(l)t} \geq \lambda + \mu] - \mathbb{P}[\{B_{D(l)t} \geq \lambda + \mu\} \cap \{|E_t| > \mu\}] \\ &\geq \mathbb{P}[B_{D(l)t} \geq \lambda + \mu] - \mathbb{P}[|E_t| > \mu]\end{aligned}$$

Now write $Q_t = \phi(y_t) + M_t$, then for $\nu > 0$,

$$\begin{aligned}\mathbb{P}[|E_t| \geq \mu] &= \mathbb{P}[\{|E_t| \geq \mu\} \cap \{Q_t \geq \nu\}] + \mathbb{P}[\{|E_t| \geq \mu\} \cap \{Q_t < \nu\}] \\ &\leq \mathbb{P}[|Q_t| \geq \nu] + \mathbb{P}[\sup_{|z|<\nu} |B_{D(l)t+z} - B_{D(l)t}| \geq \mu]\end{aligned}$$

It follows that

$$\mathbb{P}[|E_t| \geq \mu] \leq \mathbb{P}[|Q_t| \geq \nu] + 2\mathbb{P}[|B_\nu| > \mu]$$

Now observe that

$$\mathbb{P}[|Q_t| \geq \nu] \leq \mathbb{P}[|M_t| \geq \nu - \|\phi_l\|_\infty]$$

□

Finally for

$$\nu > \|\phi_l\|_\infty \tag{12.66}$$

$$\begin{aligned}\mathbb{P}[y_t.l \geq \lambda] \geq &\mathbb{P}[B_{D(l)t} \geq \lambda + \|\chi_l\|_\infty + \mu] - 2\mathbb{P}[|B_\nu| > \mu] - \mathbb{P}[|M_t| \geq \nu - \|\phi_l\|_\infty] \\ \geq &\mathbb{P}[B_{D(l)t} \geq \lambda + \|\chi_l\|_\infty + \mu] \\ &- 4\mathbb{P}[B_{D(l)t} \geq \mu\sqrt{\frac{D(l)t}{\nu}}] - \mathbb{P}[|M_t| \geq \nu - \|\phi_l\|_\infty]\end{aligned}$$

Write

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-x^2/2} dx$$

and note that

$$\frac{g(x)}{g(y)} \leq C_1 e^{y^2 - x^2/2}$$

It follows that under the additional condition

$$\lambda + \|\chi_l\|_\infty + \mu \leq C_2 \mu \sqrt{\frac{D(l)t}{\nu}} \tag{12.67}$$

one has

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{2}\mathbb{P}[B_{D(l)t} \geq \lambda + \|\chi_l\|_\infty + \mu] - \mathbb{P}[|M_t| \geq \nu - \|\phi_l\|_\infty] \tag{12.68}$$

Now consider

$$< M, M >_t = \int_0^t |\nabla \phi_l|^2 (y_s) ds$$

Write H_l the periodic solution of ($H_l(0)=0$)

$$(\frac{1}{2}\Delta - \nabla U \nabla)H_l = |\nabla \phi_l|^2 - m_U(|\nabla \phi_l|^2)$$

Observe that M_t satisfies the conditions of the corollary 12.1.1 with $f_2 = m_U(|\nabla\phi_l|^2)$, $f_1 = |\nabla \phi_l|^2_\infty$ and $t_0 = \|H_l\|_\infty/(f_1 - f_2)$.
(Note that $\|H_l\|_\infty \leq C_d e^{C_d \operatorname{Osc}(U)}$, indeed see theorem B.2.2 and observe that since with $G(x) = \frac{1}{2}\phi_l^2 + \|\phi_l\|_\infty (F_l^2 + \psi_l)$ one has $L_U G \geq L_U H_l$ and $G \geq H_l$ over $[0,1]^d$) Observe also that (simple integration by parts)

$$m_U(|\nabla \phi_l|^2) \leq 4\|\phi_l\|_\infty D(l)$$

It follows from the theorem B.2.2 that f_2 is bounded by a constant which depends only on d and $\operatorname{Osc}(U)$, by shifting f_1 one can take f_2 equal to that constant, it won't change the proof. Thus for $C_M x < t$ one has

$$\mathbb{P}(M_t \geq x) \leq 3\exp\big(-\frac{x^2}{f_2 t}\big) \tag{12.69}$$

where C_M is a constant depending only on d and $\operatorname{Osc}(U)$. It follows from the equation 12.68 that under the additional conditions,

$$C_M(\nu - \|\phi_l\|_\infty) < t \tag{12.70}$$

and

$$\lambda + \|\chi_l\|_\infty + \mu < C_3(\nu - \|\phi_l\|_\infty) \tag{12.71}$$

(where C_3 depends only on d and $\operatorname{Osc}(U)$) one has

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{4}\mathbb{P}[B_{D(l)t} \geq \lambda + \|\chi_l\|_\infty + \mu] \tag{12.72}$$

Now choose

$$\nu = \|\phi_l\|_\infty + \frac{2}{C_3}(\lambda + \|\chi_l\|_\infty + \mu) \tag{12.73}$$

with

$$\mu = 2\frac{\lambda + \|\chi_l\|_\infty}{C_2\sqrt{D(l)}}\sqrt{\frac{4(\lambda + \|\chi_l\|_\infty)}{C_3 t}} \tag{12.74}$$

Then for $\lambda > \|\chi_l\|_\infty$ and $t > C_4(d, \operatorname{Osc}(U))\lambda$ the conditions 12.70, 12.70 and 12.71 are satisfied and

$$\mu < C_5(d, \operatorname{Osc}(U))\lambda\sqrt{\frac{\lambda}{t}} \leq \frac{\lambda}{10} \tag{12.75}$$

and it follows from 12.72 that

$$\mathbb{P}[y_t.l \geq \lambda] \geq \frac{1}{4}\mathbb{P}[B_{D(l)t} \geq \lambda(1 + C_5\sqrt{\frac{\lambda}{t}}) + \|\chi_l\|_\infty] \tag{12.76}$$

13. SUB HARMONIC INEQUALITIES

13.1 General considerations

Let Ω be an open bounded subset of $\mathbb{R}^d$ with smooth boundary. Let M be a strictly coercive matrix with smooth coefficients on $\bar{\Omega}$ ($M \in I^\infty(\Omega)$): there exists $\nu > 0$ such that for all $\xi \in \mathbb{R}^d$, ${}^t\xi M \xi \geq \nu \xi^2$. Consider in Ω the operator

$$L_M = -\nabla M \nabla \tag{13.1}$$

with Dirichlet conditions on $\partial\Omega$.
Write $G_M(x, y) = G_M^x(y)$ the Green functions associated to L_M:

$$L_M G_M(x, y) = \delta(x - y) \tag{13.2}$$

Physical Interpretation In this chapter a physical interpretation of each mathematical object introduced will be given to help the intuition in terms of electrostatics. For a good introduction to the subject see [LL90]. Ω represents a Dielectric cavity with conducting boundary $\partial\Omega$ on which the electrostatic potential is imposed to be equal to 0. The dielectric constant is a measure of how well electromagnetic waves couple with the material. The relative dielectric constant has a real part that is the permittivity of the material, and an imaginary part called the loss factor. Since this section focus only on electrostatics, only the real part will be considered. Moreover this section focus on anisotropic inhomogeneous dielectric materials, thus the permittivity is a second order tensor, here it is the matrix M. In a real material the permittivity tensor reflects an equilibrium condition in an electrostatic field of the molecular dipoles composing the material, thus it is obtained by the second order partial derivatives of a thermodynamic function $M_{ij} \propto \partial_i \partial_j \tilde{F}$ associated to the free energy, which imposes the symmetry of M. Thus in all the physical interpretations given here in terms of electrostatics, M will be assumed to be symmetric.
Consider in this cavity an electrostatic potential ϕ and a density of charges g, its associated electrostatic field is given by $-\nabla\phi$ and the electrostatic displacement by $-M\nabla\phi$. Then the equation

$$L_M \phi = g \tag{13.3}$$

is the standard Poisson's equation relating the electric displacement with the density of charges, (up to the proportionality constant 4π). Moreover $G_M(x, y)$ is the electrostatic field created at the point x by a punctual positive charge placed at the point y.

13.2 A sign inequality

13.2.1 Definitions and forms

13.2.1.i Extension of the notion of Dirichlet form

For $\phi, \psi \in H_0^1(\Omega)$ and $B \in \mathcal{B}(\Omega)$ a Borel subset of Ω consider the following form

$$\begin{aligned} \mathcal{E} : (H_0^1(\Omega))^2 \times \mathcal{B}(\Omega) &\longrightarrow \mathbb{R} \\ \phi, \psi, B &\longrightarrow \int_B \nabla\phi(x) M(x) \nabla\psi(x)\, dx \end{aligned} \tag{13.4}$$

Observe that $\mathcal{E}(\phi, \psi, B)$ is bilinear in ϕ and ψ, it is symmetric if M is symmetric. Moreover for fixed ϕ and ψ it is a finite signed measure in B, thus it is natural to extend the argument B appearing in $\mathcal{E}$ to the space $L^\infty(\Omega)$ of measurable bounded functions on Ω.

$$\begin{aligned} \mathcal{E} : (H_0^1(\Omega))^2 \times L^\infty(\Omega) &\longrightarrow \mathbb{R} \\ \phi, \psi, h &\longrightarrow \int_\Omega \nabla\phi(x) M(x) \nabla\psi(x)\, h(x)\, dx \end{aligned} \tag{13.5}$$

Physical interpretation $\mathcal{E}(\phi, \psi, \Omega)$ has a good physical interpretation in the sense that it represents the energy of interaction between the charges $L_M\phi$ and $L_M\psi$, that is to say if one consider the cavity filled with charges $L_M\phi$ and one adds the charges $L_M\psi$, the total energy of the system will increase by the energy of the system $L_M\psi$ plus the energy of interaction $\mathcal{E}(\phi, \psi, \Omega)$ between those two family of charges.
This is clear, however if one wants to give a physical interpretation to $\mathcal{E}(\phi, \psi, B)$ for $B \in \mathcal{B}(\Omega)$ then one is lead to introduce the concept of the localization of the energy. This concept will be discussed here in a formal way but it should be handled with caution if one wants to give it an experimental physical meaning.
Indeed, according to R. P. Feynman ([Fey79] page 142) "asking where the electrostatic energy is localized is an interesting question but not necessary", however " ... it is natural to say that the electrostatic energy is localized in the space, where an electrostatic field can be found because one knows that when charges are accelerated they radiate an electrical field (see for instance the notion of the Poynting's vector)". Now observe that the energy associated to the system of charges $L_M\phi$ is equal to

$$\frac{1}{2}\int_\Omega \phi(x) L_M\phi(x)\, dx = \frac{1}{2}\int_\Omega {}^t\nabla\phi M \nabla\phi\, dx \tag{13.6}$$

In this formula the first member is the standard formula for the definition of the electrostatic energy, the second member is interpreted (in a formal way) by saying that the energy has the density

$${}^t\nabla\phi M \nabla\phi/2 \tag{13.7}$$

with respect to the Lebesgue measure. However Feynman concludes by observing that since the electrostatic energy of a single particle is infinite "the idea of locating the energy in the field is incompatible with the assumption of existence of punctual charges. One way out of the difficult would be to say that elementary charges, such as an electron, are not points but are really small distribution of charge. Alternatively, we could say that there is something wrong in our theory of electricity at very small distances, or with the idea of the local conservation of the energy. There are difficulties with either point of view. These difficulties have never been overcome; there exists to this day."

According to Landau-Lifchitz (see [LL90] page 13) the second member in the equation 13.6 is more a formal condition than a physical one.
According to J.D. Jackson ([Jac62] page 22) the expression 13.7 for energy density "is intuitively reasonable, since regions of high fields must contain considerable energy".
According to G. Goudet (author of a classical treaty on electricity: [Gou67] page 157) the expression 13.7 "is more than a mathematical identity because modern theories show that the energy is localized in the space where the electrostatic field is acting ... an electrical charge placed at a point of this space is submitted to a force; actual science is based on the negation of the idea of action at distance: one should admit that this force is due to the very action of the electrostatic field, even if the dielectric is the void. The space where an electrostatic field does exist is modified, and does posses an energy, which can be taken from where it is localized.".
According to Mason and Weaver ([MW29] page 266), "it is more sensible to inquire about the location of energy than to declare that the beauty of a painting is distributed over the canvas in a

specified manner."
According to J. A. Stratton ([Str41] page 110) "it may be questioned whether the term "energy density" has any physical significance ... it is difficult to either justify or disprove such a hypothesis".

Because of these difficulties the physical interpretation in terms of energy localization should be taken on a formal point of view and with lots of caution: for $B \in \mathcal{B}(\Omega)$, $\mathcal{E}(\phi, \psi, B)$ is the electrostatic interaction energy contained in the region B between the system of charges $L_M\phi$ and $L_M\psi$.

13.2.2 The energy fluctuation form

For $\phi, \psi \in H_0^1(\Omega)$ consider the following form

$$\begin{aligned} \mathcal{V} : (H_0^1(\Omega))^2 &\longrightarrow \mathbb{R}_+ \\ \phi, \psi &\longrightarrow \int_\Omega \left|\nabla\phi(x) M(x) \nabla\psi(x)\right| dx \end{aligned} \tag{13.8}$$

Observe that for $\phi \in H_0^1(\Omega)$, $\mathcal{V}(\phi, \phi) = \mathcal{E}(\phi, \phi, \Omega)$ but $\mathcal{V}(\phi, \psi)$ is not linear in ϕ or ψ, $\mathcal{V}$ is symmetric if M is symmetric. Observe also that $\mathcal{V}$ is sub additive in its arguments.

Physical interpretation Imagine that one adds to a system of positive charges $L_M\phi$, an other distribution of positive charges $L_M\psi$. The energy of interaction is $\mathcal{E}(\phi, \psi, \Omega)$ is positive because one has to work to add charges of the same sign, however the energy interaction density 13.7 is negative in a region Ω_0 and positive in the region Ω/Ω_0, since adding positive charges to positive only increase the total energy, it is natural to interpret $-\mathcal{E}(\phi, \psi, \Omega_0)$ as an energy that has been displaced in Ω. Then $\mathcal{V}(\phi, \psi)$ is equal to "energy imported + 2× energy displaced".

13.2.3 The polarization form

For $\phi, \psi \in H_0^1(\Omega) \times H_0^2(\Omega)$ consider the following form

$$\begin{aligned} \mathcal{P} : H_0^1(\Omega) \times H_0^2(\Omega) \times \mathcal{B}(\Omega) &\longrightarrow \mathbb{R} \\ \phi, \psi, B &\longrightarrow \int_B \phi(x)\big(\nabla M(x)\nabla\psi(x)\big)\, dx + \int_B \nabla\phi(x) M(x) \nabla\psi(x)\, dx \end{aligned} \tag{13.9}$$

Observe that $\mathcal{P}(\phi, \psi, B)$ is bilinear in ϕ and ψ and is a finite signed measure in B. Note also that $\mathcal{P}(\phi, \psi, \Omega) = \mathcal{E}(\phi, \psi, \Omega)$ and

$$\mathcal{P}(\phi, \psi, B) = -\mathcal{P}(\phi, \psi, B^c) \tag{13.10}$$

Write $C_D^2(\Omega)$ for the space of C^2 functions on Ω null on the boundary $\partial\Omega$ (D for Dirichlet condition). Observe that if B is an open subset of Ω with smooth boundary then by the Green formula for $\phi, \psi \in C_D^2(\Omega)$

$$\mathcal{P}(\phi, \psi, B) = \int_{\partial B} \big(n_{ext}.M.\nabla\psi\big)\phi\, d\sigma_B \tag{13.11}$$

where n_{ext} is the exterior unit vector normal to the surface ∂B at the point x and $d\sigma_B$ is the Lebesgue surface measure associated to ∂B.

13.2.4 The polarization measure

Write $\mathcal{R}(\Omega)$ the set of open subsets of Ω with smooth boundary.
For $B \in \mathcal{R}(\Omega)$ and $\psi \in C_D^2(\Omega)$ and consider the following application

$$\begin{aligned} \sigma_p : C_D^2(\Omega) \times \mathcal{R}(\Omega) &\longrightarrow \mathcal{M}(\bar{\Omega}) \\ \psi, B &\longrightarrow \Big(f \rightarrow \int_{\partial B} \big(n_{ext}.M.\nabla\psi\big) f\, d\sigma_B\Big) \end{aligned} \tag{13.12}$$

The measure

$$\sigma_p(\psi, B) = \big(n_{ext}.M.\nabla\psi\big)\, d\sigma_B \tag{13.13}$$

is a finite measure with support ∂B called to the polarization measure of the subset B generated by the potential ψ.
Write $E_x(b)$ the equipotentials of $G_M(x,y)$

$$E_x(b) = \{z \in \Omega \,:\, G(x,z) = b\} \tag{13.14}$$

and $B_x(b)$ the regions delimited by those equipotentials:

$$B_x(b) = \{z \in \Omega \,:\, G(x,z) \geq b\} \tag{13.15}$$

Observe that if B is the set $B_x(b)$ then for $x \in \text{int}(B_x(b))$, $\sigma_p(G_M^x(.), B_x(b))$ is, up to a multiplicative constant, the harmonic measure associated to the generator $-L_M$ and the boundary $E_x(b)$. In particular, $\sigma_p(G_M^x(.), \Omega)$ is the harmonic measure associated to the point x and the process generated by $-L_M$ and killed while exiting Ω.
For a finite measure signed measure μ on Ω write $\chi = G_M\mu$ the weak solution of $L_M\chi = \mu$ with Dirichlet condition on $\partial\Omega$. Observe that

$$\mathcal{E}(\phi, G_M\sigma_p(\psi, B), \Omega) = \mathcal{P}(\phi, \psi, B) \tag{13.16}$$

Physical interpretation In the presence of a potential field ψ, let's isolate (in our mind) a dielectric portion $B \in \mathcal{R}(\Omega)$ of the cavity Ω, then the electrical action of the polarized dielectric volume B plus the free charges inside it is the same as a superficial distribution of charges corresponding to the polarization measure $\sigma_p(\psi, B)$. Now "fix" this distribution of charges and "forget" the initial potential field ψ, then $\mathcal{P}(\phi, \psi, B)$ represents the energy of interaction of the potential field ϕ with the abstract superficial distribution of charges $\sigma_p(\psi, B)$ induced by ψ (if M has a discontinuity at the boundary of ∂B then this superficial distribution of charges can be observed). $-\sigma_p(\psi, \Omega)$ represents the density of charges induced on the conducting boundary $\partial\Omega$.

13.2.5 Definition of some conditions

The operator L_M is said to verify the condition 13.2.1 if and only if the following condition is true.

