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Rigorous uncertainty quantification without integral testing

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ABSTRACT

We describe a rigorous approach for certifying the safe operation of complex systems that bypasses the need for integral testing. We specifically consider systems that have a modular structure. These systems are composed of subsystems, or components, that interact through unidirectional interfaces. We show that, for systems that have the structure of an acyclic graph, it is possible to obtain rigorous upper bounds on the probability of failure of the entire system from an uncertainty analysis of the individual components and their interfaces and without the need for integral testing. Certification is then achieved if the probability of failure upper bound is below an acceptable failure tolerance. We demonstrate the approach by means of an example concerned with the performance of a fractal electric circuit.

1. Introduction

This paper describes a rigorous approach for certifying the safe operation of complex systems that bypasses the need for integral testing. Certification, as understood here, is a process that seeks to establish whether the probability of failure of a system is below an acceptable tolerance. Such a determination is particularly critical for systems whose failure may have severe consequences, be them economic, loss of life, or others. Within this framework, certification may be regarded as a tool for making high-consequence decisions regarding the deployment of high-value assets and systems (for background on the question of certification from a national security perspective see, e.g., [26,4,23,3,7]). In the present work, we specifically consider systems that have a modular structure, i.e., are composed of subsystems, or *components*, that interact with each other through *unidirectional interfaces*. We show that, for certain classes of systems having the structure of an acyclic graph, it is possible to obtain rigorous upper bounds on the probability of failure (PoF) of the entire system from an uncertainty analysis of the individual components and their interfaces and without the need for integral testing. Certification is then achieved if the PoF upper bound is below an acceptable failure tolerance.

Following [13], we specifically consider PoF upper bounds of the *concentration of measure* type [2,18,14,12]. In their simplest version, such bounds pertain to a system characterized by N uncorrelated real random inputs $X=(X_1,\dots,X_N)\in E\subseteq\mathbb{R}^N$ and a single real performance measure $Y\in\mathbb{R}$. Suppose that the function $F:\mathbb{R}^N\to\mathbb{R}$ describes the response function of the system. Suppose that the system fails when $Y\le a$, where a is a threshold for the safe operation of the system. Then, a direct application of McDiarmid's inequality [18,6] gives the following upper bound on the PoF of the system:

$$\mathbb{P}[F\le a]\le\exp\left(-2\frac{M^2}{U^2}\right),\tag{1}$$

where

$$M=(\mathbb{E}[F]-a)_+\tag{2}$$

is the *design margin* and

$$U=D_F=\left\{\sum_{i=1}^{N}\sup_{x_1,\dots,x_N,x_i'}|F(x_1,\dots,x_i,\dots,x_N)-F(x_1,\dots,x_i',\dots,x_N)|^2\right\}^{1/2}\tag{3}$$

is the *system uncertainty*. In this latter expression, the supremum is taken over the entire range of variation of the inputs. From (1) it follows that the system is certified if:

$$\exp\left(-2\frac{M^2}{U^2}\right)\le\varepsilon,\tag{4}$$

where ε is the PoF tolerance, or, equivalently, if

$$\mathrm{CF}=\frac{M}{U}\ge\sqrt{\log\sqrt{\frac{1}{\varepsilon}}},\tag{5}$$

where CF is the *confidence factor* (cf. [4,26,23]). In writing (2) and subsequently, we use the function $x_+=\max(0,x)$. We see from the preceding expressions that McDiarmid's inequality supplies

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rigorous quantitative definitions of design margin and system uncertainty. In particular, the latter is measured by the *system diameter* D_F, which measures the largest deviation in performance resulting from arbitrarily large perturbations of one input parameter at a time. We note from definition (3) that

$$D_F = \left\{ \sum_{i=1}^{N} D_{F,i}^2 \right\}^{1/2}, \tag{6}$$

where

$$D_{F,i} = \sup_{x_1,\ldots,x_N,x_i'} |F(x_1,\ldots,x_i,\ldots,x_N) - F(x_1,\ldots,x_i',\ldots,x_N)|^2 \tag{7}$$

is the *subdiameter* corresponding to variable X_i. The subdiameter $D_{F,i}$ may be regarded as a measure of the uncertainty contributed by the variable X_i to the total uncertainty of the system.