Condition 13.2.1. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C_D^2(\Omega)$ Sub harmonic with respect to L_M ($L_M\phi \geq 0$ and $L_M\phi \geq 0$)*

$$\mathcal{V}(\phi, \psi) \leq C_{\mathcal{V}}\mathcal{E}(\phi, \psi, \Omega) \tag{13.17}$$

Define also the following conditions

Condition 13.2.2. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C_D^2(\Omega)$ super harmonic with respect to L_M ($L_M\phi \leq 0$ and $L_M\phi \leq 0$)*

$$\mathcal{V}(\phi, \psi) \leq C_{\mathcal{V}}\mathcal{E}(\phi, \psi, \Omega) \tag{13.18}$$

Condition 13.2.3. *There exists $C_{\mathcal{V}} > 1$ such that for all $x, y \in \Omega$, $x \neq y$*

$$\int_\Omega \big|\nabla_z G_M(z,x) M \nabla_z G_M(z,y)\big|\, dz \leq C_{\mathcal{V}} G_M(y,x) \tag{13.19}$$

Condition 13.2.4. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C_D^2(\Omega)$*

$$\mathcal{V}(\phi, \psi) \leq C_{\mathcal{V}}\mathcal{E}\Big(G_M|L_M\phi|, G_M|L_M\psi|, \Omega\Big) \tag{13.20}$$

Condition 13.2.5. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M and all $B \in \mathcal{B}(\Omega)$*

$$-\mathcal{E}(\phi, \psi, B) \leq \frac{C_{\mathcal{V}} - 1}{2} \mathcal{E}(\phi, \psi, \Omega) \tag{13.21}$$

This condition says that the energy that has been displaced is always less than $(C_{\mathcal{V}} - 1)/2$ times the energy that has been imported.

Condition 13.2.6. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M and all $B \in \mathcal{B}(\Omega)$*

$$\mathcal{E}(\phi, \psi, B) \leq \frac{C_{\mathcal{V}} + 1}{2} \mathcal{E}(\phi, \psi, \Omega) \tag{13.22}$$

Condition 13.2.7. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M and all $h \in L^\infty(\Omega)$*

$$\mathcal{E}(\phi, \psi, h) \leq C_{\mathcal{V}} \|h\|_\infty \mathcal{E}(\phi, \psi, \Omega) \tag{13.23}$$

Condition 13.2.8. *There exists $C_{\mathcal{V}} > 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M and all $h \in L^\infty(\Omega)$ such that*

$$1 \leq h \leq \frac{C_{\mathcal{V}} + 1}{C_{\mathcal{V}} - 1} \tag{13.24}$$

one has

$$\mathcal{E}(\phi, \psi, h) \geq 0 \tag{13.25}$$

Condition 13.2.9. *There exists $C_{\mathcal{P}} \geq 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M, and all $B \in \mathcal{B}(\Omega)$ one has*

$$\mathcal{P}(\phi, \psi, B) \leq C_{\mathcal{P}} \mathcal{E}(\phi, \psi, \Omega) \tag{13.26}$$

This condition says that the energy of interaction of $L_M \phi$ with the superficial polarization charges $\sigma_{\mathcal{P}}(\psi, B)$ is always less than $C_{\mathcal{P}}$ times the energy of interaction of $L_M \phi$ with the electrostatic potential ψ which is at the origin of the superficial distribution of charges.

Condition 13.2.10. *There exists $C_{\mathcal{P}} \geq 1$ such that for all $\psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M, and all $B \in \mathcal{R}(\Omega)$, for all $x \in \Omega$ (except for a subset of 0 Lebesgue measure)*

$$G_M \sigma_p(\psi, B)(x) \leq C_{\mathcal{P}} \psi(x) \tag{13.27}$$

This condition says that the potential field $G_M \sigma_p(\psi, B)$ created by the superficial polarization measure $\sigma_p(\psi, B)$ is always less or equal to $C_{\mathcal{P}}$ times the electrostatic potential ψ which is at the origin of the superficial distribution of charges.

Theorem 13.2.1. *The conditions 13.2.1, 13.2.2, 13.2.3, 13.2.4, 13.2.5, 13.2.6, 13.2.7 and 13.2.8 are equivalent with the same constant $C_{\mathcal{V}}$. Moreover if the condition 13.2.1 is true for M it is also true for ${}^tM$.*

Proof. The equivalence between 13.2.1 and 13.2.2 is trivial.
13.2.1 ⇒ 13.2.4: Decompose $L_M \phi$ into its negative part $(L_M \phi)_-$ and positive part $(L_M \phi)_+$, do the same with ψ to obtain (using the sub additivity condition of $\mathcal{V}$)

$$\begin{aligned}\mathcal{V}(\phi, \psi) \leq & \mathcal{V}(G_M(L_M \phi)_+, G_M(L_M \psi)_+) + \mathcal{V}(G_M(L_M \phi)_-, G_M(L_M \psi)_-) \\ & + \mathcal{V}(G_M(L_M \phi)_+, G_M(L_M \psi)_-) + \mathcal{V}(G_M(L_M \phi)_-, G_M(L_M \psi)_+)\end{aligned}$$

Then use 13.2.1 and 13.2.2 to obtain

$$\begin{aligned}\frac{1}{C_{\mathcal{V}}}\mathcal{V}(\phi,\psi) \leq& \mathcal{E}\big(G_M(L_M\phi)_+, G_M(L_M\psi)_+, \Omega\big) + \mathcal{E}\big(G_M(L_M\phi)_-, G_M(L_M\psi)_-, \Omega\big) \\ &- \mathcal{E}\big(G_M(L_M\phi)_+, G_M(L_M\psi)_-, \Omega\big) - \mathcal{E}\big(G_M(L_M\phi)_-, G_M(L_M\psi)_+, \Omega\big)\end{aligned}$$

□

13.2.4 ⇒ 13.2.1 is trivial.
13.2.1 ⇔ 13.2.5 ⇔ 13.2.6: Fix ϕ and ψ, decompose Ω in Ω^+ which is the region where $\nabla\phi\nabla\psi > 0$ and Ω^- which is the region where $\nabla\phi\nabla\psi \leq 0$. Observe that

$$\mathcal{V}(\phi,\psi) = \mathcal{E}(\phi,\psi,\Omega^+) - \mathcal{E}(\phi,\psi,\Omega^-)$$

this directly leads to the proof.
13.2.1⇐ 13.2.3: Let $\phi,\psi \in C^2_D(\Omega)$, Sub harmonic. Observe that

$$\begin{aligned}\mathcal{V}(\phi,\psi) &\leq \int_{\Omega^3} L_M\phi(x)\big|\nabla_z G_M(z,x) M \nabla_z G_M(z,y)\big| L_M\psi(y)\,dz\,dy\,dx \\ &\leq \int_{\Omega^2} L_M\phi(x) G_M(y,x) L_M\psi(y)\,dy\,dx \\ &\leq \int_{\Omega} \phi(y) L_M\psi(y)\,dy\end{aligned} \tag{13.28}$$

which implies condition 13.2.1.
13.2.6⇒ 13.2.3: Fix $a,b \in \Omega^2$, $a \neq b$. Choose for $\epsilon > 0$, $g_\epsilon, f_\epsilon \in C^\infty(\Omega)$ such that $g_\epsilon, f_\epsilon > \epsilon$ and as $\epsilon \downarrow 0$, $g_\epsilon(z)$ and $f_\epsilon(z)$ weakly converge to $\delta(a-z)$ and $\delta(b-z)$.
Then by the condition 13.2.6, for $\epsilon,\epsilon' > 0$, and $B \in \mathcal{B}(\Omega)$

$$\begin{aligned}&\int_{(z,x,y)\in B\times\Omega^2} g_\epsilon(x)\nabla_z G_M(z,x) M \nabla_z G_M(z,y) f_{\epsilon'}(y)\,dz\,dx\,dy \\ &\leq \frac{C_{\mathcal{V}}+1}{2}\int_{(z,x,y)\in\Omega^3} g_\epsilon(x)\nabla_z G_M(z,x) M \nabla_z G_M(z,y) f_{\epsilon'}(y)\,dz\,dx\,dy\end{aligned} \tag{13.29}$$

Then let $\epsilon' \downarrow 0$, next $\epsilon \downarrow 0$ to obtain

$$\int_{z\in B} \nabla_z G_M(z,a) M \nabla_z G_M(z,b)\,dz \leq \frac{C_{\mathcal{V}}+1}{2} G_M(b,a) \tag{13.30}$$

which leads to the proof since B is arbitrary.
13.2.1 ⇔ 13.2.7: straightforward computation.
13.2.1 ⇔ 13.2.8: Fix ϕ and ψ, decompose Ω in Ω^+ and Ω^- and use 13.2.5 and 13.2.6.
Finally it is straightforward to see that the validity of these conditions for the matrix M is equivalent to their validity for the transposed matrix ${}^tM$ by observing that $\nabla\phi M \nabla\psi = \nabla\psi {}^tM \nabla\phi$

Theorem 13.2.2. *The condition 13.2.1 implies 13.2.9 with the constant*

$$C_{\mathcal{P}} = \frac{C_{\mathcal{V}}+1}{2} \tag{13.31}$$

If M is symmetric then 13.2.9 implies 13.2.10 with the same constant $C_{\mathcal{P}}$.
The condition 13.2.9 implies 13.2.1 with the constant

$$C_{\mathcal{V}} = 2C_{\mathcal{P}} + 1 \tag{13.32}$$

If M is symmetric then 13.2.10 implies 13.2.1 with the constant above.

Remark 13.2.1. Note that if M is symmetric then 13.2.10 implies all the conditions from 13.2.1 to 13.2.9. Moreover if $C_{\mathcal{P}} = 1$ then the formula 13.32 gives $C_{\mathcal{V}} = 3$. However the formula 13.31 with $C_{\mathcal{V}} = 3$ gives $C_{\mathcal{P}} = 2$, thus either a better proof can be found, either the condition 13.2.10 with an optimal constant $C_{\mathcal{P}}$ reflects a stronger phenomenon than the sign inequality 13.2.1 with an optimal constant $C_{\mathcal{V}}$.

Proof. 13.2.1 $\Rightarrow$ 13.2.9 $\Rightarrow$ 13.2.10 : Fix ϕ and ψ in $C^2_D(\Omega)$ and $B \in \mathcal{B}(\Omega)$, observe that by the Green Formula

$$\mathcal{P}(\phi,\psi,B) = \mathcal{E}(\phi,\psi,B) + \int_B \phi\big(\nabla M \nabla \psi\big)\, dx \tag{13.33}$$

Using the equivalence of 13.2.1 with 13.2.6 and observing that the second term in the above equation is negative if ϕ and ψ are Sub harmonic, this proves that 13.2.1 implies 13.2.9 with $C_{\mathcal{P}} = (C_{\mathcal{V}}+1)/2$.
13.2.9 $\Rightarrow$ 13.2.10 Now if B belongs to $\mathcal{R}(\Omega)$, observe that

$$\int_{\partial B} \phi(x) d\sigma_p(\psi, B) = \mathcal{P}(\phi,\psi,B) \tag{13.34}$$

thus for all $\phi \in C^2_D(\Omega)$, Sub harmonic (assume M to be symmetric)

$$\begin{aligned}\int_\Omega L_M\phi(x) G_M \sigma_p(\psi,B)\, dx &= \mathcal{P}(\phi,\psi,B)\\ &\leq C_{\mathcal{P}} \int_\Omega \psi(x) L_M \phi(x)\, dx\end{aligned} \tag{13.35}$$

then choose for $L_M\phi$ a positive approximation of the identity to obtain that 13.2.9 implies 13.2.10 with the same constant $C_{\mathcal{P}}$. 13.2.9 $\Rightarrow$ 13.2.1: this is straightforward by the identity 13.33 and the equivalence of 13.2.1 with 13.2.6.
13.2.10 $\Rightarrow$ 13.2.1: Choose ϕ and ψ smooth and Sub harmonic with Dirichlet condition on $\partial\Omega$. Decompose Ω in

$$\Omega^+_\lambda = \{x \in \Omega \,:\, \nabla\phi.M.\nabla\psi > \lambda\}$$

and

$$\Omega^-_\lambda = \{x \in \Omega \,:\, \nabla\phi.M.\nabla\psi \leq \lambda\}$$

Let $\epsilon > 0$. By Sard's theorem one can find $|\lambda| < \epsilon$ such that $\partial\Omega^+_\lambda$ has a smooth boundary. Then by the equation 13.35 and the condition 13.2.10 if M is symmetric

$$\mathcal{P}(\phi,\psi,\Omega^+_\lambda) \leq C_{\mathcal{P}}\mathcal{E}(\phi,\psi,\Omega) \tag{13.36}$$

it follows by making ϵ converge towards 0, by the identity 13.33 that

$$\mathcal{E}(\phi,\psi,\Omega^+_0) \leq (C_{\mathcal{P}}+1)\mathcal{E}(\phi,\psi,\Omega) \tag{13.37}$$

this proves the condition 13.2.6 with $C_{\mathcal{V}} = 2C_{\mathcal{P}} + 1$ which ends the proof by the equivalence with 13.2.1. (if ϕ and ψ are only C^2 regularize $L_M\phi$ and $L_M\psi$ and take the image of those regularizations by G_M to obtain the proof) □

13.3 General Sub harmonic inequalities

13.3.1 The energy stress form

For $\phi, \psi \in H_0^1(\Omega)$ consider the following form

$$\begin{aligned} \mathcal{S} : (H_0^1(\Omega))^2 &\longrightarrow \mathbb{R}_+ \\ \phi, \psi &\longrightarrow \int_\Omega \|\nabla \phi(x)\| \|M(x) \nabla \psi(x)\| \, dx \end{aligned} \tag{13.38}$$

Observe that for $\phi \in H_0^1(\Omega)$,

$$\mathcal{E}(\phi, \phi, \Omega) \leq \mathcal{S}(\phi, \phi) \tag{13.39}$$

and $\mathcal{E}(\phi, \phi, \Omega) = \mathcal{S}(\phi, \phi)$ if M is isotropic (a multiple of the identity matrix). Moreover $\mathcal{S}(\phi, \psi)$ is not linear in ϕ or ψ, $\mathcal{S}$ is symmetric if M is isotropic (a multiple of the identity matrix). Observe also that $\mathcal{S}$ is sub additive in its arguments.

13.3.2 The permittivity left deformation energy form

For $\phi, \psi \in H_0^1(\Omega)$ consider the following form

$$\begin{aligned} \mathcal{D} : (H_0^1(\Omega))^2 \times (C^\infty(\bar{\Omega}))^{d \times d} &\longrightarrow \mathbb{R} \\ \phi, \psi, N &\longrightarrow \int_\Omega \nabla \phi(x) N(x) M(x) \nabla \psi(x) \, dx \end{aligned} \tag{13.40}$$

Where $(C^\infty(\bar{\Omega}))^{d \times d}$ is the set of smooth $d \times d$ matrices on $\bar{\Omega}$. Observe that

$$\mathcal{D}(\phi, \psi, I_d) = \mathcal{E}(\phi, \psi, \Omega) \tag{13.41}$$

and

$$\mathcal{D}(\phi, \psi, N) \leq \|N\|_\infty \mathcal{S}(\phi, \psi) \tag{13.42}$$

where

$$\|N\|_\infty = \sup_{x, \nu, \xi \in \Omega \times \mathbb{S}^2} {}^t\xi N(x) \nu \tag{13.43}$$

It is also important and easy to observe that

$$\mathcal{S}(\phi, \psi) = \sup_{N \in (C^\infty(\bar{\Omega}))^{d \times d},\, N \neq 0} \frac{1}{\|N\|_\infty} \mathcal{D}(\phi, \psi, N) \tag{13.44}$$

So the energy stress form corresponds to the supremum of the permittivity left energy deformation taken on the unit sphere in the space of smooth matrices.
Observe also that $\mathcal{D}$ is linear in all its arguments and for Q smooth and coercive.

$$\mathcal{D}_M(\phi, \psi, QM^{-1}) = \mathcal{E}_Q(\phi, \psi, \Omega) \tag{13.45}$$

13.3.3 Some conditions

Condition 13.3.1. *There exists $C_\mathcal{S} > 1$ such that for all $\phi, \psi \in C_D^2(\Omega)$ Sub harmonic with respect to L_M ($L_M \phi \geq 0$ and $L_M \phi \geq 0$)*

$$\mathcal{S}(\phi, \psi) \leq C_\mathcal{S} \mathcal{E}(\phi, \psi, \Omega) \tag{13.46}$$

Condition 13.3.2. *M is symmetric and there exists $C_{\mathcal{S}} > 1$ such that if $x \to \{e_1(x), \dots, e_d(x)\}$ is a map from Ω to an orthonormal basis of $\mathbb{R}^d$ diagonalizing M at the point x then for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M, for all $(i,j) \in \{1, \dots, d\}^2$*

$$\int_\Omega \lambda(M)_{e_i}(x)\left|\partial_{e_j}\phi(x)\partial_{e_i}\psi(x)\right| dx \leq C_{\mathcal{S},2}\mathcal{E}(\phi, \psi, \Omega) \tag{13.47}$$

Where $\lambda(M)_{e_i}(x)$ is the eigenvalue corresponding to the eigenvector e_i of M at the point x.

Condition 13.3.3. *M is isotropic ($M = m(x)I_d$) and there exists $C_{\mathcal{S},3} > 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M, for all $(i,j) \in \{1, \dots, d\}^2$*

$$\int_\Omega m(x)\left|\partial_j\phi(x)\partial_i\psi(x)\right| dx \leq C_{\mathcal{S},3}\mathcal{E}(\phi, \psi, \Omega) \tag{13.48}$$

Condition 13.3.4. *There exists a constant $C_{\mathcal{S}} > 1$ such that for all $\phi, \psi \in C^2_D(\Omega)$ Sub harmonic with respect to L_M, for all $N \in (C^\infty(\bar{\Omega}))^{d\times d}$.*

$$\mathcal{D}(\phi, \psi, N) \leq C_{\mathcal{S}}\|N\|_\infty\mathcal{E}(\phi, \psi, \Omega) \tag{13.49}$$

Condition 13.3.5. *There exists a constant $C_{\mathcal{S}} > 1$ such that for all $x, y \in \Omega$, $x \neq y$,*

$$\mathcal{S}(G_M(., x), G_M(., y)) \leq C_{\mathcal{S}}G_M(y, x) \tag{13.50}$$

and for all $N \in (C^\infty(\bar{\Omega}))^{d\times d}$.

$$\mathcal{D}(G_M(., x), G_M(., y), N) \leq C_{\mathcal{S}}\|N\|_\infty G_M(y, x) \tag{13.51}$$

Theorem 13.3.1. *The condition 13.3.1 implies the condition 13.2.1 with $C_{\mathcal{V}} = C_{\mathcal{S}}$.*
If M is symmetric, the condition 13.3.1 implies the condition 13.3.2 with $C_{\mathcal{S},2} = dC_{\mathcal{S}}$.
The condition 13.3.2 implies the condition 13.3.1 with $C_{\mathcal{S}} = d^2C_{\mathcal{S},2}$.
If M is isotropic then the condition 13.3.1 implies the condition 13.3.3 with $C_{\mathcal{S},3} = dC_{\mathcal{S}}$.
The condition 13.3.3 implies the condition 13.3.1 with $C_{\mathcal{S}} = d^2C_{\mathcal{S},3}$.
The conditions 13.3.1, 13.3.4 and 13.3.5 are equivalent with the same constant $C_{\mathcal{S}}$.