We note that, in the preceding framework, the quantification of system uncertainties, as measured by the system diameter D_F, Eq. (3), entails the solution of a global optimization problem. Each objective function evaluation in that optimization problem requires the execution of two integral tests for the evaluation of $F(X_1,\ldots,X_i,\ldots,X_N)$ and $F(X_1,\ldots,X_i',\ldots,X_N)$, respectively. For some systems, integral tests are prohibitively expensive, outside the scope of laboratory testing, or otherwise unfeasible. In this paper, we show how subsystem uncertainties, measured by the corresponding subsystem diameters, can be compounded to obtain rigorous upper bounds of the probability of failure of modular systems. Evidently, the composition of subsystem uncertainties requires a quantitative understanding of the interfaces through which the subsystems interact. In the present work, the strength of those interactions is measured by the *moduli of continuity* of the interfaces (cf., e.g., [5,27]). We specifically show that, once the individual subsystem uncertainties—the subsystem diameters—and the interaction strengths—the interface moduli of continuity—are known, a rigorous upper bound on the entire system diameter D_F can be computed. Since the McDiarmid inequality (1) is monotonic in D_F, the upper bound on D_F in turn provides a conservative upper bound on the PoF of the system and, by extension, a conservative certification criterion.

2. Moduli of continuity

The concept of *modulus of continuity* plays a pivotal role in the uncertainty analysis of modular systems presented in this paper. We recall that, given a function $f:\mathbb{R}^n \to \mathbb{R}^m$, a real number $\delta > 0$ and a subset $A \subset \mathbb{R}^n$, the modulus of continuity $\omega_{ij}(f,\delta,A)$ of $f_i(x)$ with respect to x_j over A is defined as [5,27]

$$\omega_{ij}(f,\delta,A) = \sup\{|f_i(x) - f_i(x')| \;:\; x,x' \in A,\ x_k = x_k' \text{ for } k \neq j,\ |x_j - x_j'| \leq \delta\}. \tag{8}$$

We note that the computation of $\omega_{ij}(f,\delta,A)$ requires the solution of a global optimization problem over a set of dimension $n+1$. It follows from definition (8) that

$$|f_i(x_1,\ldots,x_j,\ldots,x_n) - f_i(x_1,\ldots,x_j',\ldots,x_n)| \leq \omega_{ij}(f,\delta,A) \quad \text{if } |x_j - x_j'| \leq \delta. \tag{9}$$

Thus, $\omega_{ij}(f,\delta,A)$ measures the variation of the function $f_i(x)$ over A when the variable x_j is allowed to deviate by less than δ. We note that this component-wise definition of the modulus of continuity does not require the range or image of the function f to be a normed space. This is important in practice, since the inputs and outputs of subsystems often comprise variables of varying physical origins with diverse units which therefore define vector spaces with no natural norm. The definition of the modulus of continuity and its properties can naturally be extended to situations where f is a function from a Cartesian product of metric spaces onto another Cartesian product of metric spaces. In this case, the definition of the modulus of continuity is not independent from the metrics used on input and output spaces. We also note that the modulus of continuity is always well-defined, and its continuity at $\delta = 0$ is equivalent to the uniform continuity of the function, which places rather modest regularity requirements on the response functions of the interfaces. Evidently, the modulus of continuity is monotonic on δ and A, i.e.,

$$\omega_{ij}(f,\delta,A) \leq \omega_{ij}(f,\delta',A), \quad \text{if } \delta \leq \delta', \tag{10a}$$

$$\omega_{ij}(f,\delta,A) \leq \omega_{ij}(f,\delta,A'), \quad \text{if } A \subseteq A'. \tag{10b}$$

Suppose that A is a hyper-rectangle, i.e., $A = \prod_{i=1}^{n} [a_i,b_i]$. Let $\delta_j > 0$, $j = 1,\ldots,n$, and let $x,x' \in \mathbb{R}^n$ be such that $|x_j - x_j'| \leq \delta_j$. Then, writing

$$\begin{aligned} f_i(x) - f_i(x') = {} & f_i(x_1,x_2,\ldots,x_{n-1},x_n) - f_i(x_1',x_2,\ldots,x_{n-1},x_n) \\ & + f_i(x_1',x_2,\ldots,x_{n-1},x_n) - f_i(x_1',x_2',\ldots,x_{n-1},x_n) + \cdots \\ & + f_i(x_1',x_2',\ldots,x_{n-1},x_n) - f_i(x_1',x_2',\ldots,x_{n-1}',x_n) \\ & + f_i(x_1',x_2',\ldots,x_{n-1}',x_n) - f_i(x_1',x_2',\ldots,x_{n-1}',x_n') \end{aligned} \tag{11}$$

and using the triangular inequality, it follows that:

$$|f_i(x) - f_i(x')| \leq \sum_{j=1}^{n} \omega_{ij}(f,\delta_j,A). \tag{12}$$

A fundamental property of the moduli of continuity is that they are natural with respect to composition of functions in the following sense. Consider two functions $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g: B \subset \mathbb{R}^m \to \mathbb{R}^p$, with B a hyper-rectangle such that $f(A) \subset B$, and let $\delta > 0$. Let $g \circ f: A \subset \mathbb{R}^n \to \mathbb{R}^p$ be the composition of the f and g, i.e., $(g \circ f)(x) = g(f(x))$. Then, by (12) and monotonicity we have

$$\begin{aligned} & |g_i(f(x_1,\ldots,x_j,\ldots,x_n)) - g_i(f(x_1,\ldots,x_j',\ldots,x_n))| \\ & \quad \leq \sum_{k=1}^{m} \omega_{ik}(g,\omega_{kj}(f,\delta,A),B). \end{aligned} \tag{13}$$