Proof. 13.3.1 ⇒ 13.2.1 is trivial.
13.3.1 ⇒ 13.3.2: Assume M to be symmetric. Observe that at the point x

$$\|\nabla\phi(x)\|\|M\nabla\psi(x)\| \geq \frac{1}{d}\sum_{i,j}\lambda(M)_{e_i}(x)\left|\partial_{e_j}\phi(x)\partial_{e_i}\psi(x)\right| \tag{13.52}$$

which leads to the proof.
13.3.2 ⇒ 13.3.1: Observe that at the point x

$$\|\nabla\phi(x)\|\|M\nabla\psi(x)\| \leq \sum_{i,j}\lambda(M)_{e_i}(x)\left|\partial_{e_j}\phi(x)\partial_{e_i}\psi(x)\right| \tag{13.53}$$

which leads to the proof.
The equivalence between 13.3.1 and 13.3.3 when M is isotropic is now trivial.
The equivalence between 13.3.1 and 13.3.4 is a direct consequence of the equations 13.42 and 13.44.
The proof of the equivalence between 13.3.4 and 13.3.5 is similar to the proof of the equivalence between 13.2.3 and 13.2.1 in the theorem 13.2.1. □

13.4 Local stability and Sub harmonic inequalities

13.4.1 Smooth admissible perturbation

Write $\mathcal{I}^\infty(\Omega)$ the space of strictly coercitive $d \times d$ matrices with coefficients in $C^\infty(\bar{\Omega})$. Let $\epsilon \to M_\epsilon$ ($0 \leq \epsilon \leq 1$) be a smooth application from $[0,1]$ to $\mathcal{I}^\infty(\Omega)$ (smooth in the sense that $\partial_\epsilon M_\epsilon$ exists and is a C^∞, $d \times d$ matrix on $\bar{\Omega}$).
Let $\epsilon \to g_\epsilon$ be a smooth map from $[0,1]$ to $C^\infty(\bar{\Omega})$, $g \geq 0$ (smooth in the sense that $\partial_\epsilon g$ exists and is an element of $C^\infty(\bar{\Omega})$).
For $\epsilon \in [0,1]$ write ψ_ϵ be the solution of

$$L_{M_\epsilon} \psi_\epsilon = g_\epsilon \tag{13.54}$$

with Dirichlet condition on $\partial\Omega$.
Observe that

$$L_{M_\epsilon} \frac{\partial}{\partial \epsilon} \psi_\epsilon = \partial_\epsilon g_\epsilon - L_{\partial_\epsilon M_\epsilon} \psi_\epsilon \tag{13.55}$$

The family $(M_\epsilon, g_\epsilon, \psi_\epsilon)$ is called a smooth admissible perturbation of the operator L_{M_0} and the following equation is available:

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \psi_\epsilon(x) &= G_{M_\epsilon} \big(\partial_\epsilon g_\epsilon - L_{\partial_\epsilon M_\epsilon} \psi_\epsilon\big)(x) \\ &= G_{M_\epsilon} \partial_\epsilon g_\epsilon(x) - \int_\Omega \nabla_y G_{M_\epsilon}(x,y) \partial_\epsilon M_\epsilon(y) \nabla_y G_{M_\epsilon}(y,z) g_\epsilon(z)\, dy\, dz \end{aligned} \tag{13.56}$$

The perturbation is called symmetric if M_0 is symmetric and for all ϵ, M_ϵ remains symmetric.

13.4.1.i Left perturbation

Let $\epsilon_0 > 0$, consider for $\epsilon \in [0, \epsilon_0]$, $\epsilon \to N_\epsilon^L$ a bounded (in the sense that the coefficients of N_ϵ are uniformly bounded in the L^∞ norm) smooth map from $[0, \epsilon_0]$ to $(C^\infty(\bar{\Omega}))^{d \times d}$ the set of smooth $d \times d$ matrices.
Assume that for $\epsilon \in [0, \epsilon_0]$, M_ϵ is the solution of

$$\begin{cases} \frac{dM_\epsilon}{d\epsilon} = N_\epsilon^L M_\epsilon \\ M_0 \in \mathcal{I}^\infty(\Omega) \end{cases} \tag{13.57}$$

and remains in $\mathcal{I}^\infty(\Omega)$ for $\epsilon \in [0, \epsilon_0]$ (which is always true since N_ϵ^L remains bounded). Then

$$N_\epsilon^L = (\partial_\epsilon M_\epsilon) M_\epsilon^{-1} \tag{13.58}$$

is called the left perturbation of the operator L_{M_0} and the following equation is available

$$\frac{\partial}{\partial \epsilon} \psi_\epsilon(x) = G_{M_\epsilon} \partial_\epsilon g_\epsilon(x) - \int_\Omega \nabla_y G_{M_\epsilon}(x,y) N_\epsilon^L(y) M_\epsilon(y) \nabla_y G_{M_\epsilon}(y,z) g_\epsilon(z)\, dy\, dz \tag{13.59}$$

13.4.1.ii Right perturbation

Let $\epsilon_0 > 0$, consider for $\epsilon \in [0, \epsilon_0]$, $\epsilon \to N_\epsilon^R$ a bounded smooth map from $[0, \epsilon_0]$ to $(C^\infty(\bar{\Omega}))^{d \times d}$ the set of smooth $d \times d$ matrices.
Assume that for $\epsilon \in [0, \epsilon_0]$, M_ϵ is the solution of

$$\begin{cases} \frac{dM_\epsilon}{d\epsilon} = N_\epsilon^L M_\epsilon \\ M_0 \in \mathcal{I}^\infty(\Omega) \end{cases} \tag{13.60}$$

and remains in $\mathcal{I}^\infty(\Omega)$ for $\epsilon \in [0, \epsilon_0]$. Then

$$N^R_\epsilon = (\partial_\epsilon M_\epsilon) M_\epsilon^{-1} \tag{13.61}$$

is called the right perturbation of the operator L_{M_0} and the following equation is available

$$\frac{\partial}{\partial \epsilon}\psi_\epsilon(x) = G_{M_\epsilon}\partial_\epsilon g_\epsilon(x) - \int_\Omega \nabla_y G_{M_\epsilon}(x,y) M_\epsilon(y) N^R_\epsilon(y) \nabla_y G_{M_\epsilon}(y,z) g_\epsilon(z)\, dy\, dz \tag{13.62}$$

13.4.1.iii Isotropic perturbation

Let $\epsilon_0 > 0$, consider for $\epsilon \in [0, \epsilon_0]$, $\epsilon \to n_\epsilon$ a bounded smooth map from $[0, \epsilon_0]$ to $C^\infty(\bar{\Omega})$. Assume that for $\epsilon \in [0, \epsilon_0]$, M_ϵ is the solution of

$$\begin{cases} \frac{dM_\epsilon}{d\epsilon} = n_\epsilon M_\epsilon \\ M_0 \in \mathcal{I}^\infty(\Omega) \end{cases} \tag{13.63}$$

and remains in $\mathcal{I}^\infty(\Omega)$ for $\epsilon \in [0, \epsilon_0]$. Then

$$n_\epsilon = (\partial_\epsilon M_\epsilon) M_\epsilon^{-1} \tag{13.64}$$

is called the isotropic perturbation of the operator L_{M_0} and the following equation is available

$$\frac{\partial}{\partial \epsilon}\psi_\epsilon(x) = G_{M_\epsilon}\partial_\epsilon g_\epsilon(x) - \int_\Omega n_\epsilon(y) \nabla_y G_{M_\epsilon}(x,y) M_\epsilon(y) \nabla_y G_{M_\epsilon}(y,z) g_\epsilon(z)\, dy\, dz \tag{13.65}$$

13.4.2 Operator perturbation and maximal stress vortex

For $M \in \mathcal{I}^\infty(\Omega)$, write

$$C_{\mathcal{V}}(M) = \sup_{x,\in\Omega, x\neq y} \frac{\mathcal{V}(G_M(.,x), G_M(.,y))}{G_M(y,x)} \tag{13.66}$$

The permittivity isotropic deformation constant $C_{\mathcal{V}}(M)$ is the optimal constant appearing in the condition 13.2.1 (by the equivalence with 13.2.3), it might be infinite.
Define also

$$C_{\mathcal{S}}(M) = \sup_{x,\in\Omega, x\neq y} \frac{\mathcal{S}(G_M(.,x), G_M(.,y))}{G_M(y,x)} \tag{13.67}$$

The permittivity anisotropic deformation constant $C_{\mathcal{S}}(M)$ is the optimal constant appearing in the condition 13.3.1 (by the equivalence with 13.3.5), it might be infinite.

13.4.2.i Integration along maximal stress vortices

Theorem 13.4.1. *Let $(M_\epsilon, g_\epsilon, \psi_\epsilon)_{\epsilon\in[0,1]}$ be a smooth admissible isotropic symmetric perturbation of the operator L_{M_0}, write n_ϵ its isotropic perturbation. Assume that $g_\epsilon(x) = g_0(x) > 0$ ($g_\epsilon(x)$ does not depend on ϵ) then*

$$e^{-\int_0^1 C_{\mathcal{V}}(M_\epsilon)\|n_\epsilon\|_\infty\, d\epsilon} \leq \frac{\psi_1(x)}{\psi_0(x)} \leq e^{\int_0^1 C_{\mathcal{V}}(M_\epsilon)\|n_\epsilon\|_\infty\, d\epsilon} \tag{13.68}$$

Proof. By using the equation 13.65 the symmetry of M_ϵ and the definition of the permittivity isotropic deformation constant it follows that:

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\psi_\epsilon(x) &\leq \|n_\epsilon\|_\infty C_{\mathcal{V}}(M_\epsilon)\psi_\epsilon(x) \\ &\geq -\|n_\epsilon\|_\infty C_{\mathcal{V}}(M_\epsilon)\psi_\epsilon(x) \end{aligned} \tag{13.69}$$

An the proof follows by the positivity of ψ_ϵ and a simple integration. ◻

Corollary 13.4.1. *Let $(M_\epsilon)_{\epsilon\in[0,1]}$ be a smooth admissible isotropic symmetric perturbation of the operator L_{M_0}, write n_ϵ its isotropic perturbation. Then for all $x, y \in \Omega$, $x \neq y$*

$$e^{-\int_0^1 C_{\mathcal{V}}(M_\epsilon)\|n_\epsilon\|_\infty \, d\epsilon} \leq \frac{G_{M_1}(x,y)}{G_{M_0}(x,y)} \leq e^{\int_0^1 C_{\mathcal{V}}(M_\epsilon)\|n_\epsilon\|_\infty \, d\epsilon} \tag{13.70}$$

Proof. Straightforward since the previous theorem is valid for all $g > 0$. □

Theorem 13.4.2. *Let $(M_\epsilon, g_\epsilon, \psi_\epsilon)_{\epsilon\in[0,1]}$ be a smooth admissible anisotropic symmetric perturbation of the operator L_{M_0}, write N_ϵ its left perturbation. Assume that $g_\epsilon = g_0 > 0$ then*

$$e^{-\int_0^1 C_{\mathcal{S}}(M_\epsilon)\|N_\epsilon\|_\infty \, d\epsilon} \leq \frac{\psi_1(x)}{\psi_0(x)} \leq e^{\int_0^1 C_{\mathcal{S}}(M_\epsilon)\|N_\epsilon\|_\infty \, d\epsilon} \tag{13.71}$$

Proof. By using the equation 13.59 the symmetry of M_ϵ and the definition of the permittivity anisotropic deformation constant it follows that:

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\psi_\epsilon(x) &\leq \|N_\epsilon\|_\infty C_{\mathcal{S}}(M_\epsilon)\psi_\epsilon(x) \\ &\geq -\|N_\epsilon\|_\infty C_{\mathcal{S}}(M_\epsilon)\psi_\epsilon(x) \end{aligned} \tag{13.72}$$

An the proof follows by the positivity of ψ_ϵ and a simple integration. □

Corollary 13.4.2. *Let $(M_\epsilon)_{\epsilon\in[0,1]}$ be a smooth admissible anisotropic symmetric perturbation of the operator L_{M_0}, write N_ϵ its anisotropic perturbation. Then for all $x, y \in \Omega$, $x \neq y$*

$$e^{-\int_0^1 C_{\mathcal{S}}(M_\epsilon)\|N_\epsilon\|_\infty \, d\epsilon} \leq \frac{G_{M_1}(x,y)}{G_{M_0}(x,y)} \leq e^{\int_0^1 C_{\mathcal{S}}(M_\epsilon)\|N_\epsilon\|_\infty \, d\epsilon} \tag{13.73}$$

Proof. Straightforward since the previous theorem is valid for all $g > 0$. □

13.4.2.ii Deformation Vortex

Let $M_0 \in I^\infty(\Omega)$. Write $\mathbb{S}\big((C^\infty(\bar{\Omega}))^{d\times d}\big)$ the elements $N \in (C^\infty(\bar{\Omega}))^{d\times d}$ such that $\|N\|_\infty = 1$. Consider for $\epsilon > 0$ the solution of

$$\frac{dM_\epsilon}{d\epsilon} = N M_\epsilon \tag{13.74}$$

with initial condition M_0. Note that

$$M_\epsilon = e^{\epsilon N} M_0 \tag{13.75}$$

With this definition of M_ϵ, consider the deformation operator

$$\begin{aligned} \mathbb{D} : C(\bar{\Omega}) \times \mathbb{S}\big((C^\infty(\bar{\Omega}))^{d\times d}\big) &\longrightarrow C^2_D(\Omega) \\ g, N &\longrightarrow \frac{\partial\big(G_{M_\epsilon} g\big)}{\partial \epsilon} \end{aligned} \tag{13.76}$$

$\mathbb{D}(., N)$ is called the deformation vortex of the operator L_{M_0} in the stress direction N. If N is isotropic, the vortex is said to be isotropic.

Theorem 13.4.3. *For $M_0 \in I^\infty(\Omega)$, M_0 symmetric*

$$C_{\mathcal{V}}(M_0) = \sup_{n\in\mathbb{S}(C^\infty(\bar{\Omega})), g\in C^(\bar{\Omega}), g>0, x\in\Omega} \frac{\mathbb{D}(g, nI_d)(x)}{\big(G_{M_0} g\big)(x)} \tag{13.77}$$

and with $M_\epsilon = e^{\epsilon n} I_d$

$$C_{\mathcal{V}}(M_0) = \sup_{n\in\mathbb{S}(C^\infty(\bar{\Omega})), x,y\in\Omega, x\neq y} \left|\frac{d}{d\epsilon}\ln\big(G_{M_\epsilon}(x,y)\big)\right| \tag{13.78}$$

Remark 13.4.1. This theorem is important because it says that the stability of the operator L_{M_0} under a small isotropic perturbation $dM_\epsilon = d\epsilon\, nM$ is completely and exactly reflected in the permittivity deformation constant $C_{\mathcal{V}}(M)$.
In other words, consider a dielectric material with permittivity M and a fixed distribution of charges g, this distribution of charges creates an electrostatic potential field ψ by the equation $L_M\psi = g$, now one can wonder how much ψ will be perturbed under a small isotropic perturbation of M, then the sharp upper bound to the question "how much" is completely reflected in the constant $C_{\mathcal{V}}(M)$.
Note that if $C_{\mathcal{V}}(M_0) = \infty$, with the material associated dielectric constant M_0 one can find bounded distribution of charges $0 \leq g \leq 1$ and small perturbations of the permittivity tensor $dM_\epsilon = d\epsilon\, nM$ producing a deformation of the potential electrostatic field $G_{M_0}(g)$ without a priori bound.
Note also that if $\sup_{M\in I^\infty(\Omega)} C_{\mathcal{V}}(M) = \infty$ (M symmetric in the supremum) then one can find dielectric materials reacting to the perturbation of its dielectric constants by a very strong deformation of the potential electrostatic field (without a priori bound).

Proof. Straightforward by using the equation 13.65 the symmetry of M_ϵ and the definition of the permittivity isotropic deformation constant. □

Theorem 13.4.4. *For $M_0 \in I^\infty(\Omega)$, M_0 symmetric*

$$C_{\mathcal{S}}(M_0) = \sup_{N\in\mathbb{S}\left((C^\infty(\Omega))^{d\times d}\right), g\in C(\bar{\Omega}), g>0, x\in\Omega} \frac{\mathbb{D}(g,N)(x)}{(G_{M_0}g)(x)} \tag{13.79}$$

and with $M_\epsilon = e^{\epsilon N} I_d$

$$C_{\mathcal{S}}(M_0) = \sup_{N\in\mathbb{S}\left((C^\infty(\Omega))^{d\times d}\right), x,y\in\Omega, x\neq y} \left|\frac{d}{d\epsilon}\ln\left(G_{M_\epsilon}(x,y)\right)\right| \tag{13.80}$$

Remark 13.4.2. This theorem is also important because it says that the stability of the operator L_{M_0} under a small anisotropic perturbation $dM_\epsilon = d\epsilon\, NM$ is completely and exactly reflected in the permittivity anisotropic deformation constant $C_{\mathcal{S}}(M)$.
In other words, consider a dielectric material with permittivity M and a fixed distribution of charges g, this distribution of charges creates an electrostatic potential field ψ by the equation $L_M\psi = g$, now one can wonder how much ψ will be perturbed under a small anisotropic perturbation of M, then the sharp upper bound to the question "how much" is completely reflected in the constant $C_{\mathcal{S}}(M)$.
Note that if $C_{\mathcal{S}}(M_0) = \infty$, with the material associated dielectric constant M_0 one can find bounded distribution of charges $0 \leq g \leq 1$ and small perturbations of the permittivity tensor $dM_\epsilon = d\epsilon\, NM$ producing a deformation of the potential electrostatic field $G_{M_0}(g)$ without a priori bound.
Note also that if $\sup_{M\in I^\infty(\Omega)} C_{\mathcal{S}}(M) = \infty$ (M symmetric in the supremum) then one can find dielectric materials reacting to the anisotropic perturbation of its dielectric constants by a very strong deformation of the potential electrostatic field (without a priori bound).