Therefore, by the least-upper-bound property of the supremum it follows that

$$\omega_{ij}(g \circ f,\delta,A) \leq \sum_{k=1}^{m} \omega_{ik}(g,\omega_{kj}(f,\delta,A),B). \tag{14}$$

This inequality shows that the moduli of continuity of a composite function $g \circ f$ can be estimated conservatively from the moduli of continuity of the individual functions. This property of the moduli of continuity in turn enables uncertainties of the integral system to be bounded once the uncertainties of the subsystems and their interfaces are known, as shown next.

3. Uncertainty analysis of modular systems

We consider modular systems consisting of subsystems connected by unidirectional interfaces. The system may be regarded as an oriented graph $G(V,E)$ whose nodes V are the subsystems and whose edges E are the interfaces between the subsystems, Fig. 1. Specifically, a node a is an *ancestor* of a node b—and hence the graph contains and oriented edge from b to a—if the state of the subsystem a depends on the state of the subsystem b, i.e., if the outputs of subsystem b figure among the inputs of system a. Here and subsequently we adopt standard conventions and terminology from graph theory. In particular, we write $a \prec b$ if node a is an ancestor of node b, and $b \succ a$ if b is a descendant of a. Thus, $a \prec b$ if a takes input from b, and $b \succ a$ if b feeds into a.

We assume that the response of every subsystem is characterized by a function $F_a: \mathcal{X}_a \to \mathcal{Y}_a$ that maps vectors of input parameters $X_a \in \mathcal{X}_a$ to vectors of output performance measures $Y_a \in \mathcal{Y}_a$. Then, a node a is an ancestor of b in the graph $G(V,E)$ if $\mathcal{Y}_b$ is a subspace of $\mathcal{X}_a$. For purposes of analysis, we assume that

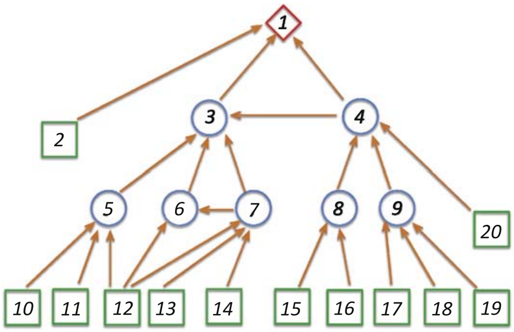

Fig. 1. Graph representation of the input–output relations between the subsystems of a modular system. The nodes of the graph represent the subsystems of the system. An arrow indicates that the outputs of the subsystem at the beginning of the arrow are among the inputs of the subsystem at the end of the arrow. The subsystems represented by square boxes do not take input from other systems and are referred to as *fundamental*. The system has one single *root* subsystem, i.e., a subsystem that does not feed into any other subsystem and through which the system output takes place.

the graph $G(V,E)$ is *acyclic*, i.e., it contains no closed-loop paths. We define the *fundamental* subsystems—respectively, *leaves* of the graph—as those subsystems that take no input from other subsystems—respectively, those nodes that have no descendants. We assume that the system contains a single *root subsystem*—respectively, *root node*—i.e., a subsystem that does not feed into any other subsystem—respectively, a node with no ancestors. We identify the output of the system with the output of its root subsystem. We designate by R the root node of the graph and by V_L the set of leaves of the graph. With these conventions, the space of inputs of the system is $\mathcal{X}=\prod_{a\in V_L}\mathcal{X}_a$, i.e., $\mathcal{X}$ is the Cartesian product of the input spaces of the fundamental subsystems, and the space of outputs is $\mathcal{Y}=\mathcal{Y}_R$. In addition, for all subsystems other than the fundamental ones, $a\notin V_L$, we assume the relation $\mathcal{X}_a=\prod_{b\succ a}\mathcal{Y}_b$, i.e., the input space of subsystems i is the Cartesian product of the output spaces of all its descendants. We note that non-fundamental subsystems could in principle have inputs of their own, not provided by any descendant subsystem. We can accommodate such cases within the present framework simply by adding a fundamental subsystem whose response function is the identity mapping and which supplies the requisite additional inputs. The function $F:\mathcal{X}\to\mathcal{Y}$ that describes the response of the integrated system is defined from the recursive algorithm:

(i) Input $X\in\mathcal{X}$. Set $V_0=V_L$, $k=0$.
(ii) Reset $V_{k+1}=\{a\in V:\ b\in\cup_{l=0}^{k}V_l,\ \forall b\succ a\}$.
(iii) For $a\in V_{k+1}$, compute $X_a=(F_b(X_b),\ b\succ a)$.
(iv) If $V_{k+1}=\{R\}$, output $Y=F_R(X_R)$, exit. Otherwise, reset k to $k+1$, go to (ii).