Proof. Straightforward by using the equation 13.59 the symmetry of M_ϵ and the definition of the permittivity anisotropic deformation constant. □

13.5 Strong Sub harmonic functions and sign inequality

13.5.1 Some tools

Lemma 13.5.1. *For all $M \in I^\infty(\Omega)$, $b > 0$, $x, y \in \Omega$, $x \neq y$*

$$\begin{split}\int_{G_M(z,x)>b} \nabla_z G_M(z,x) M \nabla_z G_M(z,y)\, dz = \\ \left(G_M(y,x) - b\right) 1\left(G_M(y,x) > b\right)\end{split} \tag{13.81}$$

Proof. Write for $x \in \Omega$, $b > 0$

$$B_x^t(b) = \{z \in \Omega \,:\, G_M(z,x) > b\} \tag{13.82}$$

Observe that by Sard's theorem complementary of the set $S(B_x^t)$ of $b \geq 0$ such that $B_x^t(b)$ has a smooth boundary is of Lebesgue measure 0.
N Now let $x, y \in \Omega$, $x \neq y$ and choose $b \in S(B_x^t)$ such that $b \neq G_M(y,x)$. Observe that by the Green formula

$$\begin{aligned}\int_{B_x^t(b)} \nabla_z G_M(z,x) M \nabla_z G_M(z,y)\, dz &= \int_{B_x^t(b)} G_M(z,x)\delta(z-y)\, dz \\ &+ b \int_{\partial B_x^t(b)} n_{ext}.M\nabla_z G_M(z,y) d\sigma(z) \\ &= \big(G_M(y,x) - b\big)\, 1(y \in B_x^t(b))\end{aligned} \tag{13.83}$$

Now since the equation 13.83 is valid for all $b \in S(B_x^t)$ which complementary is of 0, Lebesgue measure and for $x \neq y$,

$$\int_\Omega |\nabla_z G_M(z,x) M \nabla_z G_M(z,y)|\, dz < \infty \tag{13.84}$$

it follows that for all $c > 0$

$$\begin{aligned}\int_{G_M(z,x)>c} \nabla_z G_M(z,x) M \nabla_z G_M(z,y)\, dz = \\ \lim_{b\downarrow c, b \in S(B_x^t)} \int_{B_x^t(b)} \nabla_z G_M(z,x) M \nabla_z G_M(z,y)\, dz\end{aligned} \tag{13.85}$$

which proves the result □

13.5.2 Proof of the sign inequality in dimension one

Theorem 13.5.1. *For $d = 1$, $C_{\mathcal{V}} = C_{\mathcal{S}}$ and $C_{\mathcal{V}}$ is a homotopy invariant: for all $M \in I^\infty(\Omega)$*

$$C_{\mathcal{V}}(M) = 3 \tag{13.86}$$

Proof. There is no loss of generality by assuming that Ω is the segment $(0,1)$. Observe that $G_M(x,y)$ is symmetric, moreover $G_M(x,z)$ is increasing from 0 to x and decreasing from x to 1. It follows that for $x, y \in \Omega$, $\nabla_z G_M(x,z) \nabla_z G_M(y,z)$ is negative in (x,y) and positive in $(0,x) \cup (y,1)$. It follows that

$$\begin{aligned}\mathcal{V}\big(G_M(x,.), G_M(y,.)\big) &= \mathcal{E}\big(G_M(x,.), G_M(y,.), (0,x)\big) \\ &- \mathcal{E}\big(G_M(x,.), G_M(y,.), (x,y)\big) + \mathcal{E}\big(G_M(x,.), G_M(y,.), (y,1)\big)\end{aligned} \tag{13.87}$$

but by the lemma 13.5.1,

$$\mathcal{E}\big(G_M(x,.), G_M(y,.), G_M(x,.) > G_M(x,y)\big) = 0 \tag{13.88}$$

But observe that

$$\{z \in (0,1) \,:\, G_M(x,z) > G_M(x,y)\} = (x,y) + A_x \tag{13.89}$$

where A_x is a subset of $(0,x)$ It follows that

$$\mathcal{E}\big(G_M(x,.), G_M(y,.), (0,x)\big) \geq -\mathcal{E}\big(G_M(x,.), G_M(y,.), (x,y)\big) \tag{13.90}$$

Similarly

$$\mathcal{E}\big(G_M(x,.),G_M(y,.),(y,1)\big) \geq -\mathcal{E}\big(G_M(x,.),G_M(y,.),(x,y)\big) \tag{13.91}$$

From the equations 13.87, 13.90, 13.91 and

$$\begin{aligned} G_M(x,y) = \mathcal{E}\big(G_M(x,.),G_M(y,.),(0,x)\big) \\ + \mathcal{E}\big(G_M(x,.),G_M(y,.),(x,y)\big) + \mathcal{E}\big(G_M(x,.),G_M(y,.),(y,1)\big) \end{aligned} \tag{13.92}$$

it follows that

$$\mathcal{V}\big(G_M(x,.),G_M(y,.)\big) \leq 3G_M(x,y) \tag{13.93}$$

which proves that $C_{\mathcal{V}}(M) \leq 3$.
Now by computing the precise value of $G_M(x,y)$ (easy task left to the reader) one sees that $C_V(M)$ is the optimal constant, moreover for all $x,y \in (0,1)$

$$\mathcal{V}\big(G_M(x,.),G_M(y,.)\big) < 3G_M(x,y) \tag{13.94}$$

and for $x \downarrow 0$ and $y \uparrow 1$

$$\frac{\mathcal{V}\big(G_M(x,.),G_M(y,.)\big)}{G_M(x,y)} \to 3 \tag{13.95}$$

□

Remark 13.5.1. The proof given above is geometrical and explains why it is natural to expect that $C_{\mathcal{V}}(M) = 3$ is also an homotopy invariant for all dimensions.

13.5.2.i Consequence on the Green functions

For $d = 1$, $U \in C^\infty(\bar{\Omega})$ write $G_{e^{-2(U)}}$ the Green function associated to $M = e^{-2U}$ with Dirichlet conditions on $\partial\Omega$.

Corollary 13.5.1. *For $d = 1$ and $U, P \in C^\infty(\bar{\Omega})$ one has*

$$e^{-6\|P\|_\infty} \leq \frac{G_{e^{-2(U+P)}}(x,y)}{G_{e^{-2(U)}}(x,y)} \leq e^{6\|P\|_\infty} \tag{13.96}$$

Proof. This is a direct consequence of the theorem 13.5.1 and the corollary 13.4.1 by considering the following isotropic deformation path: $M_\epsilon = e^{2(U+\epsilon P)}$ for $\epsilon \in [0,1]$. □

13.5.2.ii Consequence on the exit times

For $U \in C^\infty(\bar{\Omega})$ write $\mathbb{E}^U$ the expectation associated to the diffusion generated by the operator and τ its exit time from Ω.

$$L_U = \frac{1}{2}\Delta - \nabla U \nabla \tag{13.97}$$

Corollary 13.5.2. *For $U, P \in C^\infty(\bar{\Omega})$, $x \in \Omega$*

$$e^{-4\,\mathrm{Osc}(P)} \leq \frac{\mathbb{E}_x^{U+P}[\tau]}{\mathbb{E}_x^U[\tau]} \leq e^{4\,\mathrm{Osc}(P)} \tag{13.98}$$

Proof. This is a direct consequence of the corollary 13.5.1 by observing that

$$\mathbb{E}_x^U[\tau] = 2\int_\Omega G_{e^{-2U}}(x,y)e^{-2U(y)}\,dy \tag{13.99}$$

□

13.5.2.iii Geometric point of view

Corollary 13.5.3. *For $d = 1$ and $\lambda \in C^\infty([a,b])$ ($a < b$ and $\lambda > 0$) one has for all $f, g \in C^1([a,b])$ such that λf and $g\lambda$ are both increasing or decreasing functions.*

$$\begin{aligned}\int_{[a,b]} \Big| \big(f(x) - \frac{1}{\lambda(x)} \frac{\int_{[a,b]} f(y)dy}{\int_{[a,b]} \frac{1}{\lambda(y)}dy}\big)\lambda(x)\big(g(x) - \frac{1}{\lambda(x)} \frac{\int_{[a,b]} g(y)dy}{\int_{[a,b]} \frac{1}{\lambda(y)}dy}\big)\Big|\,dx \\ \leq 3\int_{[a,b]} \big(f(x) - \frac{1}{\lambda(x)} \frac{\int_{[a,b]} f(y)dy}{\int_{[a,b]} \frac{1}{\lambda(y)}dy}\big)\lambda(x)\big(g(x) - \frac{1}{\lambda(x)} \frac{\int_{[a,b]} g(y)dy}{\int_{[a,b]} \frac{1}{\lambda(y)}dy}\big)\,dx\end{aligned} \tag{13.100}$$

Proof. This is a direct consequence of the theorem 13.5.1. Indeed observe that for $a \leq x \leq b$ and

$$\phi(z) = \int_a^z \big(f(x) - \frac{1}{\lambda(x)} \frac{\int_{[a,b]} f(y)dy}{\int_{[a,b]} \frac{1}{\lambda(y)}dy}\big)\,dx \tag{13.101}$$

$$\psi(z) = \int_a^z \big(g(x) - \frac{1}{\lambda(x)} \frac{\int_{[a,b]} g(y)dy}{\int_{[a,b]} \frac{1}{\lambda(y)}dy}\big)\,dx \tag{13.102}$$

ψ and ϕ are both null on $z = a$ and $z = b$, moreover, there are both Sub harmonic or both super harmonic with respect to $L_M = -\nabla\lambda\nabla$ □

13.5.3 Extension to dimension $d \geq 2$

Theorem 13.5.2. *Let $\phi, \psi \in C^2_D(\Omega)$, $M \in I^\infty(\Omega)$. Assume that there exists a $Z : \Omega \to W$ a diffeomorphism from Ω to a smooth bounded open set W of $\mathbb{R}^d$ such that $F^{-1}(\partial W) \subset \partial\Omega$ and*

$$(\frac{DZ}{Dx})M(Z)^t(\frac{DZ}{Dx})\big|\frac{Dx}{DZ}\big| = k(Z)I_d \tag{13.103}$$

such that for all $i \in \{1, \dots, d\}$, for all $(Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_d) \in \mathbb{R}^{d-1}$, ϕ, ψ (as functions of Z_i) are both Sub harmonic or super harmonic with respect to the operator
$-\frac{\partial}{\partial Z_i}\big(k(Z)\frac{\partial}{\partial Z_i}\big)$ on $W_i = \{Z_i \in \mathbb{R} \,:\, Z \in W\}$ then

$$\int_\Omega |\nabla_x\phi(x)M(x)\nabla_x\psi(x)|\,dx \leq 3\int_\Omega \nabla_x\phi(x)M(x)\nabla_x\psi(x)\,dx \tag{13.104}$$

Proof. Observe that for $h \in C^\infty(\bar{\Omega})$ one has,

$$\int_\Omega h(x)\nabla_x\phi(x)M(x)\nabla_x\psi(x)\,dx = \int_W h(Z)k(Z)\nabla_Z\phi(Z)\nabla_Z\psi(Z)\,dZ \tag{13.105}$$

with

$$k(Z)I_d = (\frac{DZ}{Dx})M(Z)^t(\frac{DZ}{Dx})\big|\frac{Dx}{DZ}\big| \tag{13.106}$$

It follows from the theorem 13.5.1 the theorem 13.2.1 and the property 13.2.7 that

$$
\begin{aligned}
\int_W h(Z)k(Z)\nabla_Z\phi(Z)\nabla_Z\psi(Z)\,dZ &= \sum_{i=1}^{d}\int_{\mathbb{R}^{d-1}} dZ_1\dots dZ_{i-1}dZi+1\dots dZ_d \\
&\int_{Z_i\in W_i} h(Z)k(Z)\frac{\partial\phi(Z)}{\partial Z_i}\frac{\partial\psi(Z)}{\partial Z_i}\,dZ_i \\
&\le \sum_{i=1}^{d}\int_{\mathbb{R}^{d-1}} dZ_1\dots dZ_{i-1}dZi+1\dots dZ_d \\
&3\|h\|_\infty\int_{Z_i\in W_i} k(Z)\frac{\partial\phi(Z)}{\partial Z_i}\frac{\partial\psi(Z)}{\partial Z_i}\,dZ_i \\
&\le \int_W k(Z)\nabla_Z\phi(Z)\nabla_Z\psi(Z)\,dZ \\
&\le 3\|h\|_\infty\int_\Omega \nabla_x\phi(x)M(x)\nabla_x\psi(x)\,dx
\end{aligned}
\tag{13.107}
$$

Thus by the theorem 13.2.1

$$
\int_\Omega |\nabla_x\phi(x)M(x)\nabla_x\psi(x)|\,dx \le 3\int_\Omega \nabla_x\phi(x)M(x)\nabla_x\psi(x)\,dx \tag{13.108}
$$

□

Let $\lambda\in C^\infty(\bar\Omega)$ such that $\lambda>0$ on $\bar\Omega$, then $\psi\in C^2_D(\Omega)$ is said to be strongly Sub harmonic (resp. strongly super harmonic) with respect to the operator $-\nabla(\lambda\nabla)$ if for all $x\in\Omega$, all $e\in\mathbb{S}^d$ $-\frac{\partial}{\partial e}\big(\lambda(x)\frac{\partial}{\partial e}\phi\big)\ge 0$ (resp ≤ 0)

Theorem 13.5.3. *For all $\psi,\phi\in C^2_D(\Omega)$ strongly Sub harmonic or super harmonic with respect to the operator $-\nabla(\lambda\nabla)$ one has*

$$
\int_\Omega |\nabla_x\phi(x)\lambda(x)\nabla_x\psi(x)|\,dx \le 3\int_\Omega \nabla_x\phi(x)\lambda(x)\nabla_x\psi(x)\,dx \tag{13.109}
$$

Proof. As for the theorem 13.5.2 it is a direct consequence of the theorem 13.5.1. □

Corollary 13.5.4. *Assume that ϕ,ψ are both convex or both concave and null on $\partial\Omega$, then*

$$
\int_\Omega |\nabla_x\phi(x).\nabla_x\psi(x)|\,dx \le 3\int_\Omega \nabla_x\phi(x).\nabla_x\psi(x)\,dx \tag{13.110}
$$

13.5.4 Weak stability results

Theorem 13.5.4. *Assume that M,Q are symmetric smooth coercive matrices on $\bar\Omega$ and $d\ge 1$. Assume $M\le\lambda Q$ with $\lambda>0$, then for all $f\in C^0(\Omega)$,*

$$
\int_\Omega G_Q(x,y)f(y)f(x)\,dx\,dy \le \lambda\int_\Omega G_M(x,y)f(y)f(x)\,dx\,dy \tag{13.111}
$$

Proof. Let $f\in C^0(\bar\Omega)$. Write ψ_M , ψ_Q the solutions of $L_M\psi_M=f$ and $L_Q\psi_Q=f$ with Dirichlet conditions on $\partial\Omega$. Observe that ψ_M and ψ_Q are the unique minimizer of the following variational formulae

$$
I_M(h,f)=\frac{1}{2}\mathcal{E}_M(h,h,\Omega)-\int_\Omega h(x)f(x)\,dx \tag{13.112}
$$

$$I_Q(h,f) = \frac{1}{2}\mathcal{E}_Q(h,h,\Omega) - \int_\Omega h(x)f(x)\,dx \tag{13.113}$$

and at the minimum

$$I_M(\psi_M, f) = -\frac{1}{2}\int_\Omega \psi_M(x)f(x)\,dx \tag{13.114}$$

$$I_Q(\psi_Q, f) = -\frac{1}{2}\int_\Omega \psi_Q(x)f(x)\,dx \tag{13.115}$$

Moreover observe that

$$\begin{aligned} I_M(h,f) &\leq \frac{\lambda}{2}\mathcal{E}_Q(h,h,\Omega) - \int_\Omega h(x)f(x)\,dx \\ &\leq \lambda I_Q(h, \frac{f}{\lambda}) \end{aligned} \tag{13.116}$$

and the minimum of the right member in the equation 13.116 is reached at ψ_Q/λ. It follows that

$$-\frac{1}{2}\int_\Omega \psi_M(x)f(x)\,dx \leq -\frac{1}{2\lambda}\int_\Omega \psi_Q(x)f(x) \tag{13.117}$$

thus

$$\int_\Omega \psi_Q(x)f(x) \leq \lambda \int_\Omega \psi_M(x)f(x) \tag{13.118}$$

which proves the result □

APPENDIX

A. ONE OBJECT - DIFFERENT FACES -ABSTRACT CONNECTIONS

It had been known for a long time that there exist deep and close connections between the Dirichlet integral, the Laplace Operator, the heat equation, and Brownian motion. With the notion of Dirichlet forms, Beurling and Deny ([BD58], [BD59]) provided the proper axiomatic setting for exploring these connections abstractly. In subsequent developments (Fukushima, Silverstein and others), Markovian semigroups and probabilistic potential theory based on symmetric Markov processes played a prominent role (see the monograph of M. Fukushima, Y. Oshima and M. Takeda [FOT94]). The analysis of these connections is an active field of research [JKM+98]. Indeed Dirichlet forms play a prominent role in various fields of mathematics, this is mainly due to the fact that they allow the development of highly nontrivial extensions of classical theories under minimal regularity hypotheses.

In this chapter some abstract connections given in [FOT94] will be summed up.
Let H be an abstract real Hilbert space with inner product $(.,.)$

A.1 Semigroup

A family $\{P_t, t > 0\}$ of linear operators on H is called a semigroup (of symmetric operators) on H if it satisfies the following conditions:

- each P_t is a symmetric operator with domain $\mathcal{D}(P_t) = H$.
- semigroup property: $P_t P_s = P_{t+s},\ \ t, s > 0$
- contraction property $(P_t u, P_t u) \leq (u, u),\ \ t > 0,\ u \in H$

It is called strongly continuous if in addition

- for $u \in H,\ \ \lim_{t\to 0}(P_t u - u, P_t u - u) = 0$

A.2 Resolvent

A family $\{G_\alpha, \alpha > 0\}$ of linear operators on H is called a resolvent on H if it satisfies the following conditions:

- each G_α is a symmetric operator with domain $\mathcal{D}(G_\alpha) = H$.
- resolvent equation: $G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0$.
- contraction property $(\alpha G_\alpha u, \alpha G_\alpha u) \leq (u, u),\ \ \alpha > 0,\ u \in H$

It is called strongly continuous if in addition

- for $u \in H,\ \ \lim_{\alpha\to+\infty}(\alpha G_\alpha u, \alpha G_\alpha u) = 0$

Given a strongly continuous semigroup $\{P_t,\, t > 0\}$, the strong limit of the Riemann sum

$$G_\alpha u = \int_0^\infty e^{-\alpha t} P_t u dt \tag{A.1}$$

determines a strongly continuous resolvent $\{G_\alpha, \alpha > 0\}$ on H called the resolvent of the given semigroup.
Given a strongly continuous resolvent $\{G_\alpha, \alpha > 0\}$, the limit

$$P_t u = \lim_{\beta\to\infty} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} (\beta G_\beta)^n u \quad u \in H \tag{A.2}$$

determines a strongly continuous semigroup $\{P_t,\, t > 0\}$ on H called the semigroup of the given resolvent.

The operations A.1 and A.2 are injective, inverse of each other and put into one to one correspondences the family of strongly continuous semigroups with the family of strongly continuous resolvents.

A.3 Generator

The generator L of a strongly continuous semigroup $\{P_t,\, t > 0\}$ on H is an operator on H defined by

$$\begin{cases} Lu = \lim_{t\to 0} \frac{P_t u - u}{t} \\ \mathcal{D}(L) = \{u \in H \,:\, Lu \text{ exists as a strong limit}\} \end{cases} \tag{A.3}$$

Given a strongly continuous resolvent $\{G_\alpha,\, \alpha > 0\}$ on H, for $\alpha > 0$, G_α is invertible and the operator defined by

$$\begin{cases} Lu = \alpha u - G_\alpha^{-1} u \\ \mathcal{D}(L) = G_\alpha(H) \end{cases} \tag{A.4}$$

is independent of $\alpha > 0$, called the generator of the resolvent $\{G_\alpha,\, \alpha > 0\}$ and is a non-positive definite self-adjoint operator. Moreover the generator A.3 of a strongly continuous semigroup on H coincides with the generator A.4 of its resolvent A.1.
Let $-L$ be an non-negative definite self-adjoint operator on H. Then

$$P_t = \exp(tL), \quad t > 0 \tag{A.5}$$

$$G_\alpha = (\alpha - L)^{-1},\, \alpha > 0 \tag{A.6}$$

are a strongly continuous semigroup and a strongly continuous resolvent on H respectively.

The operations A.5 and A.3 are injective, inverse of each other and put into one to one correspondences the family of strongly continuous semigroups with the family of non-negative definite self-adjoint operators on H.

The operations A.6 and A.4 are injective, inverse of each other and put into one to one correspondences the family of strongly continuous semigroups with the family of non-negative definite self-adjoint operators on H.

A.4 Symmetric forms and Dirichlet forms

$\mathcal{E}$ is called a symmetric form on H if it is a non-negative definite symmetric form on H, that is to say if the following conditions are satisfied:

- $\mathcal{E}$ is defined on $\mathcal{D}[\mathcal{E}] \times \mathcal{D}[\mathcal{E}]$ with values in $\mathbb{R}$, $\mathcal{D}[\mathcal{E}]$being a dense linear subspace of H called the domain of $\mathcal{E}$.
- for $a \in \mathbb{R}$ and $u, v, w \in \mathcal{D}[\mathcal{E}]$
  $\mathcal{E}(u,v) = \mathcal{E}(v,u), \quad \mathcal{E}(u+v,w) = \mathcal{E}(u,w) + \mathcal{E}(v,w),$
  $a\mathcal{E}(u,v) = \mathcal{E}(au,v), \quad \mathcal{E}(u,u) \geq 0$

For each $\alpha > 0$ and a given symmetric form $\mathcal{E}$ on H,

$$\begin{aligned} \mathcal{E}_\alpha(u,v) &= \mathcal{E}(u,v) + \alpha(u,v), \quad u,v \in \mathcal{D}[\mathcal{E}] \\ \mathcal{D}[\mathcal{E}_\alpha] &= \mathcal{D}[\mathcal{E}] \end{aligned} \tag{A.7}$$

defines a new symmetric form on H and a symmetric form $\mathcal{E}$ is said to be closed if

- $u_n \in \mathcal{D}[\mathcal{E}]$, $\lim_{n,m\to\infty} \mathcal{E}_1(u_n - u_m, u_n - u_m) = 0$
  $\Rightarrow \exists u \in \mathcal{D}[\mathcal{E}], \quad \lim_{n\to\infty} \mathcal{E}_1(u_n - u, u_n - u) = 0$

Let $-L$ be a non-negative definite self-adjoint operator on H, write $\{E_\lambda\}$ the spectral family associated with $-L$. Then the expression

$$\begin{cases} \mathcal{D}[\mathcal{E}] = \mathcal{D}(\sqrt{-L}) = \{u \in H \,:\, \int_{[0,\infty)} \lambda d(E_\lambda u, u) < \infty\} \\ \mathcal{E}(u,v) = (\sqrt{-L}u, \sqrt{-L}v) = \int_{[0,\infty)} \lambda d(E_\lambda u, v) \end{cases} \tag{A.8}$$

defines a closed symmetric form on H.
Conversely, given a closed symmetric form $\mathcal{E}$ on H, there exists by the Riesz representation theorem a unique element $G_\alpha u \in \mathcal{D}[\mathcal{E}]$ such that

$$\mathcal{E}_\alpha(G_\alpha u, v) = (u,v), \quad \forall v \in \mathcal{D}[\mathcal{E}] \tag{A.9}$$

for each $\alpha > 0$ and $u \in H$. The family $\{G_\alpha,\, \alpha > 0\}$ defined in this way is a strongly continuous resolvent and allows to associate to $\mathcal{E}$ a unique non-positive definite self-adjoint operator L on H.

The operations A.8 and A.9 are injective, inverse of each other and put into one to one correspondence the family of closed symmetric forms on H and the family of non-positive definite self-adjoint operators L on H.
Thus the family of closed symmetric forms on H is also into a one to one correspondence with the family of strongly continuous semigroup and resolvent on H and the associated applications are given by the following operations A.10 and A.11:
For any $u \in H$, $\frac{(u - P_t u, u)}{t}$ is non-decreasing as $t \downarrow 0$ and

$$\begin{cases} \mathcal{D}[\mathcal{E}] = \{u \in H \,:\, \lim_{t\to 0} \frac{(u-P_t u, u)}{t} < \infty\} \\ \mathcal{E}(u,v) = \lim_{t\to 0} \frac{(u-P_t u, v)}{t} \quad u,v \in \mathcal{D}[\mathcal{E}] \end{cases} \tag{A.10}$$

For any $u \in H$, $\beta(u - \beta G_\beta u, u)$ is non-decreasing as $t \uparrow \infty$ and

$$\begin{cases} \mathcal{D}[\mathcal{E}] = \{u \in H \,:\, \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, u) < \infty\} \\ \mathcal{E}(u,v) = \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, v) \quad u,v \in \mathcal{D}[\mathcal{E}] \end{cases} \tag{A.11}$$

If H is the L^2-space $L^2(X,m)$ associated to a σ-finite measure space $(X,\mathcal{B},m)$; consisting of square integrable m-measurable extended real valued functions on X and endowed with the inner product

$$(u,v) = \int_X u(x)v(x)dm(x), \quad u,v \in L^2(X,m) \tag{A.12}$$

then a closed symmetric form $\mathcal{E}$ on $L^2(X,m)$ is called Markovian if the unit contraction operates on $\mathcal{E}$; that is to say

- $u \in \mathcal{D}[\mathcal{E}]$, $v = (0 \vee u) \wedge 1 \Rightarrow v \in \mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(v,v) \leq \mathcal{E}(u,u)$

A closed, Markovian symmetric form is called a Dirichlet form. Then the couple $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ is called a Dirichlet space relative to $L^2(X,m)$.
To say that a closed symmetric form is Markovian is equivalent to say that its associated semigroup P_t or resolvent G_α are Markovian, a linear operator S on $L^2(X,m)$ with $\mathcal{D}(S) = L^2(X,m)$ being called Markovian if $u \in L^2(X,m)$ and $0 \leq u \leq 1$ m-a.e.imply $0 \leq Su \leq 1$ m-a.e.

A.5 Symmetric Markov Process - Hunt Process - Diffusion and Dirichlet Space

Let X be a locally compact separable metric space and m be an everywhere dense positive Radon measure on X. A Markov process (see [FOT94] for a definition) $\mathbf{M}$ on $(X,\mathcal{B}(X))$ is called m-symmetric if the transition function p_t of M is m-symmetric in the sense that for all non-negative measurable functions u and v.

$$\int_X u(x)(p_t v)(x)m(dx) = \int_X (p_t u)(x)v(x)m(dx) \tag{A.13}$$

A Hunt process (for a definition see [FOT94] page 314) is a special Markov process that possesses useful properties such as the right continuity of sample paths, the quasi left continuity and the strong Markov property.
A Hunt process on X is called a diffusion if for every $x \in X$, $\mathbb{P}_x$ a.s., X_t is continuous in t until it reaches its cemetery state.
The transition function $\{p_t,\, t > 0\}$ of a m-symmetric Hunt process $\mathbf{M}$ on X uniquely determines a strongly continuous Markovian semigroup $\{P_t,\, t > 0\}$ on $L^2(X,m)$ and thus uniquely a Dirichlet space $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ on $L^2(X,m)$.
A symmetric form $\mathcal{E}$ is said to be regular if and only if

- $\mathcal{D}[\mathcal{E}] \cap C_0(X)$ is dense in $\mathcal{D}[\mathcal{E}]$ with $\mathcal{E}_1$ norm and dense in $C_0(X)$ (the set of continuous functions on X with compact support) with uniform norm.

Conversely, given a regular Dirichlet form $\mathcal{E}$ on $L^2(X,m)$, there exists an m-symmetric Hunt process M on $(X,\mathcal{B}(X))$ whose Dirichlet form is the given one $\mathcal{E}$. This Hunt process is unique (up to an equivalence) in the sense that if two m-symmetric Hunt processes have a common regular Dirichlet space on $L^2(X,M)$ then they possess a common properly exceptional set outside which their transition functions coincide. Moreover this Hunt process is equivalent to an m-symmetric diffusion if and only if its associated Dirichlet form $\mathcal{E}$ possesses the local property that is to say if

- $u,v \in \mathcal{D}[\mathcal{E}]$, Supp$[u]$ and Supp$[v]$ are disjoint compact sets
  $\Rightarrow \mathcal{E}(u,v) = 0$

B. ANALYTICAL TOOLS

B.1 Sharp estimates from the theory of elliptic operators

This section introduce some sharp estimates from the theory of divergence form and non divergence form elliptic operators. For a good introduction to the subject see the books of M. Giaquinta [Gia83], [Gia93]; D. Gilbarg and N.S. Trudinger [GT83]; D. Kinderlehrer and G. Stampacchia [KS80] and although the course of G. Stampacchia on elliptic equations with discontinuous coefficients [Sta66] was published in 1966, it is still interesting and contains powerful and beautiful proofs.

B.1.1 Divergence form operator with bounded coefficients

Throughout this subsection, the operator (considered in the weak sense) on which the results will be given is

$$L = \nabla(A\nabla) \tag{B.1}$$

defined on some open set $\Omega \subset \mathbb{R}^d$ (for $d \geq 3$) with smooth boundary $\partial\Omega$. A is a $d \times d$ matrix with bounded coefficients in $L^\infty(\Omega)$ such that for all $\xi \in \mathbb{R}^d$.

$$\lambda|\xi|^2 \leq {}^t\xi A \xi \tag{B.2}$$

and for all i, j

$$|A_{ij}| \leq M \tag{B.3}$$

for some positive constant $0 < \lambda, M < \infty$.
Here a theorem concerning elliptic equation with discontinuous coefficient from G. Stampacchia is presented. Its proof in a more general form can be found in [Sta66], chapter 5, theorem 5.4 (see also [Sta65])
The explicit dependence of the constants in M and λ have been obtained by following the proof of G. Stampacchia [Sta66]

B.1.1.i Sharp estimates of the L^∞ norm

Let $p > d \geq 3$
For $1 \leq i \leq d$ let $f_i \in L^p(\Omega)$
if $\chi \in H^1_{loc}(\Omega)$ is a local (weak) solution of the equation

$$\nabla\big(A\nabla\chi\big) = -\sum_{i=1}^{d} \partial_i f_i \tag{B.4}$$

where A is coercive B.2 and bounded B.3 then χ is in $L^\infty(\Omega)$ and if $x_0 \in \Omega$ and $R > 0$

Theorem B.1.1. *The solution of B.4 verify the following inequality (in the essential supremum sense with $\Omega(x_0,R) = \Omega \cap B(x_0,R)$)*

$$\begin{aligned}\max_{\Omega(x_0,\frac{R}{2})} |\chi| \leq & K\Big[\big\{\frac{1}{R^d}\int_{\Omega(x_0,R)} \|\chi\|^2\big\}^{\frac{1}{2}} \\ & + \sum_{i=1}^{d} \|f_i\|_{L^p(\Omega(x_0,R))} \frac{R^{1-\frac{d}{p}}}{\lambda}\Big]\end{aligned} \tag{B.5}$$

with

$$K = C_d(\frac{M}{\lambda})^{\frac{3d}{2}} \tag{B.6}$$

B.1.2 A short reminder on Laplace operator

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary and $\chi \in C^2(\Omega)$ the solution of

$$\Delta\chi = f \tag{B.7}$$

with f bounded.

B.1.2.i Gradient estimates for Poisson's equation

Write $\rho_x = \operatorname{dist}(x, \partial\Omega)$ and $\rho_{xy} = \min(\rho_x, \rho_y)$ where dist is the Euclidean distance. Then the following theorem is proven in [GT83] (theorem 3.9, page 41)

Theorem B.1.2. *The solution of Poisson's equation B.7 verify the following gradient estimates*

$$\sup_{x\in\Omega} \rho_x |\nabla\chi(x)| \leq C_d\big[\sup_{\Omega}|\chi| + \sup_{\Omega}\rho_x^2|f(x)|\big] \tag{B.8}$$

and for all $x, y \in \Omega$, $x \neq y$

$$\rho_{xy}^2 \frac{|\nabla\chi(x) - \nabla\chi(y)|}{|x-y|} \leq C_d\big[\sup_{\Omega}|\chi| + \sup_{\Omega}\rho_x^2|f(x)|\big]\big[1 + |\ln\frac{\rho_{xy}}{|x-y|}|\big] \tag{B.9}$$

It is interesting to notice that this theorem is essentially sharp and the estimate B.9 cannot be improved without further continuity assumptions on f ([GT83] page 41).

B.1.2.ii Estimates of the Hölder continuity of the second derivates of the solution of the Poisson equation

Assume that the second member f in B.7 is uniformly Hölder continuous with exponent α that is to say $[f]_{\alpha,\Omega} < \infty$ with

$$[f]_{\alpha,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, \quad 0 < \alpha \leq 1 \tag{B.10}$$

Define also the following norms to control χ

$$|g|_{0,\alpha,\Omega} = \sup_{\Omega}|g| + (\operatorname{diam}\Omega)^\alpha [g]_{\alpha,\Omega} \tag{B.11}$$

$$\begin{aligned}|g|_{2,\alpha,\Omega} = & \sup_{\Omega}|g| + (\operatorname{diam}\Omega)\sup_{\Omega,i}|\partial_i g| + (\operatorname{diam}\Omega)^2 \sup_{\Omega,i,j}|\partial_i\partial_j g| \\ & + (\operatorname{diam}\Omega)^{2+\alpha}\sup_{i,j}[\partial_i\partial_j g]_{\alpha,\Omega}\end{aligned} \tag{B.12}$$

Then it is proven in [GT83] (theorem 4.6, page 60) that the second derivates of the solution χ of B.7 are uniformly Hölder continuous with exponent α and the following estimate is available.

Theorem B.1.3. *Let χ be the solution of B.7. Let $x_0 \in \Omega$ and $R > 0$ such that $B_2 = B(x_0, 2R) \subset \Omega$, write $B_1 = B(x_0, R)$ then*

$$|\chi|_{2,\alpha,B_1} \le C(d,\alpha)\big(|\chi|_{0,B_2} + R^2|f|_{0,\alpha,B_2}\big) \tag{B.13}$$

B.1.3 Application to a particular case

Let $\chi \in C^2(\Omega)$ such that $|\nabla\chi|$ is bounded, be the solution of

$$\big(\frac{1}{2}\Delta + b\nabla\big)\chi = g \tag{B.14}$$

where g and b are continuous and bounded.

B.1.3.i Control of the gradient

Lemma B.1.1. *Let χ be the solution of B.14 and assume that $x_0 \in \Omega$ is such that $|\nabla\chi(x_0)| = \|\nabla\chi\|_\infty$. Then for $\rho < \min(\frac{1}{C_d\|b\|_\infty}, \mathrm{dist}(x_0, \partial\Omega))$*

$$|\nabla\chi|_\infty \le C_d\Big[\frac{\sup_{B(x_0,\rho)}}{\rho}|\chi| + \rho \sup_{B(x_0,\rho)} |g|\Big] \tag{B.15}$$

Proof. Observe that

$$\Delta\chi = 2(g - b\nabla\chi) \tag{B.16}$$

Since $\rho < \mathrm{dist}(x_0, \Omega)$ an application of the control B.8 in theorem B.1.2 to the ball $B(x_0, \rho)$ gives

$$\rho|\nabla\chi(x_0)| \le C_d\big[\sup_{B(x_0,\rho)} |\chi| + \rho^2(\|g\|_\infty + \|b\|_\infty\|\nabla\chi\|_\infty)\big] \tag{B.17}$$

Thus for $\rho < 1/(C_d\|b\|_\infty)$

$$\|\nabla\chi\|_\infty \le \frac{\sup_{B(x_0,\rho)} |\chi| + \rho^2 \sup_{B(x_0,\rho)} |g|}{\rho - C_d\|b\|_\infty\rho^2} \tag{B.18}$$

Thus for $\rho < 1/(2C_d\|b\|_\infty)$

$$|\nabla\chi|_\infty \le 2C_d\Big[\frac{\sup_{B(x_0,\rho)}}{\rho}|\chi| + \rho \sup_{B(x_0,\rho)} |g|\Big] \tag{B.19}$$

□

B.1.3.ii Hölder continuity of the gradient

In the sequel it is shown that a bound on the L^∞ norm of b and g leads to a control on the Hölder continuity of the gradient of χ for all exponents $\alpha \in [0, 1)$.

Lemma B.1.2. *Let χ be the solution of B.14 and $R > 0$ such that $2R < \mathrm{dist}(x_0, \Omega)$. Then for $\alpha \in [0, 1)$ and $x, y \in B(x_0, R)$, $x \neq y$*

$$\begin{aligned} R^2\frac{|\nabla\chi(x) - \nabla\chi(y)|}{|x-y|^\alpha} \le & C(d,\alpha)\big[\sup_{B(x_0,2R)} |\chi| + R^2 \sup_{B(x_0,2R)} (|g| + |b||\nabla\chi|)\big] \\ & \cdot \big[1 + R^{1-\alpha}\big] \end{aligned} \tag{B.20}$$

Proof. Since $2R < \mathrm{dist}(x_0, \Omega)$ an application of the control B.9 in theorem B.1.2 to the ball $B(x_0, 2R)$ gives for $x, y \in B(x_0, R)$, $x \neq y$ and $f = 2(g - b.\nabla\chi)$

$$R^2\frac{|\nabla\chi(x) - \nabla\chi(y)|}{|x-y|} \le C_d\big[\sup_{B(x_0,2R)} |\chi| + R^2 \sup_{B(x_0,2R)} |f(x)|\big]\big[3 + |\ln\frac{2R}{|x-y|}|\big] \tag{B.21}$$

which leads to the proof. □

B.1.3.iii Application to linear harmonic functions in periodic medium

Let $F \in C^2(\mathbb{R}^2)$ such that be the solution of

$$\big(\frac{1}{2}\Delta + b\nabla\big)F = 0 \tag{B.22}$$

where b is continuous and periodic on $\mathbb{R}^d$ of period T_1^d. Assume also that

$$F(x) = l.x - \chi_l(x) \tag{B.23}$$

where $\chi_l \in C^2(T_1^d)$ and $l \in \mathbb{R}^d$ is such that $|l| = 1$.