Thus, in the example of Fig. 1, the sequence of active subsystems is V_0={2,10,11,12,13,14,15,16,17,18,19,20}, V_1={5,7,8,9}, V_2={6,4}, V_3={3} and V_4={1}. Correspondingly, the sequence of subsystem outputs is (Y_5, Y_7, Y_8, Y_9), (Y_6, Y_4), Y_3 and $Y=Y_1$.

We now proceed to derive an upper bound on the system diameter D_F, cf. (3), by *propagating uncertainties* along the graph $G(V,E)$. As noted in the introduction, an upper bound on the system diameter immediately translates, through McDiarmid's inequality (1), on an upper bound on the PoF of the system and, by extension, a conservative certification criterion. Let $\{V_k,k=0,\dots,N\}$ be the sequence of nodal sets generated during the recursive definition of the integrated response function $F(X)$, with $V_0=V_L$ and V_N={R}. Define $\mathcal{X}_k=\prod_{a\in V_k}\mathcal{X}_a$ and $\mathcal{Y}_k=\prod_{a\in V_{k+1}}\mathcal{X}_a$. We note that $\mathcal{X}=\mathcal{X}_0$, $\mathcal{Y}_k=\mathcal{X}_{k+1}$ and, by assumption, $\dim\mathcal{Y}_N=\dim\mathcal{Y}=1$. Define further $F_k:\mathcal{X}_k\to\mathcal{Y}_k$ as the Cartesian product $F_k(X_k)=(F_a(X_a),a\in V_k)$, $X_k\in\mathcal{X}_k$. We then have the composition rule

$$F=F_N\circ\cdots\circ F_0. \tag{15}$$

With this notation and conventions, we may now proceed to state the main result of this paper.

Theorem 3.1. *Suppose that the system inputs are known to take values in a set $A\subset\mathcal{X}$. Let*

$$D_i=\sup\{|x_i-x_i'|:\ x,x'\in A,\ x_j=x_j',\ \text{for } j\neq i\}. \tag{16}$$

For $k=1,\dots,N$, let $B_k\subset\mathcal{X}_k$ be hyper-rectangles containing $(F_k\circ\cdots\circ F_0)(A)$. For $j=1,\dots,\dim\mathcal{X}_0$ let

$$D_{ij}^{(0)}=\omega_{ij}(F_0,D_j,A),\quad i=1,\dots,\dim\mathcal{Y}_0 \tag{17}$$

and for $k=1,\dots,N$ define the sequence

$$D_{ij}^{(k)}=\sum_{l=1}^{\dim\mathcal{X}_k}\omega_{il}(F_k,D_{lj}^{(k-1)},B_k),\quad i=1,\dots,\dim\mathcal{Y}_k. \tag{18}$$

Then,

$$D_{F,i}\leq D_{1i}^{(N)}. \tag{19}$$

Proof. The proof of inequality (19) is an immediate consequence of the definitions and the monotonicity and composition properties of the modulus of continuity. Thus, from (14) and monotonicity we have

$$\begin{aligned}\omega_{ij}(F_1\circ F_0,D_j,A)&\leq\sum_{l=1}^{\dim\mathcal{X}_1}\omega_{il}(F_1,\omega_{lj}(F_0,D_j,A),B_1)\\&=\sum_{l=1}^{\dim\mathcal{X}_1}\omega_{il}(F_1,D_{lj}^{(0)},B_1)=D_{ij}^{(1)}\end{aligned} \tag{20}$$

and

$$\begin{aligned}\omega_{ij}(F_2\circ F_1\circ F_0,D_j,A)&\leq\sum_{l=1}^{\dim\mathcal{X}_2}\omega_{il}(F_2,\omega_{lj}(F_1\circ F_0,\omega_{ij}(F_1\circ F_0,D_j,A),B_2)\\&\leq\sum_{l=1}^{\dim\mathcal{X}_2}\omega_{il}(F_2,D_{lj}^{(1)},B_2)=D_{ij}^{(2)}.\end{aligned} \tag{21}$$

Proceeding by induction we arrive at the inequality

$$\omega_{ij}(F_N\circ\cdots\circ F_0,D_j,A)\leq D_{ij}^{(N)}. \tag{22}$$

The bound (19) follows from the definition (7) of the subdiameters and the lowest-upper-bound property (9) of the moduli of continuity. $\square$