Lemma B.1.3. *Let $l.x - \chi_l(x)$ be the solution of B.22 then*

$$\|\nabla\chi_l\|_\infty \leq C_d\big[1 + \|b\|_\infty\|\chi_l\|_\infty\big] \tag{B.24}$$

and for all $\alpha \in [0,1)$, $x, y \in \mathbb{R}^d$

$$\frac{|\nabla\chi_l(x) - \nabla\chi_l(y)|}{|x-y|^\alpha} \leq C(d,\alpha)(1 + \|\chi_l\|_\infty)(1 + \|b\|_\infty)^2 \tag{B.25}$$

and if $b \in C^1(T_1^d)$ for all $i, j \in \{1, \dots, d\}$

$$\|\partial_i\partial_j\chi_l\|_\infty \leq C_d(1 + \|\chi_l\|_\infty)(1 + \|b\|_\infty)^3(1 + \|\sup_k \partial_k b\|_\infty) \tag{B.26}$$

Proof. By the lemma B.1.1

$$\|\nabla F\|_\infty \leq C_d\big[1 + \|b\|_\infty\|\chi_l\|_\infty\big] \tag{B.27}$$

which leads to

$$\|\nabla\chi_l\|_\infty \leq C_d\big[1 + \|b\|_\infty\|\chi_l\|_\infty\big] \tag{B.28}$$

Thus by the lemma B.1.2 for all $x, y \in \mathbb{R}^2$, $x \neq y$

$$\begin{aligned}\frac{|\nabla F(x) - \nabla F(y)|}{|x-y|^\alpha} &\leq C(d,\alpha)\big[1 + \|\chi_l\|_\infty + \|b\|_\infty(1 + \|b\|_\infty\|\chi_l\|_\infty)\big] \\ &\leq C(d,\alpha)(1 + \|\chi_l\|_\infty)(1 + \|b\|_\infty)^2\end{aligned} \tag{B.29}$$

which leads to

$$\frac{|\nabla\chi_l(x) - \nabla\chi_l(y)|}{|x-y|^\alpha} \leq C(d,\alpha)(1 + \|\chi_l\|_\infty)(1 + \|b\|_\infty)^2 \tag{B.30}$$

Assume that $b \in C^1(T_1^d)$ (b Hölder continuous is sufficient) then by the theorem B.1.3 for all $i, j \in \{1, \dots, d\}$

$$\|\partial_i\partial_j\chi_l\|_\infty \leq C(d,1/2)\big(1 + \|\chi_l\|_\infty + \|\sup_k \partial_k b\|_\infty\|\nabla F\|_\infty + \|b\|_\infty|\nabla F|_{0,1/2}\big) \tag{B.31}$$

which leads to

$$\|\partial_i\partial_j\chi_l\|_\infty \leq C_d(1 + \|\chi_l\|_\infty)(1 + \|b\|_\infty)^3(1 + \|\sup_k \partial_k b\|_\infty) \tag{B.32}$$

□

B.2 Application to the Cell problem

B.2.1 Potential Diffusion

Let $d \geq 3$, $U \in C^\infty(T_1^d)$ and L_U be the operator

$$L_U = \frac{1}{2}\Delta - \nabla U.\nabla \tag{B.33}$$

B.2.1.i Control of the Cell problem

Let $\chi_l \in C^\infty(T_1^d)$ be the solution of the cell problem

$$L_U \chi_l = -l.\nabla U \tag{B.34}$$

with $\chi_l(0) = 0$, $l \in \mathbb{R}^d$, $|l| = 1$, then the following theorem gives a control on this solution.

Theorem B.2.1. *The solution of the above cell problem satisfies the following inequalities:*

1.

$$\|\chi_l\|_\infty \leq C_d \exp\big((3d+2)\operatorname{Osc}(U)\big) \tag{B.35}$$

2.

$$\|\nabla\chi_l\|_\infty \leq C_d(1 + \|\nabla U\|_\infty)\exp\big((3d+2)\operatorname{Osc}(U)\big) \tag{B.36}$$

3. *for all* $\alpha \in [0,1)$, $x, y \in \mathbb{R}^d$

$$\frac{|\nabla\chi_l(x) - \nabla\chi_l(y)|}{|x-y|^\alpha} \leq C(d,\alpha)(1 + \|\nabla U\|_\infty)^2 \exp\big((3d+2)\operatorname{Osc}(U)\big) \tag{B.37}$$

4. *for all* $i, j \in \{1, \dots, d\}$

$$\|\partial_i\partial_j\chi_l\|_\infty \leq C_d(1 + \|\nabla U\|_\infty)^3(1 + \|\sup_{kp}\partial_k\partial_p U\|_\infty)\exp\big((3d+2)\operatorname{Osc}(U)\big) \tag{B.38}$$

Remark B.2.1. For $d = 1$, this theorem is trivial, for $d = 2$, consider $U(x_1, x_2)$ as a function on T_1^3 to obtain the same results (just change in all the constants d by $d+1$).

Proof. χ_l satisfies

$$\nabla\big(\exp(-2U)\nabla\chi_l\big) = l.\nabla\exp(-2U)$$

then by the theorem B.1.1 for $x_0 \in [0,1]^d$

$$\max_{B(x_0,\frac{1}{2})}|\chi_l| \leq C_d \exp(3\operatorname{Osc}(U)d)\big[\big(\int_{B(x_0,1)}|\chi_l|^2\big)^{\frac{1}{2}} + |l|\exp(2\operatorname{Osc}(U))\big]$$

Now by periodicity

$$\int_{B(x_0,1)}|\chi_l|^2 dx \leq \int_{T_1^d}|\chi_l|^2 dx$$

and by the Poincaré inequality

$$\int_{T_1^d}|\chi_l|^2 dx \leq C_d\int_{T_1^d}|\nabla\chi_l|^2 dx$$

thus (see 5.1 for the definition of the measure m_U)

$$\int_{B(x_0,1)} |\chi_l|^2 dx \leq C_d \exp(2\,\mathrm{Osc}(U)) \int_{T_1^d} |\nabla \chi_l|^2 m_U(dx)$$

And since

$$\int_{T_1^d} |l - \nabla \chi_l|^2 m_U(dx) = l^2 - \int_{T_1^d} |\nabla \chi_l|^2 m_U(dx)$$

one has

$$\int_{T_1^d} |\nabla \chi_l|^2 m_U(dx) \leq l^2$$

and the bound on $\|\chi_l\|_\infty$ is proven. Now the controls on $\nabla \chi_l$ and $\partial_i \partial_j \chi_l$ are a direct consequence of the lemma B.1.3 by noticing that $l.x - \chi_l(x)$ is harmonic with respect to L_U. □

B.2.1.ii Control of the ergodicity

Let $\phi_l \in C^\infty(T_1^d)$ be the solution of

$$L_U \phi_l = |l - \nabla \chi_l|^2 - \int_{T_1^d} |l - \nabla \chi_l|^2 m_U(dx) \tag{B.39}$$

with $l \in \mathbb{R}^d$, $|l| = 1$, $\phi_l(0) = 0$. Remember that ϕ_l reflects the speed at which the diffusion associated to L_U converges towards its homogenized behavior, that's why in the following theorem a control on ϕ_l will be given.

Theorem B.2.2. *Let ϕ_l be the above solution of B.39. Then*

1.

$$\|\phi_l\|_\infty \leq C_d \exp\big((9d+4)\,\mathrm{Osc}(U)\big) \tag{B.40}$$

$$\|\nabla \phi_l\|_\infty \leq C_d \exp\big((9d+4)\,\mathrm{Osc}(U)\big)(1 + \|\nabla U\|_\infty) \tag{B.41}$$

Remark B.2.2. For $d = 1$, this theorem is trivial, for $d = 2$, consider $U(x_1, x_2)$ as a function on T_1^3 to obtain the same results (just change in all the constants d by $d + 1$).

Proof. Write

$$F_l = l.x - \chi_l$$

Since

$$L_U(F_l^2) = |l - \nabla \chi_l|^2$$

if one writes $\psi_l = F_l^2 - \phi_l$ then

$$\nabla\big(\exp(-2U)\nabla \psi_l\big) = 2\exp(-2U) \int_{T_1^d} |l - \nabla \chi_l|^2 m_U(dx)$$

Thus by the theorem B.1.1 for $x_0 \in [0,1]^d$

$$\begin{aligned} \max_{B(x_0,\frac{1}{2})} |\psi(x)| \leq & C_d \exp\big(3\,\mathrm{Osc}(U) d\big) \Big[\Big(\int_{B(x_0,1)} |\psi_l|^2 dx \Big)^{\frac{1}{2}} \\ & + \exp(2\,\mathrm{Osc}(U)) \int_{T_1^d} |l - \nabla \chi_l|^2 m_U(dx) \Big] \end{aligned}$$

But

$$\Big(\int_{B(x_0,1)} \psi^2 dx\Big)^{\frac{1}{2}} \leq C\Big[\Big(\int_{B(x_0,1)} \phi^2 dx\Big)^{\frac{1}{2}} + \Big(\int_{B(x_0,1)} F_l^4 dx\Big)^{\frac{1}{2}}\Big]$$

and

$$\begin{aligned}\Big(\int_{B(x_0,1)} F_l^4 dx\Big)^{\frac{1}{2}} &\leq C\Big(\int_{B(x_0,1)} ((l.x)^4 + |\chi_l|^4) dx\Big)^{\frac{1}{2}} \\ &\leq C_d(l^2 + \|\chi_l\|_\infty^2) \\ &\leq C_d\, l^2 \exp\big((6d+4)\operatorname{Osc}(U)\big)\end{aligned}$$

where in the last inequality the theorem B.2.1 has been used Moreover, by periodicity

$$\int_{B(x_0,1)} \phi^2 dx \leq \int_{T_1^d} \phi^2 dx$$

and by the Poincaré inequality

$$\int_{T_1^d} \phi^2 dx \leq C_d \int_{T_1^d} |\nabla \phi|^2 dx$$

but

$$\begin{aligned}\frac{1}{2}\int_{T_1^d} |\nabla\phi|^2 m_U(dx) &= -\int_{T_1^d} \phi_l L_U \phi_l m_U(dx) \\ &= -\int_{T_1^d} \phi_l \big(|l - \nabla\chi_l|^2 - \int_{T_1^d} |l - \nabla\chi_l|^2 m_U(dy)\big) m_U(dx) \\ &\leq 2\|\phi_l\|_\infty \int_{T_1^d} |l - \nabla\chi_l|^2 m_U(dy)\end{aligned}$$

And since

$$\|\phi_l\|_\infty \leq \max_{x\in[0,1]^d} |F_l(x)|^2 + \max_{x_0\in[0,1]^d} |\psi_l(x)|$$

one obtains

$$\|\phi_l\|_\infty \leq C_d\, l^2 \exp\big((9d+4)\operatorname{Osc}(U)\big) + C_d|l| \exp\big(3d\operatorname{Osc}(U)\big)\|\phi_l\|_\infty^{\frac{1}{2}}$$

and since

$$\begin{cases} x^2 \leq a + bx \\ x \geq 0 \\ a, b > 0 \end{cases} \Rightarrow x \leq \frac{-b + \sqrt{b^2 + 4a}}{2} \Rightarrow x^2 \leq (b^2 + 4a)$$

One obtains

$$\|\phi_l\|_\infty \leq C_d \exp\big((9d+4)\operatorname{Osc}(U)\big)|l|^2$$

□

Then the bound B.41 results from an application of the lemma B.1.1 and an optimization on the choice of ρ.

B.2.2 Scaling Behavior

Let $U \in C^\infty(T_1^d)$, $l \in \mathbb{R}^d$
and $\chi_l \in \mathcal{C}^\infty(T_1^d)$ and $\phi_l \in \mathcal{C}^\infty(T_1^d)$ be the solutions of B.34 and B.39 For $R > 0$ write

$$V(x) = U(\frac{x}{R})$$

$$\chi_{l,R}(x) = R\chi_l(\frac{x}{R})$$

$$\phi_{l,R}(x) = R^2\phi_l(\frac{x}{R})$$

Then by a straightforward computation

Lemma B.2.1.

$$(\frac{1}{2}\Delta - \nabla V.\nabla)\chi_{l,R} = -l.\nabla V$$

$$(\frac{1}{2}\Delta - \nabla V.\nabla)\phi_{l,R} = |l - \nabla\chi_{l,R}|^2 - \int_{T_1^d} |l - \nabla\chi_{l,R}|^2 m_U(dx)$$

C. PROBABILISTIC TOOLS

C.1 Thermodynamics and mixing tools

The purpose of this section is to apply the thermodynamic formalism and the theory of level 3-large deviations to a particular case. For an introduction to the subject see [Rue78], [Ell85] and [Kel98]. The main result is the following theorem

Theorem C.1.1. *Let $V \in C^{\alpha}(T_1^d)$ (Hölder continuous with exponent $\alpha > 0$). Let $R \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{T_1^d} \exp\big(\sum_{k=0}^{n-1} V(R^k x)\big) dx = \mathcal{P}_R(V) \tag{C.1}$$

Where $\mathcal{P}_R$ is the pressure associated to the scaling shift induced by R on the torus, it will be studied and characterized below.
One of the useful property of this pressure, is given in the following theorem

Theorem C.1.2. *Let $V \in C(T_1^d)$, and $\mathcal{P}_R$ the pressure associated to the shift induced by $R \in \mathbb{N}$. Then*

$$\mathcal{P}(V) + \mathcal{P}(-V) \geq 0 \tag{C.2}$$

and

$$\mathcal{P}(V) + \mathcal{P}(-V) = 0 \Leftrightarrow \Big[V - \int_{T_1^d} V\Big] \in \mathcal{I}_{S_R}(T_1^d) \tag{C.3}$$

Where $\mathcal{I}_{S_R}(T_1^d)$ is the closed subspace of $\mathcal{C}(T_1^d)$ generated by the elements $U(x) - U(R^k x)$ with $U \in C(T_1^d)$ and $k \in \mathbb{N}$ (defined in subsection C.1.2).

C.1.1 Level-3 Large Deviation

Let $R \in \mathbb{N}/0, 1$ and define the shift operator s_R acting the torus T_1^d by

$$\begin{aligned} s_R : T_1^d &\longrightarrow T_1^d \\ x &\longrightarrow Rx \end{aligned} \tag{C.4}$$

To s_R is associated a scaling operator S_R acting on the periodic continuous functions on T_1^d

$$\begin{aligned} S_R : C(T_1^d) &\longrightarrow C(T_1^d) \\ \big(x \to f(x)\big) &\longrightarrow \big(x \to f(s_R x) = f(Rx)\big) \end{aligned} \tag{C.5}$$

Thus one can see the torus as a shift space equipped with the transformation s_R

$$\begin{aligned} s_R : T_1^d &\longrightarrow T_1^d \\ x = \sum_{k=1}^{\infty} \frac{x^k}{R^k} &\longrightarrow Rx = \sum_{k=1}^{\infty} \frac{x^{k+1}}{R^k} \end{aligned} \tag{C.6}$$

where for each k, x^k is a vector in $B = \{0, 1, \dots, R-1\}^d$ and for each $i \in \{1, \dots, d\}$ $\sum_{k=1}^{\infty} \frac{x_i^k}{R^k}$ is the expression of x_i in base R ($x_i^k \in \{0, \dots, R-1\}$)
Gift B with the discrete topology and $B^{\mathbb{N}^*}$ with the product topology. Write ν the probability measure on B affecting identical weight $1/R^d$ to each element of B and write $\mathbb{P}_\nu$ the associated product measure on $B^{\mathbb{N}^*}$.
Then, with respect to the probability space $\big(B^{\mathbb{N}^*}, \mathcal{B}(B^{\mathbb{N}^*}), \mathbb{P}_\nu\big)$ the coordinate representation process $x = (x^1, \dots, x^p, \dots)$ is a sequence of i.i.d. random variables distributed by ν. When x is seen as an element of the torus T_1^d then the probability measure induced by ν on the torus is the Lebesgue measure.
Now define the empirical measure E_n associated to the process x by

$$E_n(x, .) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{s_R^k \,\mathrm{cycle}(x,n)} \tag{C.7}$$

where cycle(x, n) is the periodic point in $B^{\mathbb{N}^*}$ obtained by repeating $(x^1, \dots, x^n)$ periodically. For each x, $E_n(x, .)$ is an element of the space $\mathcal{M}(B^{\mathbb{N}^*})$ of measures on $B^{\mathbb{N}^*}$ and invariant by the shift s_R.
Then $\{Q_n^{(3)}\}$, the $\mathbb{P}_\nu$ distribution on $\mathcal{M}(B^{\mathbb{N}^*})$ of the empirical process $\{E_n\}$ have a large deviation property (see for instance [Ell85], theorem 9.1.1) with speed n and entropy function $I_\nu^{(3)}$ such that for $P \in \mathcal{M}(B^{\mathbb{N}^*})$

$$I_\nu^{(3)} = \int_{B^{\mathbb{N}^*}} I_\nu^{(2)}(\tilde{P}) dP \tag{C.8}$$

where $\tilde{P}$ denotes the marginal distribution of x^1 associated to P and $I_\nu^{(2)}$ is the relative entropy of $\tilde{P}$ with respect to ν

$$I_\nu^{(2)}(\mu) = \int_B \ln \frac{d\mu}{d\nu}\, d\mu \tag{C.9}$$

Now chose $V \in C(T_1^d)$, Hölder continuous with exponent α. Since $\{Q_n^{(3)}\}$ have a large deviation property by Varadhan's theorem

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{T_1^d} \exp\big(nE_n(x, V)\big) dx = \mathcal{P}_R(V) \tag{C.10}$$

Where $\mathcal{P}_R(V)$ is the pressure of V and is given by the following variational formula:

$$\mathcal{P}_R(V) = \sup_{P \in \mathcal{M}_{s_R}(B^{\mathbb{N}^*})} \{ \int V\, dP - I_\nu^{(3)}(P) \} \tag{C.11}$$

where $\mathcal{M}_{s_R}(B^{\mathbb{N}^*})$ is the space of measures on $B^{\mathbb{N}^*}$ invariant by the shift s_R.
Now since V is Hölder continuous

$$\begin{aligned} |nE_n(x, V) - \sum_{k=0}^{n-1} V(R^k x)| &\leq \sum_{k=0}^{n-1} (\frac{C_d}{R^{n-k}})^\alpha \\ &\leq C(d, \alpha) \sum_{k=0}^{\infty} \frac{1}{R^{k\alpha}} \leq C(d, \alpha, R) < \infty \end{aligned} \tag{C.12}$$

It follows from C.10 that

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{T_1^d} \exp\big(\sum_{k=0}^{n-1} V(R^k x)\big) dx = \mathcal{P}_R(V) \tag{C.13}$$

C.1.2 A reminder on the Pressure

C.1.2.i Basic Properties

The following basic properties of the pressure can be found in [Kel98] theorem 4.1.10. (except the first one, note that the definition of the pressure given here differs from the standard one of the topological pressure by a constant which is $d \ln R$)

- $\mathcal{P}_R(0) = 0$
- $\mathcal{P}_R$ is a convex function on the space of upper semi continuous functions on the torus to $[-\infty, \infty)$
- $\mathcal{P}_R$ is isotonic: $U \leq V \Rightarrow \mathcal{P}_R(U) \leq \mathcal{P}_R(V)$
- For U and V upper semi-continuous $|\mathcal{P}_R(U+V) - \mathcal{P}_R(U)| \leq \|V\|_\infty$
- For U upper semi-continuous, V continuous on the torus and $k \in \mathbb{N}$

$$\mathcal{P}_R(U + S_R^k V - V) = \mathcal{P}_R(U) \tag{C.14}$$

and hence $\mathcal{P}_R(U + S_R^k V) = \mathcal{P}_R(U + V)$

C.1.2.ii Strict convexity of the pressure

Write $\mathcal{I}_{S_R}(T_1^d)$ the closed subspace of $\mathcal{C}(T_1^d)$ generated by the elements $V - S_R^k V$ with $V \in C(T_1^d)$ and $k \in \mathbb{N}$. Write $[U]$ the equivalence class of U, then by the proposition 4.7 of [Rue78] the function

$$\begin{aligned} \mathcal{P}_R : C(T_1^d)/\mathcal{I}_{S_R}(T_1^d) &\longrightarrow [-\infty, +\infty) \\ [U] &\longrightarrow \mathcal{P}_R(U) \end{aligned} \tag{C.15}$$

is well defined on the set of equivalence classes induced by $\mathcal{I}_{S_R}(T_1^d)$ on $C(T_1^d)$. Moreover it is strictly convex on the subset

$$\{[U] \in C(T_1^d)/\mathcal{I}_{S_R}(T_1^d) \ : \ \int_{T_1^d} U(x)dx = 0\} \tag{C.16}$$

In other words, for $U, V \in C(T_1^d)$

$$W \in \mathcal{I}_{S_R}(T_1^d) \Rightarrow \mathcal{P}_R(U + W) = \mathcal{P}_R(U) \tag{C.17}$$

And for $0 < t < 1$

$$\begin{aligned} t\mathcal{P}_R(U) + (1-t)\mathcal{P}_R(V) =& \mathcal{P}_R(tU + (1-t)V) \\ &\Leftrightarrow \left[U - V - \int_{T_1^d} (U - V)\right] \in \mathcal{I}_{S_R}(T_1^d) \end{aligned} \tag{C.18}$$

C.1.3 Gibbs measure

Let $U \in C^\alpha(T_1^d)$ ($\alpha > 0$) then U induces a regular local energy function and there exists a unique shift-invariant ergodic measure μ_U minimizing the variational formulation C.11 of the pressure $\mathcal{P}(U)$ (see corollary 5.3.2 of [Kel98]). This measure is called equilibrium state and is a Gibbs measure (see [Kel98] chapter 5).