The sequence $D_{ij}^{(k)}$ may be regarded as a measure of uncertainty in the ith output variable due to the variability of the jth input variable after k levels of operation of the system. We note that the proof of the theorem is constructive in nature. Consequently, a natural implementation of the approach is simply to follow the constructions of the proof. The first step in the application of the approach to a specific system is to identify its graph structure and the inputs and outputs of each subsystem. In this work, we assume that the response of the subsystems can be determined experimentally. The next step in the analysis is to compute the ranges D_i of the input variables, Eq. (16). For every input variable, this operation requires the solution of a global optimization problem. However, this optimization problem becomes trivial if the range $A\subset\mathcal{X}$ of the input parameters is a hyper-rectangle, which corresponds to the case in which ranges are known independently for each input variable. Once the ranges D_i are known, we may proceed to the computation of the level-zero uncertainties. This computation entails the solution of a global optimization problem of the form (8) yielding the level-zero moduli of continuity $D_{ij}^{(0)}$, Eq. (17). In order to propagate uncertainties further, we need to determine a hyper-rectangle $B_1\subset\mathcal{Y}_0=\mathcal{X}_1$ that bounds the range of variation of the level-zero output variables, which in turn equal the level-one input variables. The smallest such hyper-rectangle is $B_1 = \prod_{i=1}^{\dim \mathcal{Y}_0} [\min F_{0i}, \max F_{0i}]$. Again, the computation of these ranges requires the solution of two global optimization problems in each variable. These operations can then be iterated until the root subsystem is reached. Uncertainties in the input variables associated with the leaf i are propagated through possibly multiple *paths* connecting the node i with the root 1. It is interesting to note that the computation of $D_{1i}^{(N)}$ additionally results in the identification of the path responsible for most of the *sensitivity* of the variable 1 with respect to the variable i, i.e., the path with the highest *flow of uncertainty*. It should be carefully noted that each level of analysis requires the execution of subsystem tests only, and that at no time during the analysis an integral test is required.

4. Example of application: fractal electrical circuit

In this section, we apply the preceding approach to a fractal electrical circuit. Specifically, we consider an LC electrical circuit consisting of a capacitance C connected to a network of random inductors placed on the edges of a Sierpinski triangle, Fig. 2. The Sierpinski triangle is a fractal set that arises in many applications, including multiband fractal antenna technologies [1]. The fractal geometry of the Sierpinski triangle is self-similar at all scales, which lends itself ideally to the type of graph-oriented hierarchical analysis formulated in the preceding section.

The equivalent inductance L_{eq} of the circuit may be written as a function

$$L_{eq} = F(L_1, \ldots, L_N). \tag{23}$$

of the individual inductances L_i of the inductors. We assume that the operation of the circuit requires that L_{eq} be above a threshold value. In this manner, L_{eq} is identified as a performance measure of interest. We recall that when a capacitor of capacitance C and an inductive circuit with an equivalent inductance L_{eq} are connected, as in the example under consideration, an electrical current can alternate between them at the resonant frequency $\Omega_r = 1/\sqrt{CL_{eq}}$ of the circuit. This resonance has applications in tuning and antenna technologies, voltage or current amplification, and induction heating [36,1]. For instance, tuning a radio set to a particular frequency can be achieved by adjusting the resonant frequency of an LC circuit.

The effective inductance can conveniently be computed by recourse to the $\Delta - Y$ transform [9], Fig. 3. We recall that the $\Delta - Y$ transform is an invertible map which translates inductive circuit elements from the Δ configuration to the Y (or star) configuration.

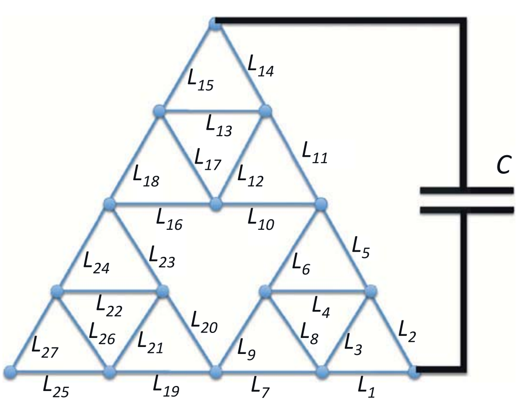

Fig. 2. An LC electrical circuit with inductors placed on the edges of a Sierpinski triangle of depth 3.

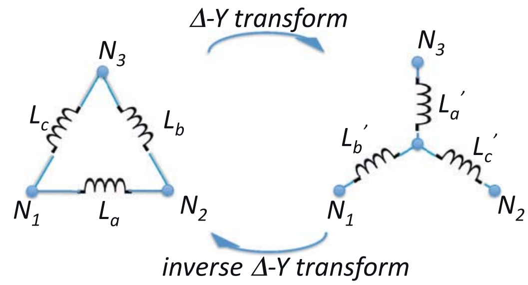

Fig. 3. The $\Delta - Y$ transform for the determination of the equivalent inductance L_{eq} of the Sierpinski LC circuit. The invertible map ($\Delta - Y$ transform) between (L_a, L_b, L_c) and (L'_a, L'_b, L'_c) is $L'_a = L_b L_c/(L_a + L_b + L_c)$, $L'_b = L_c L_a/(L_a + L_b + L_c)$, and $L'_c = L_a L_b/(L_a + L_b + L_c)$.

We note from Fig. 3 that the $\Delta - Y$ transform effectively removes the center node from the Y configuration. Thus, a recursive application of the $\Delta - Y$ transform generates a series of increasingly simpler LC circuits with the same equivalent inductance. The recursion ends in a circuit of one single element, whose inductance is precisely L_{eq}.

We define a hierarchical modular structure on the circuit by the natural grouping of elements shown in Fig. 4, consisting of replacing nine inductances by three equivalent ones. In order to fit the uncertainty analysis into the framework of Section 3, we regard every such grouping as a subsystem having the original nine inductances as inputs and the resulting three effective inductances as outputs. The inductances of the original circuit at the inputs of the total integrated system, whereas the equivalent inductance L_{eq} is its output. For instance, the depth-three Sierpinski triangle of Fig. 2 has 27 input parameters $(L_1, \ldots, L_{27})$, and the hierarchy of response functions corresponding to the scheme of Fig. 4 is

$$L_{eq} = F_2(L_{2,1}, L_{2,2}, L_{2,3}),$$

$$L_{2,1} = F_{1,1}(L_{1,1}, \ldots, L_{1,9}),$$

$$L_{2,2} = F_{1,2}(L_{1,1}, \ldots, L_{1,9}),$$

$$L_{2,3} = F_{1,3}(L_{1,1}, \ldots, L_{1,9}),$$

$$L_{1,1} = F_{0,1}(L_1, \ldots, L_9),$$

$$L_{1,2} = F_{0,2}(L_1, \ldots, L_9),$$

$$L_{1,3} = F_{0,3}(L_1, \ldots, L_9),$$

$$L_{1,4} = F_{0,4}(L_{10}, \ldots, L_{18}),$$

$$L_{1,5} = F_{0,5}(L_{10}, \ldots, L_{18}),$$

$$L_{1,6} = F_{0,6}(L_{10}, \ldots, L_{18}),$$

$$L_{1,7} = F_{0,7}(L_{19}, \ldots, L_{27}),$$

$$L_{1,8} = F_{0,8}(L_{19}, \ldots, L_{27}),$$

$$L_{1,9} = F_{0,9}(L_{19}, \ldots, L_{27}).$$

The graph structure of the resulting modular system is shown in Fig. 5.

The ranges $[\ell_i, u_i]$ of the input inductances L_i are chosen such that $u_i - \ell_i = 0.1(\ell_i + u_i)$ and such that the equivalent inductance of a circuit with $L_i = (\ell_i + u_i)/2$ is 1. Two cases are considered:

(i) Homogeneous input ranges: In this case, the ranges of all input inductances are taken to be identical.

(ii) Inhomogeneous input ranges: In this case, we consider three groups of identical inductance ranges over each one of the main subtriangles of the circuit.

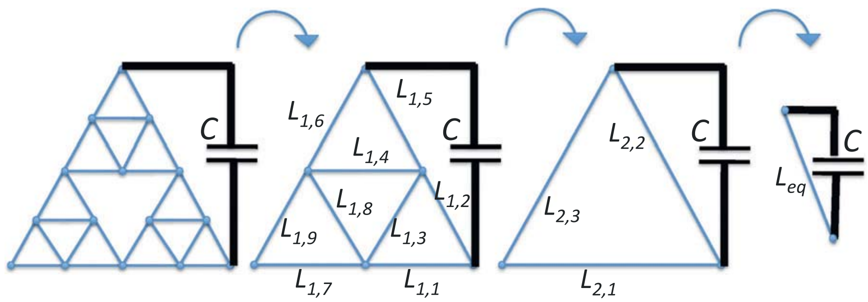

Fig. 4. Decomposition of the Sierpinski triangle of depth 3 into subsystems for purposes of modular UQ analysis.