C.1.4 A functional mixing property

In the subsection a technical functional mixing lemma will be introduced,it is quite simple but will proves efficient and useful when it will be combined to a cohomological framework to deduce sharp estimates on the effective diffusivities in a multi-scale medium.
First define the translation operator θ_y ($y \in \mathbb{R}^d$) acting the torus T_1^d by

$$\begin{aligned} \theta_y : T_1^d &\longrightarrow T_1^d \\ x &\longrightarrow x + y \end{aligned} \tag{C.19}$$

To θ_y is associated a translation operator Θ_y acting on the space of continuous periodic functions.

$$\begin{aligned} \Theta_y : C(T_1^d) &\longrightarrow C(T_1^d) \\ \big(x \to f(x)\big) &\longrightarrow \big(x \to f(\theta_y x) = f(x+y)\big) \end{aligned} \tag{C.20}$$

Then the following link the mixing property of the scaling operator S_R with the translation operator Θ

Lemma C.1.1. *Let* $(g, f) \in \big(C(T_1^d)\big)^2$ *and* $R \in \mathbb{N}^*$ *Then*

$$\begin{aligned} \int_{T_1^d} g(x) S_R f(x) dx = &\int_{T_1^d} g(x)dx \int_{T_1^d} f(x)dx \\ &+ \iint_{\left(T_1^d\right)^2} \Theta_y S_R f(z)\big(\Theta_{\frac{y}{R}} g(z) - g(z)\big)\big) dy\, dz \end{aligned} \tag{C.21}$$

Proof. By a straightforward computation

$$\begin{aligned} \iint_{\left(T_1^d\right)^2} & f(Rz+y)\big(g(z+\frac{y}{R}) - g(z))\big) dy\, dz \\ &= \int_{y \in T_1^d} dy \int_{z \in T_1^d} f(Rz+y) g(z+\frac{y}{R}) dz \\ &\quad - \int_{z \in [0,1]^d} g(z)dz \int_{y \in T_1^d} f(Rz+y)dy \\ &= \int_{T_1^d} g(x) f(Rx)dx - \int_{T_1^d} g(x)dx \int_{T_1^d} f(x)dx \end{aligned}$$

In the last equality the periodicity of the functions has been used. Thus

$$\begin{aligned} \int_{T_1^d} g(x) f(Rx) dx = &\int_{T_1^d} g(x)dx \int_{T_1^d} f(x)dx \\ &+ \iint_{\left(T_1^d\right)^2} f(Rz+y)\big(g(z+\frac{y}{R}) - g(z))\big) dy\, dz \end{aligned}$$

□

From this lemma, the following corollary directly follows.

Corollary C.1.1. *Let* $(g, f) \in \big(C^1(T_1^d)\big)^2$ *and* $R \in \mathbb{N}^*$ *Then for* $\alpha \in (0,1]$

$$\Big| \int_{T_1^d} g(x) S_R f(x) dx - \int_{T_1^d} g(x)dx \int_{T_1^d} f(x)dx \Big| \leq \frac{\|g\|_\alpha}{R^\alpha} \int_{T_1^d} |f| dx \tag{C.22}$$

with $\|g\|_\alpha = \sup_{x \neq y} |g(x) - g(y)|/|x-y|^\alpha$

Corollary C.1.2. *Let $g, f, V, T \in C^1(T_1^d$ and $R \in \mathbb{N}^*$ Then for $\alpha \in (0,1]$ and $U = S_R V + T$*

$$\begin{aligned} m_U(g(S_R f)) =& m_V(f) m_T(g) \\ &+ \frac{m_V(|f|)}{R^\alpha} e^{2\operatorname{Osc}(T)}(4\|g\|_\infty \|\nabla T\|_\infty + \|g\|_\alpha) \end{aligned} \tag{C.23}$$

Proof. Observe that

$$m_U(g(S_R f)) = m_V(f) m_T(g) \frac{\int_{T_1^d} e^{-2V} dx \int_{T_1^d} e^{-2T} dx}{\int_{T_1^d} e^{-2U} dx} + \frac{I_1}{R^\alpha}$$

with

$$I_1 \leq m_V(|f|) \|g e^{-2T}\|_\alpha \frac{\int_{T_1^d} e^{-2V} dx}{\int_{T_1^d} e^{-2U} dx}$$

but it follows from the corollary C.1.1 that

$$\begin{aligned} &\Big| \frac{\int_{T_1^d} e^{-2V} dx \int_{T_1^d} e^{-2T} dx}{\int_{T_1^d} e^{-2U} dx} - 1 \Big| \\ &\qquad \leq \frac{2}{R} \|\nabla T\|_\infty e^{2\operatorname{Osc}(T)} \end{aligned} \tag{C.24}$$

observe also that

$$\|g e^{-2T}\|_\alpha \leq e^{-2\inf T}(\|g\|_\alpha + 2\|\nabla T\|_\infty \|g\|_\infty)$$

it follows that

$$m_U(g(S_R f)) = m_V(f) m_T(g) + \frac{m_V(|f|)}{R^\alpha} e^{2\operatorname{Osc}(T)}(4\|g\|_\infty \|\nabla T\|_\infty + \|g\|_\alpha)$$

□

Corollary C.1.3. *Let $f, g \in C^\infty(T_1^1)$ and $\phi \in C^\infty(\mathbb{R})$ such that $f, g, \phi > 0$. Let $R \in \mathbb{N}^*$.*

$$\Big| \frac{\int_0^x g(y) f(Ry) \phi(y) dy}{\int_0^1 g(y) f(Ry) dy} - \frac{\int_0^x g(y) \phi(y) dy}{\int_0^1 g(y) dy \int_0^1 f(y) dy} \Big| \leq \frac{I}{R} \tag{C.25}$$

with

$$\begin{aligned} I =& \frac{\int_0^x g(y)\phi(y) dy}{\int_0^1 g(y) dy} 2 \frac{\|\nabla g\|_\infty}{\inf g} + \frac{2}{R} (\sup_{y \in [x-1/R, x]} \phi(y)) \frac{\sup \phi}{\inf \phi} \\ &+ 2 \frac{\|\nabla g\|_\infty}{\inf g} \sum_{k=0}^{[Rx]-1} \sup_{y \in [0,1]} \phi(\frac{k}{R} + \frac{y}{R}) + \|\nabla g\|_\infty \sum_{k=0}^{[Rx]-1} \sup_{y \in [0,1]} |\nabla \phi(\frac{k}{R} + \frac{y}{R})| \end{aligned}$$

Proof. Let $x \in \mathbb{R}$, observe that

$$\int_0^x g(y) f(Ry) \phi(y) dy = \int_0^{\frac{[xR]}{R}} g(y) f(Ry) \phi(y) dy + \int_{\frac{[xR]}{R}}^x g(y) f(Ry) \phi(y) dy$$

But

$$\int_0^{\frac{[xR]}{R}} g(y) f(Ry) \phi(y) dy = \frac{[xR]}{R} \int_0^1 f(Rz) g(z) \phi(z) dz$$

Note also that, by the corollary C.1.1

$$\left|\frac{1}{\int_0^1 g(y)f(Ry)dy}-\frac{1}{\int_0^1 g(y)dy\int_0^1 f(y)dy}\right|\leq\frac{1}{R}\frac{2\|\nabla g\|_\infty}{\int_0^1 g(y)f(Ry)dy\int_0^1 g(y)dy}$$

It follows by the corollary C.1.1 after some straightforward computation that

$$\left|\frac{\int_0^x g(y)f(Ry)\phi(y)dy}{\int_0^1 g(y)f(Ry)dy}-\frac{\int_0^x g(y)\phi(y)dy}{\int_0^1 g(y)dy\int_0^1 f(y)dy}\right|\leq\frac{I}{R}$$

with

$$\begin{aligned}I=&\frac{\int_0^x g(y)\phi(y)dy}{\int_0^1 g(y)dy}2\frac{\|\nabla g\|_\infty}{\inf g}+\frac{2}{R}(\sup_{y\in[x-1/R,x]}\phi(y))\frac{\sup\phi}{\inf\phi}\\&+2\frac{\|\nabla g\|_\infty}{\inf g}\sum_{k=0}^{[Rx]-1}\sup_{y\in[0,1]}\phi(\frac{k}{R}+\frac{y}{R})+\|\nabla g\|_\infty\sum_{k=0}^{[Rx]-1}\sup_{y\in[0,1]}|\nabla\phi(\frac{k}{R}+\frac{y}{R})|\end{aligned}$$

□

Corollary C.1.4. *Let $f,g\in C^\infty(T_1^1)$. Let $R\in\mathbb{N}^*$.*

$$\left|\int_0^x g(y)f(Ry)dy-\int_0^x g(y)dy\int_0^1 f(y)dy\right|\leq\frac{I}{R}\tag{C.26}$$

with

$$I=\|f\|_\infty(2\|g\|_\infty+|x|\|\nabla g\|_\infty)$$

Proof. Let $x\in\mathbb{R}$, observe that

$$\int_0^x g(y)f(Ry)dy=\int_0^{\frac{[xR]}{R}}g(y)f(Ry)dy+\int_{\frac{[xR]}{R}}^x g(y)f(Ry)dy$$

But

$$\int_0^{\frac{[xR]}{R}}g(y)f(Ry)dy=\frac{[xR]}{R}\int_0^1 f(Rz)g(z)dz$$

Then the result follows by the corollary C.1.1 after some straightforward computation. □

C.2 Smooth pre-fractal measure

In the section the notion of smooth pre-fractal measure will be introduced and analyzed (this name is given according the name "Sierpinski pre-Carpet" introduced by H. Osada). It will constitute the medium on which the sub-diffusions will take place, this medium will also be called a smooth periodic pre-fractal.

C.2.1 Definitions

Definition C.2.1. A smooth pre-fractal measure is a collection $\{(r_n,U_n)_{n\in\mathbb{N}}\}$ where for each n, $r_n\in\mathbb{N}/\{0,1\}$ and $U_n\in C^\infty(T_1^d)$ such that $U_n(0)=0$ and

$$K_1=\sup_{n\in\mathbb{N}}\|\nabla U_n\|_\infty<\infty\tag{C.27}$$

Which implies that $K_0=\sup_n \operatorname{Osc}(U_n)<\infty$

Write $\rho_{min} = \min_{n\geq 1} r_n$ and $\rho_{max} = \sup r_n \leq \infty$.
A smooth pre-fractal measure will be written $SPFM(\rho_{min}, \rho_{max}, K_1)$.
It induces a potential of order $-p$ (where $p \in \mathbb{N}$)

$$U^{-p}(x) = \sum_{n=0}^{\infty} U_n(\frac{R_p x}{R_n}) \tag{C.28}$$

(where $R_n = \prod_{k=0}^{n} r_k$) which is a well defined function in $C^\infty(\mathbb{R}^d)$.

- A $SPFM$ will be said "with bounded ratio" if $\rho_{max} < \infty$.
- A $SPFM$ will be said self-similar if for all n, $r_n = \rho$ and $U_n = U_0 \in \mathcal{C}^\infty(T_1^d)$.

C.2.2 Properties of the measures induced by a SPFM

C.2.2.i Mixing properties

It is natural to associate the potential of order $-p$ induced by a SPFM with the measure on $\mathbb{R}^d$:

$$m_{U^{-p}} = \frac{e^{-2U^{-p}(x)}dx}{\int_{T_1^d} e^{-2U^{-p,0}(x)}dx} \tag{C.29}$$

where for $k \in \mathbb{Z}$, $k \geq -p$

$$U^{-p,k} = \sum_{n=0}^{k+p} U_n(\frac{R_p x}{R_n}) \tag{C.30}$$

Now by the Corollary C.1.1 and by a simple induction one obtains easily the following proposition.

Proposition C.2.1.

$$\int_{T_1^d} e^{-2U^{-p,0}(x)}dx \leq \int_{T_1^d} e^{-2U_0(x)}dx \prod_{k=1}^{p} [\int_{T_1^d} e^{-2U_k(x)}dx(1 + \frac{2K_1 e^{2K_0}}{r_k})] \tag{C.31}$$

and for $\rho_{min} > 2K_1 e^{2K_0}$

$$\int_{T_1^d} e^{-2U^{-p,0}(x)}dx \geq \int_{T_1^d} e^{-2U_0(x)}dx \prod_{k=1}^{p} [\int_{T_1^d} e^{-2U_k(x)}dx(1 - \frac{2K_1 e^{2K_0}}{r_k})] \tag{C.32}$$

C.2.2.ii Growth rate

It is then interesting to investigate on the growth rate of this measure. More precisely

Definition C.2.2. The Growth rate at infinity of a measure μ on $\mathbb{R}^d$ is the segment $[d_{f,\min}^\infty(\mu), d_{f,\max}^\infty(\mu)]$ where

$$d_{f,\min}^\infty(\mu) = \lim\inf_{r\to\infty} \frac{\mu(B(0,r))}{\ln r} \tag{C.33}$$

$$d_{f,\max}^\infty(\mu) = \lim\sup_{r\to\infty} \frac{\mu(B(0,r))}{\ln r} \tag{C.34}$$

Now it is easy to see that the growth rate at infinity of $m_{U^{-p}}$ is independent of the order $-p$ thus it will be sufficient to investigate the growth rate at infinity of m_{U^0}. Now by a straightforward computation for $r \geq 10$

$$\frac{\ln m_U^0(B(0,r))}{\ln r} = d + \frac{\ln \int_{T_1^d} \exp(-2U^{-n(r),0}(x))dx}{\ln r} + \frac{C(d, K_1, \rho_{min})}{\ln r} \tag{C.35}$$

where

$$n(r) = \sup\{n \in \mathbb{N} \,:\, R_n \leq r\} \tag{C.36}$$

It follows that

$$d_{f,\min}^{\infty}(m_U^0) = d + \lim\inf_{r\to\infty} \frac{\ln \int_{T_1^d} \exp(-2U^{-n(r),0}(x))dx}{\rho(r)n(r)} \tag{C.37}$$

$$d_{f,\min}^{\infty}(m_U^0) = d + \lim\sup_{r\to\infty} \frac{\ln \int_{T_1^d} \exp(-2U^{-n(r),0}(x))dx}{\rho(r)n(r)} \tag{C.38}$$

where

$$\rho(r) = \frac{\ln r}{n(r)} \tag{C.39}$$

is the geometric mean ratio between scales at the length r.
It follows immediately that $\rho(r) \to \infty$ imply $d_{f,\max}^{\infty}(m_U^0) = d_{f,\min}^{\infty}(m_U^0) = d$.
Moreover by the proposition C.2.1,

$$d_{f,\min}^{\infty}(m_U^0) \geq d + \lim\inf_{r\to\infty} \frac{\sum_{k=1}^{n(r)} \ln \left[\int_{T_1^d} e^{-2U_k(x)} dx \right]}{n(r)\rho(r)} + \frac{\ln(1 - \frac{2K_1 e^{2K_0}}{\rho_{min}})}{\rho(r)} \tag{C.40}$$

and if $\rho_{\min} > 2K_1 e^{2K_0}$

$$d_{f,\max}^{\infty}(m_U^0) \geq d + \lim\inf_{r\to\infty} \frac{\sum_{k=1}^{n(r)} \ln \left[\int_{T_1^d} e^{-2U_k(x)} dx \right]}{n(r)\rho(r)} + \frac{\ln(1 + \frac{2K_1 e^{2K_0}}{\rho_{min}})}{\rho(r)} \tag{C.41}$$

Growth rate at infinity of the measure associated to a self-similar SPFM From the equations C.37 and C.38 and the theorem C.1.1 it follows that for a self-similar SPFM.

$$d_{f,\max}^{\infty}(m_U^0) = d_{f,\min}^{\infty}(m_U^0) = d_f^{\infty}(m_U^0) = d + \frac{\mathcal{P}(-2U_0)}{\rho} \tag{C.42}$$

Note that this definition of growth rate at infinity dimension is not invariant under a translation of U_0 (indeed under a translation by Θ_y, U_0 should be modified to $x \to U_0(x+y) - U_0(y)$ so that U^0 is well defined). It is interesting to note that the value of $d_f^{\infty}(m_U^0)$ is fixed by the necessity of e^{-2U^0} to be a well defined density measure but it can be greater than the dimension of the space.
Thus $d_f^{\infty}(m_U^0)$ is not translation invariant and one can have $d_f^{\infty}(m_U^0) > d$ (be careful if you try to link it with a sort of Hausdorff fractal dimension of the pre-fractal).

C.2.3 Growth rate at 0

One might think, well this definition of d_f that gives back a value that can be greater than d is unsatisfactory, and may be by looking at the growth rate at 0 of the torus one might obtain a better characterization, this is the object of this subsection.
The natural way to define a growth rate at 0 is to consider the measure $m_{U^{-p,0}}$ on the torus T_1^d, observe that this measures are invariant if one add to each U_n a different constant c_n, then define the growth rate at 0 at the point x by the segment $[d^0_{f,x,min}, d^0_{f,x,max}]$ by for $0 < \alpha < 1$

$$d^0_{f,x} = \lim_{p\to\infty} \frac{-\ln\Big(m_{U^{-p,0}}\big(B(x, \frac{1}{R_{[p\alpha]}})\big)\Big)}{\ln R_{[p\alpha]}} \tag{C.43}$$

Proposition C.2.2. *$d^0_{f,x}$ does not depend on $0 < \alpha < 1$ and*

$$d^0_{f,x,\min} = d + \lim\inf_{r\to\infty} \frac{\ln \int_{T_1^d} e^{-2\left(U^{-n(r),0}(y) - U^{-n(r),0}(x)\right)} dy}{\rho(r)n(r)} \tag{C.44}$$

$$d^0_{f,x,\max} = d + \lim\sup_{r\to\infty} \frac{\ln \int_{T_1^d} e^{-2\left(U^{-n(r),0}(y) - U^{-n(r),0}(x)\right)} dy}{\rho(r)n(r)} \tag{C.45}$$

Proof. Easy to check. □

This proposition says that the growth rate at 0 at the point 0 is the same that the growth rate at infinity at the point 0, moreover it depends on the point x and $d^0_{f,x,\max}$ can be greater than d. Thus the growth rate at 0 does suffer from the same pathology.

C.2.4 From a SPFM to a fractal measure

The purpose of this subsection is to investigate on the following problem: how to build a fractal measure on the torus from a given smooth pre fractal measure.