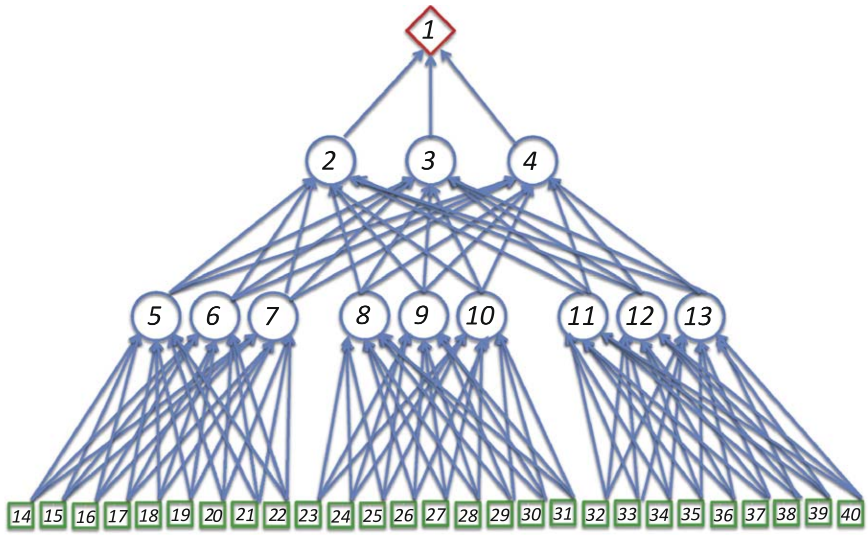

Fig. 5. Graph representation of the modular decomposition of the Sierpinski triangle of depth 3 shown in Fig. 4.

The modular uncertainty analysis formulated in Section 3 requires the solution of a number global optimization problems. In all such calculations we use a simulated annealing algorithm [10] adapted for appropriately generating random *neighbors* in the feasible set of the optimization problems. We use a default cooling schedule of $T_{new} = 0.8 \times T^{old}$ with $T_0 = 1.0$, where T is the numerical temperature. The optimization stops if $T \leq 10^{-8}$, $N > N_{max} = 2000$, or $N_R > 300$ where N be the number of function evaluations and N_R is the number of successive rejected states. Temperature decrease happens if $N_T > 30$ or $N_S > 20$, i.e., if 30 function evaluations are made or if there are 20 successive accepted optimal states found at the current temperature. In all calculations, the numerical Boltzmann constant is set to 1.0.

Numerical results for the resonance frequency diameters are collected in Fig. 6. As expected, the modular upper bound (19) lies above the system diameters. Thus, the modular upper bound (19) supplies a conservative estimate of the PoF of the system when inserted into McDiarmid's inequality (1) in place of the system diameter. In addition, we observe that the modular upper bound is tight, i.e., overestimates the system diameter by a modest amount. A cost analysis of the computation of the modular upper bound is also of interest. Thus, the direct computation of all system subdiameters of the depth-three Sierpinski triangle requires the solution of 27 global optimization problems in 28 variables each. By contrast, the computation of the modular upper bound requires the solution of a larger number of smaller global optimization problems. In addition, many of the global optimization problems are decoupled and can be solved concurrently.

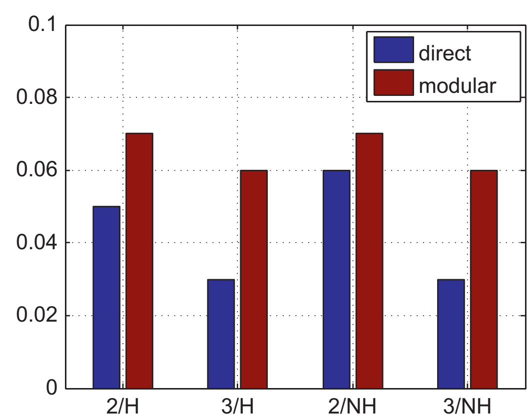

Fig. 6. Sierpinski triangles of depth $d=2$ and $d=3$ with homogeneous and inhomogeneous inductance ranges. Comparison of resonant frequency diameters and diameter upper bounds (19) computed from modular approach.

5. Concluding remarks

We have developed a modular/hierarchical uncertainty quantification framework based on concentration of measure inequalities for probability of failure upper bound calculations. In this framework, the relations between subsystems are represented by directed, acyclic graphs and the bounded uncertainty in the input variables is propagated to the output variable, the performance measure, inductively throughout the underlying graph structure. Most importantly, the approach bypasses the need to perform integral experiments, and all testing can be done at the subsystem level. In this manner, the present modular approach supplies rigorous PoF upper bounds and, by extension, conservative certification criteria, without integral testing. The approach also affords reductions in computational complexity, especially when the subsystems are small and weakly coupled. The feasibility of the approach has been demonstrated by means of an application to a fractal electrical circuit. In this particular application, the modular upper bounds of the system diameter are found to be tight.

It is important to note that the decomposition of a system into subsystems is not unique in general, and that the modular uncertainty upper bounds depend on the choice of decomposition. Evidently, the number of subsystems of a modular system can be decreased by grouping subsystems. Conversely, the number of subsystems can be increased by a further decomposition of the subsystems. An advantage of a fine-grained modular decomposition is that the resulting subsystems are simple and easy to test. However, the tightness of the modular uncertainty upper bound tends to deteriorate with an increasing number of subsystems, which results in the composition of a larger number of conservative upper-bound estimates. Conversely, grouping subsystems increases the tightness of the modular uncertainty upper bound. The tightest upper bound, namely, D_F, is obtained when the entire system is tested as a single unit. However, grouping has the disadvantage that all the subsystems in a group must be tested together. Therefore, in general the choice of modular decomposition is a compromise between the desire to avoid integral testing and the desire to obtain tight uncertainty estimates.

Several obvious extensions of the modular approach suggest themselves as the subject of further research. Thus, extensions to handle graph structures that are not necessarily acyclic, as well as fundamental variables that are not independent, would considerably broaden the applicability of the present approach. In addition, the differing levels of uncertainty attendant to the two performance measures considered in the Sierpinski triangle, namely, the equivalent inductance and the resonant frequency, illustrate the important fact that McDiarmid's inequality can be refined through an appropriate choice of coordinate transformation. For instance, using the results of the Sierpinski triangle analysis we find $\mathbb{P}[\Omega_r \geq 1.05] \leq 0.135$ from McDiarmid's inequality applied to L_{eq} and $\mathbb{P}[\Omega_r \geq 1.05] \leq 0.240$ from McDiarmid's inequality applied to Ω_r directly. The question of devising optimal coordinate transformations delivering the smallest possible PoF upper bounds remains open at present.

A question of practical concern is whether McDiarmid's inequality, which requires rather minimal knowledge of the distribution of the system inputs, supplies sharp enough PoF upper bounds in practice. Evidently, McDiarmid's inequality supplies small PoF upper bounds, thus possibly enabling certification, for systems whose diameter is small compared to the design margin, i.e., for systems whose response function has small oscillations over the operating range of the input parameters. This property may not be exhibited by systems of interest, e.g., systems whose response function exhibits *cliff behavior*, in which case the question arises as to how McDiarmid's inequality can be systematically improved upon. One QMU protocol that reuses McDiarmid's inequality while generating a convergent sequence of PoF upper bounds can be formulated through a recursive partitioning of the domain of the inputs [28]. By a convergent sequence of PoF upper bounds here we mean a sequence that converges to the exact PoF upper bound in the limit of an infinitely fine partitioning of the domain of the inputs. Because the partitioned QMU protocol is based on the use of McDiarmid's inequality over each subdomain in the partition, it is a simple matter to combine it with the modular QMU protocol presented in this paper. This combined protocol delivers rigorous and convergent PoF upper bounds without the need for integral testing.

In closing, we remark on the choice of McDiarmid's inequality as the basis of the work presented in this paper. McDiarmid's inequality is but a simple example of a large class of inequalities known collectively as 'concentration of measure' (CoM) inequalities (cf., e.g., [11] for a monograph and [2,13] for surveys). Several key properties of CoM inequalities render them attractive for purposes of QMU. Thus, as their name indicates, CoM inequalities become sharper as the dimensionality of the response function, i.e., the number of input parameters, increases. Thus, in this particular sense, CoM inequalities enjoy a *blessing of dimensionality*, as opposed to suffering from the *curse of dimensionality* that plagues other approaches. In addition, CoM inequalities are well-suited to systems for which failure is a *rare event*, e.g., as a result of a large design margin. Indeed, the evaluation of McDiarmid's upper bound requires the same amount of computation *regardless of the choice of margin*, i.e., regardless of whether or not failure is a rare event. By way of sharp contrast, we recall that Monte Carlo schemes experience great difficulty in dealing with rare events. The study of the CoM phenomenon was pioneered in the early seventies by V. Milman in his work on the asymptotic geometry of Banach spaces [21,20,19,22]. Far-reaching extensions, that in particular provide dimension-free concentration of measure inequalities in product spaces, have more recently been advanced by M. Talagrand, cf. [35,33,31,32,30,29,24]. A brief compendium of representative concentration of measure inequalities is collected in [13]. These include: convex-distance inequality [33,30,14]; CoM inequalities with correlated random variables [25,15,16,11,8]; CoM inequalities for empirical processes defined by sampling, [33,30,34,17]; and others. These more advanced inequalities supply avenues for extension of the present QMU methodology to modular systems including correlated inputs, inputs with known probability distributions, unbounded inputs and other cases of interest. However, advanced bounds such as those based on Talagrand's convex distance inequality, while possibly providing sharper bounds than McDiarmid's simple inequality, also render the propagation of uncertainties across subsystems far more costly, e.g., by coupling the computation of the subdiameters. In summary, McDiarmid's simple inequality suffices for the strict purposes of introducing the basic modular-system QMU protocol presented in this paper and, in particular, for demonstrating how complex systems can be rigorously certified without integral testing in practice, hence its choice as a convenient basis for the present work.

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