C.2.4.i Completion of a self similar SPFM

If the SPFM is self similar the problem is easy and there are basically two ways: The first one is to consider the sequence $(m_{U^{-p,0}})_{p\in\mathbb{N}}$ of probability measures on the torus T_1^d, where

$$m_{U^{-p,0}} = \frac{e^{-2U^{-p,0}(x)}dx}{\int_{T_1^d} e^{-2U^{-p,0}(x)}dx} \tag{C.46}$$

Since the torus is compact this sequence of measures in tight and one can extract a subsequence converging to a measure on the torus and call fractal measure the limit.
The second one is to consider the sequence $(m_{U^{-p,+\infty}})_{p\in\mathbb{N}}$ of probability measures on the cube $\mathbb{R}^d$, where

$$m_{U^{-p,+\infty}} = \frac{e^{-2U^{-p,+\infty}(x)}dx}{\int_{T_1^d} e^{-2U^{-p,0}(x)}dx} \tag{C.47}$$

This sequence of measures in tight and one can extract a subsequence converging to a measure on the each compact subset of $\mathbb{R}^d$ and call fractal measure the limit.
Unicity problems will be studied in a sequel work.

C.2.4.ii Completion of a non self similar SPFM

In this case the problem is more serious because it requires an a priori choice. Indeed the first way would be to consider the sequence $m_{U^{-p,0}}$ on the torus, this sequence is tight and one can call fractal measure the limits of converging subsequences. It is easy to see that with this method the limits are not unique because the scale of order 0 is always changing. The same pathology happens if on consider the sequence of probability measures $m_{U^{-p,\infty}}$ on the unit cube $[0,1]^d$
The alternative way to avoid this pathology would be to complete the non self similar SPFM by smaller scales $(U_{-k})_{k\in\mathbb{N}^*}$ $(U_{-k}\in C^\infty(T_1^d))$ and $(r_{-k})_{k\in\mathbb{N}^*}$, $(r_{-k})_{k\in\mathbb{N}^*}$. Then write

$$\frac{1}{R_{-k}} = r_{-1}\dots r_{-k} \tag{C.48}$$

and

$$V^{-m,p} = \sum_{k=-m}^{p} U^k(\frac{x}{R_k}) \tag{C.49}$$

then consider the measure

$$m_{V^{-p,0}} = \frac{e^{-2V^{-p,0}(x)}dx}{\int_{T_1^d} e^{-2V^{-p,0}(x)}dx} \tag{C.50}$$

on the torus T_1^d or the measure

$$m_{V^{-p,+\infty}} = \frac{e^{-2V^{-p,+\infty}(x)}dx}{\int_{T_1^d} e^{-2V^{-p,0}(x)}dx} \tag{C.51}$$

on R^d. With theses choices the obstacle of order $0,-1,\dots,-k$ does not change for $p\geq k$, as usual one can extract subsequences and call fractal measure the limit measure. The unicity problem is postponed to a sequel work.

C.3 Control induced by the linearity of harmonic functions

C.3.1 Quasi-harmonic functions

C.3.1.i Perturbation of the mean squared displacement

Let $U,T\in C^\infty(\mathbb{R}^d)$ be smooth potentials with bounded gradient. Write $V=U+T$ and y_t the diffusion associated to the generator L_V.

Lemma C.3.1. *Assume that F_U is smooth and harmonic with respect to L_U and ∇F_U is bounded. Then*

$$\mathbb{E}[F_U^2(y_t)] \leq 2\mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds]e^{t\|\nabla T\|_\infty^2} \tag{C.52}$$

Proof. Observe that $L_V F_U = -\nabla T\nabla F_U$. It follows by the Ito formula that

$$\begin{aligned}\mathbb{E}[F_U^2(y_t)] &= \mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds] - 2\mathbb{E}[\int_0^t (F_U\nabla T\nabla F_U)(y_s)ds]\\ &\leq \mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds] + 2\|\nabla T\|_\infty \Big[\int_0^t \mathbb{E}[F_U^2(y_s)]ds\Big]^{\frac{1}{2}}\Big[\int_0^t\\ &\mathbb{E}[|\nabla F_U(y_s)|^2]ds\Big]^{\frac{1}{2}}\\ &\leq 2\mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds] + \|\nabla T\|_\infty^2 \mathbb{E}[\int_0^t F_U^2(y_s)ds]\end{aligned}$$

It follows by Gronwall lemma that

$$\mathbb{E}[F_U^2(y_t)] \leq 2\mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds] e^{t\|\nabla T\|_\infty^2}$$

□

Lemma C.3.2. *Under the assumptions of the lemma C.3.1, for $t > 0$*

$$\mathbb{E}[F_U^2(y_t)] \geq (\frac{1}{2} - t\|\nabla T\|_\infty^2)\mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds] \tag{C.53}$$

Proof. Observe that

$$F_U(y_t) = \int_0^t \nabla F_U(y_s) d\omega_s - \int_0^t \nabla T \nabla F_U(y_s) ds$$

It follows that

$$\begin{aligned}\mathbb{E}[F_U^2(y_t)] &\geq \frac{1}{2}\mathbb{E}[\big(\int_0^t \nabla F_U(y_s) d\omega_s\big)^2] - t\|\nabla T\|_\infty^2 \mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds] \\ &\geq (\frac{1}{2} - t\|\nabla T\|_\infty^2)\mathbb{E}[\int_0^t |\nabla F_U(y_s)|^2 ds]\end{aligned}$$

Which finishes the proof □

C.3.1.ii Control induced by the linearity of harmonic functions and the quadradicity of ergodic functions

Let $W \in C^\infty(T^d_{R_W})$ of period $R_W \in \mathbb{N}/0,1$. Let $T \in C^\infty(\mathbb{R}^d)$ with bounded gradient and write $V = W + T$ and y_t the potential diffusion associated to L_V. Assume that for $l \in \mathbb{S}^d$ there exists $\chi_l^U, \chi_l^P \in C^\infty(T^d_{R_W})$,
$C_1^\chi, C_2^\chi, C^U, C_1^\phi, C_2^\phi, \zeta_1, \zeta_2, R_P > 0$ such that

$$\chi_l^W = \chi_l^U + \chi_l^P \tag{C.54}$$

and

$$C_1^\chi x^2 \leq \sum_{i=1}^d (x_i - \chi_{e_i}^P)^2 \leq C_2^\chi x^2 \tag{C.55}$$

$$\|\chi_l^U\|_\infty \leq \frac{R_W}{R_P} C^U \tag{C.56}$$

and for all $t > 0$

$$\sum_{i=1}^d \mathbb{E}[\int_0^t |\nabla F_{e_i}^W|^2(y_s) ds] \leq \zeta_2 \lambda_{\max}\big(D(W)\big)t + \frac{R_W^2}{R_P^2} C_2^\phi \tag{C.57}$$

$$\sum_{i=1}^d \mathbb{E}[\int_0^t |\nabla F_{e_i}^W|^2(y_s) ds] \geq \zeta_1 \lambda_{\max}\big(D(W)\big)t - \frac{R_W^2}{R_P^2} C_1^\phi \tag{C.58}$$

then

Lemma C.3.3. *Under the assumptions C.54, C.55, C.56, C.57 and C.58 for* $t > 0$

$$\mathbb{E}[y_t^2] \leq \frac{2d}{C_1^\chi}\big(\frac{R_W}{R_P}C^U\big)^2 + \frac{4}{C_1^\chi}e^{t\|\nabla T\|_\infty^2}\Big(\zeta_2\lambda_{\max}\big(D(W)\big)t + \frac{R_W^2}{R_P^2}C_2^\phi\Big) \tag{C.59}$$

and for $2t\|\nabla T\|_\infty^2 < 1$

$$\mathbb{E}[y_t^2] \geq \frac{1}{2C_2^\chi}\big(\frac{1}{2} - t\|\nabla T\|_\infty^2\big)\Big(\zeta_1\lambda_{\max}\big(D(W)\big)t - \frac{R_W^2}{R_P^2}C_1^\phi\Big) - \frac{d}{C_2^\chi}\big(\frac{R_W}{R_P}C^U\big)^2 \tag{C.60}$$

Proof. Upper bound: by the lemma C.3.1, for $t > 0$

$$\sum_{i=1}^d \mathbb{E}[|F_{e_i}^W|^2(y_t)] \leq 2\Big(\zeta_2\lambda_{\max}\big(D(W)\big)t + \frac{R_W^2}{R_P^2}C_2^\phi\Big)e^{t\|\nabla T\|_\infty^2} \tag{C.61}$$

Now by the assumptions C.54, C.55 and C.56:

$$\begin{aligned}
\sum_{i=1}^d \mathbb{E}[|F_{e_i}^W|^2] &= \sum_{i=1}^d \Big(x_i - \chi_{e_i}^P - \chi_{e_i}^U\Big)^2 \\
&\geq \sum_{i=1}^d \Big(\frac{1}{2}\big(x_i - \chi_{e_i}^P\big)^2 - \|\chi^U\|_{e_i}^2\Big) \\
&\geq \frac{C_1^\chi}{2}x^2 - d\big(\frac{R_W}{R_P}C^U\big)^2
\end{aligned}$$

Thus

$$\mathbb{E}[y_t^2] \leq \frac{2d}{C_1^\chi}\big(\frac{R_W}{R_P}C^U\big)^2 + \frac{4}{C_1^\chi}e^{t\|\nabla T\|_\infty^2}\Big(\zeta_2\lambda_{\max}\big(D(W)\big)t + \frac{R_W^2}{R_P^2}C_2^\phi\Big) \tag{C.62}$$

Lower bound: by the lemma C.3.2

$$\sum_{i=1}^d \mathbb{E}[|F_{e_i}^W|^2(y_t)] \geq \big(\frac{1}{2} - t\|\nabla T\|_\infty^2\big)\Big(\zeta_1\lambda_{\max}\big(D(W)\big)t - \frac{R_W^2}{R_P^2}C_1^\phi\Big) \tag{C.63}$$

Now by the assumptions C.54, C.55 and C.56:

$$\begin{aligned}
\sum_{i=1}^d \mathbb{E}[|F_{e_i}^W|^2] &= \sum_{i=1}^d \Big(x_i - \chi_{e_i}^P - \chi_{e_i}^U\Big)^2 \\
&\leq \sum_{i=1}^d 2\Big(\big(x_i - \chi_{e_i}^P\big)^2 + \|\chi^U\|_{e_i}^2\Big) \\
&\leq 2C_2^\chi x^2 + 2d\big(\frac{R_W}{R_P}C^U\big)^2
\end{aligned}$$

It follows that

$$\mathbb{E}[y_t^2] \geq \frac{1}{2C_2^\chi}\big(\frac{1}{2} - t\|\nabla T\|_\infty^2\big)\Big(\zeta_1\lambda_{\max}\big(D(W)\big)t - \frac{R_W^2}{R_P^2}C_1^\phi\Big) - \frac{d}{C_2^\chi}\big(\frac{R_W}{R_P}C^U\big)^2 \tag{C.64}$$

□

C.4 Control induced on the upper bound of the transition probability densities

Lemma C.4.1. *Let X be a positive bounded random variable, $\mu > 0$ and $\lambda > 0$ Then*

$$\mathbb{E}[\exp(\mu\sqrt{X})] \leq 1 + \mu \exp(\frac{\mu^2}{4\lambda})\sqrt{\frac{\pi}{\lambda}}\mathbb{E}[\exp(\lambda X)]$$

Proof. Observe that

$$\begin{aligned}\mathbb{E}[\exp(\mu\sqrt{X})] &= \int_0^\infty \exp(\mu\sqrt{x})d\mathbb{P}(X \geq x) \\ &= [\exp(\mu x)\mathbb{P}(X \geq x)]_0^{+\infty} + \int_0^{+\infty} \mu\frac{\exp(\mu\sqrt{x})}{2\sqrt{x}}\mathbb{P}(X \geq x)dx \\ &\leq 1 + \int_0^{+\infty} \mu\frac{\exp(\mu\sqrt{x})}{2\sqrt{x}}\mathbb{E}[\exp(\lambda(X - x))]dx \\ &\leq 1 + \mathbb{E}[\exp(\lambda X)]\int_0^{+\infty} \mu \exp(\mu y - \lambda y^2)dy\end{aligned}$$

But

$$\exp(\mu y - \lambda y^2) = \exp\big(-(\sqrt{\lambda}y - \frac{\mu}{2\sqrt{\lambda}})^2\big)\exp(\frac{\mu^2}{4\lambda})$$

So

$$\begin{aligned}\int_0^{+\infty} \exp(\mu y - \lambda y^2) &\leq \int_{-\infty}^{+\infty} \exp(-\lambda y^2) \\ &\leq \exp(\frac{\mu^2}{4\lambda})\sqrt{\frac{\pi}{\lambda}}\end{aligned}$$

Which proves the lemma □

C.4.1 A General Lemma

Let $W \in C^\infty(T^d_{R_W})$ of period $R_W \in \mathbb{N}/0,1$. Let $T \in C^\infty(\mathbb{R}^d)$ with bounded gradient and write $V = W + T$ and y_t the potential diffusion associated to L_V. Assume that there exists $C_2^\phi, \zeta_2, R_P > 0, C^\chi$ such that for $l \in \mathbb{S}^d$ and for all $t > 0$ and all $x \in \mathbb{R}^d$

$$\mathbb{E}_x[\int_0^t |\nabla F_l^W|^2(y_s)ds] \leq \zeta_2{}^t lD(W)lt + \frac{R_W^2}{R_P^2}C_2^\phi \tag{C.65}$$

$$\|\chi_l^W\|_\infty \leq C^\chi R_W \tag{C.66}$$

Lemma C.4.2. *Assume that*

$$C^\chi R_W \leq h/2 \tag{C.67}$$

$$\|\nabla T\|_\infty 2^3\big(\zeta_2^t lD(W)l\big)^{\frac{1}{2}} \leq \frac{h}{t} \leq \frac{R_P}{R_W\sqrt{C_2^\phi}}\zeta_2{}^t lD(W)l \tag{C.68}$$

and

$$\frac{R_P}{R_W\sqrt{C_2^\phi}}\zeta_2{}^t lD(W)l e^{-\frac{h^2}{2^{11}\zeta_2{}^t lD(W)lt}} \leq \frac{h}{t} \tag{C.69}$$

then

$$\mathbb{P}[l.y_t \geq h] \leq Ce^{-\frac{h^2}{2^9\zeta_2{}^t lD(W)lt}} \tag{C.70}$$

Proof. Let $l \in S^d$ and $\lambda > 0$, then for $h > 0$

$$\mathbb{P}[l.y_t \geq h] \leq \mathbb{E}[e^{\lambda(l.y_t - h)}]$$

Observe that

$$l.y_t = \chi_l^W(y_t) + \int_0^t \nabla F_l^W(y_s)d\omega_s - \int_0^t \nabla T.\nabla F_l^W(y_s)ds$$

It follows by the Cauchy Schwartz inequality that

$$\mathbb{P}[l.y_t \geq h] \leq e^{\lambda(C^\chi R_W - h)}\mathbb{E}[e^{2\lambda\int_0^t \nabla F_l^W(y_s)d\omega_s}]^{\frac{1}{2}}\mathbb{E}[e^{2\sqrt{t}\|\nabla T\|_\infty\lambda\left(\int_0^t |\nabla F_l^W(y_s)|^2 ds\right)^{\frac{1}{2}}}]^{\frac{1}{2}}$$

Now by the lemma C.4.1 with $X = \int_0^t |\nabla F_l^W(y_s)|^2 ds$, $\lambda' = 8\lambda^2$ and $\mu' = 2\lambda\sqrt{t}\|\nabla T\|_\infty$ it follows that

$$\mathbb{E}[e^{2\sqrt{t}\|\nabla T\|_\infty\lambda\left(\int_0^t |\nabla F_l^W(y_s)|^2 ds\right)^{\frac{1}{2}}}] \leq 1 + \|\nabla T\|_\infty\sqrt{\frac{t\pi}{2}}e^{\|\nabla T\|_\infty^2 t/8}\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F_l^W(y_s)|^2 ds}]$$

Observe also that as in the proof of the lemma 12.1.1

$$\mathbb{E}[e^{2\lambda\int_0^t \nabla F_l^W(y_s)d\omega_s}] \leq \mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F_l^W(y_s)|^2 ds}]^{\frac{1}{2}}$$

It follows that

$$\mathbb{P}[l.y_t \geq h] \leq Ce^{\lambda(C^\chi R_W - h)}e^{\|\nabla T\|_\infty^2 t/4}\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F_l^W(y_s)|^2 ds}]$$

Now observe that $\int_0^t \nabla F_l^W(y_s)d\omega_s$ satisfies the conditions of the theorem 12.1.1 with $f_2 = \zeta_2{}^t lD(W)l$, and $t_0(f_1 - f_2) = \frac{R_W^2}{R_P^2}C_2^\phi$. It follows that for

$$8\lambda^2 \leq \frac{R_P^2}{2eR_W^2 C_2^\phi} \tag{C.71}$$

$$\mathbb{E}[e^{8\lambda^2\int_0^t |\nabla F_l^W(y_s)|^2 ds}] \leq CR_P^4\frac{e^{8\lambda^2\zeta_2{}^t lD(W)lt}}{\lambda^4(C_2^\phi)^2 R_W^4}$$

and

$$\mathbb{P}[l.y_t \geq h] \leq Ce^{\lambda(C^\chi R_W - h)}e^{\|\nabla T\|_\infty^2 t/4}R_P^4\frac{e^{8\lambda^2\zeta_2{}^t lD(W)lt}}{\lambda^4(C_2^\phi)^2 R_W^4}$$

assume $C^\chi R_W < h/2$ and choose

$$\lambda = \frac{h}{32\zeta_2{}^t lD(W)lt}$$

for

$$h \leq \frac{R_P}{R_W\sqrt{C_2^\phi}}\zeta_2{}^t lD(W)lt \tag{C.72}$$

the condition C.71 is satisfied and it follows that

$$\mathbb{P}[l.y_t \geq h] \leq Ce^{-\frac{h^2}{2^7\zeta_2{}^t lD(W)lt}}e^{\|\nabla T\|_\infty^2 t/4}\frac{R_P^4(\zeta_2{}^t lD(W)lt)^4}{h^4(C_2^\phi)^2 R_W^4}$$

assume now that

$$\|\nabla T\|_\infty^2 t \leq \frac{h^2}{2^6 \zeta_2{}^t l D(W) l t} \tag{C.73}$$

it follows that

$$\mathbb{P}[l.y_t \geq h] \leq C e^{-\frac{h^2}{2^8 \zeta_2{}^t l D(W) l t}} \frac{R_P^4 (\zeta_2{}^t l D(W) l t)^4}{h^4 (C_2^\phi)^2 R_W^4}$$

and under the condition

$$\frac{R_P^2 (\zeta_2{}^t l D(W) l t)}{R_W^2 C_2^\phi} \leq \frac{h^2}{\zeta_2{}^t l D(W) l t} e^{\frac{h^2}{2^{10} \zeta_2{}^t l D(W) l t}}$$

which is implied by

$$\frac{R_P}{R_W \sqrt{C_2^\phi}} \zeta_2{}^t l D(W) l e^{-\frac{h^2}{2^{11} \zeta_2{}^t l D(W) l t}} \leq \frac{h}{t} \tag{C.74}$$

it follows that

$$\mathbb{P}[l.y_t \geq h] \leq C e^{-\frac{h^2}{2^9 \zeta_2{}^t l D(W) l t}}$$

□

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